ON A COMBINATORIAL PROBLEM OF ERDŐS AND HAJNAL

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In this note we consider some problems related to the following question: What is the smallest integer $m(n)$ for which there exists a family F_n of sets $A_1, A_2, \ldots, A_{m(n)}$ with the following properties, (i) each member of F_n has n elements and (ii) if S is a set which meets each member of F_n, then S contains at least one member of F_n ?

Erdős and Hajnal [1] observed that

$$m(n) \leq \binom{2n-1}{n} < 4^n$$

and that $m(1) = 1$, $m(2) = 3$, $m(3) = 7$. We do not know the value of $m(n)$ for $n \geq 4$. Erdős [2] proved that, for all n ,

$$m(n) > 2^{n-1} ,$$

and that for $n \geq n_o(\epsilon)$,

$$m(n) > (1 - \epsilon)\, 2^n \log 2 .$$

It was communicated to us by Erdős that W. Schmidt recently proved that,

$$m(n) > 2^n (1 + 4n^{-1})^{-1} .$$

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In [2], Erdős remarks that it is not known whether or not

$$\lim_{n\to\infty} m(n)^{1/n}$$

exists. In this note we answer this question in the affirmative. In addition, we shall prove that for every $\epsilon > 0$,

$$(1) \qquad m(n) = O(\sqrt{7} + \epsilon)^n .$$

First we prove the following lemmas.

LEMMA 1. $m(ab) \leq m(a)\, m(b)^a$.

Proof. Let $F_a = \{A_1, A_2, \ldots, A_{m(a)}\}$ be a family of sets with properties (i) and (ii). Let

$$\bigcup_{i=1}^{m(a)} A_i = \{X_1, \ldots, X_\ell\}.$$ For $j = 1, 2, \ldots, \ell$, let

$F_b^j = \{B_1^j, B_2^j, \ldots, B_{m(b)}^j\}$ be families of sets, each with properties (i) and (ii). Also, suppose that no set in F_b^i meets any set in F_b^j, if $i \neq j$. Pick $A_i \in F_a$. We have $A_i = \{X_{i_1}, X_{i_2}, \ldots, X_{i_a}\}$, say. From each of the families $F_b^{i_1}, F_b^{i_2}, \ldots, F_b^{i_a}$ pick one B. The union of these B's is a set consisting of ab elements. Let F_{ab} be the family of all possible sets constructed in this way. It is clear that the number of sets in F_{ab} is $m(a)\, m(b)^a$. The proof of the lemma will be complete if we show that any set S which meets every member of F_{ab} contains at least one member of F_{ab}. Let $A_i = \{X_{i_1}, X_{i_2}, \ldots, X_{i_a}\} \in F_a$. A_i is said to have property P with respect to S if S meets every member of

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each of the families $F_b^{i_1}, F_b^{i_2}, \ldots, F_b^{i_a}$. There are two cases to consider.

Case 1. At least one $A_i \in F_a$ has property P with respect to S. Then we are finished since, if such is the case, S contains at least one member of each of the families $F_b^{i_1}, F_b^{i_2}, \ldots, F_b^{i_a}$ and hence contains at least one member of F_{ab}.

Case 2. No $A_i \in F_a$ has property P with respect to S. Then in each A_i there is an element, which we denote by $X_{A'_i}$ such that S misses at least one of the sets in $F_b^{A'_i}$. Now $T = \{X_{A'_i}; i = 1, 2, \ldots, m(a)\}$ meets every member of F_a and hence contains one of the A' s, say $A_j = \{X_{j_1}, X_{j_2}, \ldots, X_{j_a}\}$.

Hence S misses at least one set in each of the families $F_b^{j_1}$, $F_b^{j_2}, \ldots, F_b^{j_a}$ and therefore misses a set in F_{ab}, contrary to the assumption that S meets every member of F_{ab}. The proof of the lemma is complete.

LEMMA 2. $m(n+1) \geq m(n)$.

Proof. Consider a family of $m(n+1)$ sets with properties (i) and (ii). If an arbitrary element is deleted from each member of this family we get a new family of $m(n+1)$ sets each with n elements. If S meets each member of the new family then it meets each member of the old family. S thus contains a member of the old family and hence a member of the new family. The lemma follows.

We proceed to prove (1). Repeated application of lemma 1 and the fact that $m(3) = 7$ yields

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(2) $$m(3^k) \leq (\sqrt{7})^{3^k - 1} .$$

For given $\epsilon > 0$, let k be the smallest positive integer such that

(3) $$1 < \left(\frac{4}{\sqrt{7}}\right)^{\frac{1}{3^k}} \leq 1 + \frac{\epsilon}{\sqrt{7}} .$$

Then, if n is of the form $\ell \cdot 3^k$, we have

$$m(n) = m(\ell \cdot 3^k) \leq m(\ell)\, m(3^k)^{\ell}$$

$$\leq 4^{\ell} \, (\sqrt{7})^{(3^k - 1)^{\ell}} \leq (\sqrt{7} + \epsilon)^n ,$$

where we have used lemma 1, (2) and (3). If $\ell \cdot 3^k < n < (\ell + 1)3^k$, lemma 2 and some straightforward computation gives

$$m(n) \leq m((\ell + 1)3^k) < C(\sqrt{7} + \epsilon)^n .$$

Thus (1) is established.

In order to establish the existance of $\lim_{n\to\infty} m(n)^{1/n}$ we make use of the following fact; if a and b are independent positive integers (neither one is a power of the other) then the set of integers of the form $a^u b^v$, where u and v run through all positive integers, is dense, in the sense that for given $\epsilon > 0$ and n sufficiently large there is a number of the form $a^u b^v$ satisfying $n < a^u b^v < n(1+\epsilon)$. Clearly a and $a-1$ are independent.

Let

$$C_1 = \underline{\lim}_{n=\infty} m(n)^{1/n} \leq \overline{\lim}_{n=\infty} m(n)^{1/n} = C_2 .$$

For arbitrary fixed $\epsilon > 0$ let a be a positive integer satisfying

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$m(a) < (C_1 + \epsilon)^a$. Then, using lemma 1, it is not difficult to see that for u and v sufficiently large

$$(4) \qquad m(a^u(a-1)^v) \le (C_1 + 3\epsilon)^{a^u (a-1)^v} .$$

If n is sufficiently large, u and v can be determined so that $n < a^u (a-1)^v < n(1+\epsilon)$. By lemmas 1 and 2 and (4)

$$m(n) \le m(a^u(a-1)^v) \le (C_1 + 3\epsilon)^{a^u (a-1)^v} .$$

Hence

$$m(n)^{1/n} \le (C_1 + 3\epsilon)^{1+\epsilon}$$

for all sufficiently large n. It follows that $C_1 = C_2$.

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