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Branching Rules for Ramified Principal Series Representations of GL(3) over a p-adic Field

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Abstract. We decompose the restriction of ramified principal series representations of the p-adic group $\mathrm{GL}(3,\mathrm{k})$ to its maximal compact subgroup $K = \mathrm{GL}(3,\mathcal{R})$. Its decomposition is dependent on the degree of ramification of the inducing characters and can be characterized in terms of filtrations of the Iwahori subgroup in K. We establish several irreducibility results and illustrate the decomposition with some examples.

1 Introduction

The complex representations of p-adic algebraic groups are of great interest, both in their own right and in what they can reveal through the Langlands program in number theory. The representation theory of p-adic groups also often mirrors the theory for real Lie groups, and it is especially interesting to see how analogous results will develop.

To this end, one goal is to examine the finer structure of representations by considering their restrictions to compact open subgroups. The theory of types promises that one can classify representations in the Bernstein decomposition by identifying, among certain representations of compact open subgroups, which ones they contain. In contrast, in the theory of real Lie groups, the *maximal* compact subgroups have a crucial role, encoding as they do all the topology of the group, and one classifies irreducible unitary representations by classifying the irreducible Harish-Chandra modules. Our interest is to explore the extent to which information about the representations of the p-adic group resides in the maximal compact subgroup.

Representations of compact subgroups of p-adic groups are very tangible at a number of levels. First, the representations of sufficiently small (exponentiable) compact open subgroups can all be constructed using Kirillov theory, as shown by Howe [H2]. Secondly, each compact open subgroup is pro-finite and consequently its representation theory is largely determined by the representation theory of Lie groups over finite local rings. Finally, any admissible representation of a p-adic group decomposes with finite multiplicity upon restriction to a compact open subgroup and so one can expect to recover information about the original representation by examining these constituents.

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That said, the maximal compact subgroups are not exponentiable so Howe's theory does not apply. Furthermore the representation theory of Lie groups over local rings is just beginning to see significant progress (see, for example, [L]). Therefore one objective of this study is to provide some interchange between the representation theories of p-adic groups and of Lie groups over local rings.

Let k be a p-adic field (that is, a local non-archimedean field) and denote by $\mathcal{R}$ its integer ring. In this paper we consider the group $G = \mathrm{GL}(3, \mathrm{k})$ and let $K = \mathrm{GL}(3, \mathcal{R})$ be a maximal compact subgroup. In [CN], the authors considered unramified principal series representations and showed how their restriction to K decomposed as per the double cosets in K of smaller compact open subgroups C_{c} (defined in Section 2). In [CN], the added assumption that the inducing character was trivial on the compact part of the torus implied that every double coset supported an intertwining operator of the representation, an assumption we relax here.

This paper is organized as follows. In Section 2 we set our notation and recall some necessary results from [CN]. The key calculation for determining the decomposition is the determination of the double cosets in $C_{\mathsf{c}} \backslash K / C_{\mathsf{d}}$ which support intertwining operators for the restricted principal series representation; this is the main result in Section 3. We go on to consider questions of irreducibility in Section 4 and conclude with several examples to illustrate these decompositions in Section 5.

The question of parameterizing double cosets of the upper triangular subgroup B in K, and more generally of the subgroups $C_{(n,n,n)}$ in K, has been visited and solved by several authors with various goals in mind. In [OPV] the goal was to look at which Bruhat decompositions would be independent of the characteristic of the residue field; the answer was that only $\mathrm{GL}(2, \mathrm{k})$ has this property. This implies, in particular, that the decomposition of principal series is essentially independent of p for $\mathrm{GL}(2, \mathrm{k})$ (see [N, Si]) but will depend on the properties of the residue field in all other cases.

Several authors have considered related questions on the decomposition of representations of p-adic groups upon restriction to a maximal compact subgroup. These include the work of Silberger on $\mathrm{GL}(2, \mathrm{k})$ [Si], Nevins on $\mathrm{SL}(2, \mathrm{k})$ [N], and Bader and Onn [BO] on the Grassmann representation of $\mathrm{GL}(n, \mathrm{k})$. Paskunas proved that every irreducible supercuspidal representation of $\mathrm{GL}(n, \mathrm{k})$ has a unique Bushnell–Kutzko type appearing as a multiplicity 1 component of the restriction to the maximal compact subgroup $\mathrm{GL}(n, \mathcal{R})$ [P]. Gregory Hill has also constructed classes of representations of $\mathrm{GL}(n, \mathcal{R})$ [Hi]; a key part of his results was the determination of the double cosets of the subgroups $C_{(0,j,j)}$ in K.

2 Notation and Background

Let k be a local non-archimedean field of residual characteristic p. Let q denote the number of elements in the residue field of k. We assume throughout that $p > 2$ and $q > 3$. Denote the integer ring of k by $\mathcal{R}$ and the maximal ideal of $\mathcal{R}$ by $\mathcal{P}$. Choose a uniformizer π and normalize the discrete valuation on k so that $\mathrm{val}(\pi) = 1$. Define $\mathcal{U}^0 = \mathcal{R}^\times$ and $\mathcal{U}^i = 1 + \mathcal{P}^i$ for $i > 0$.

Let $G = \mathrm{GL}(3, \mathrm{k})$ and let $K = \mathrm{GL}(3, \mathcal{R})$. Write T_G for the diagonal torus in G and B_G for the upper triangular Borel subgroup. Write $T = T_G \cap K$ and $B = B_G \cap K$ for their intersections with K.

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2.1 Principal Series and Posets

Let χ_G be a character, not necessarily unitary, of the torus T_G and extend it trivially over the subgroup B_G. Then the (normalized) induced representation $\phi_G = \operatorname{Ind}_{B_G}^G \chi_G$ is a principal series representation of G. We consider its restriction to K. Writing $\chi = \chi_G|_T$ and $\phi = \phi_G|_K$, we have that $\phi = \operatorname{Ind}_B^K \chi$, since K is a good maximal compact subgroup. The principal series representation is called *ramified* if $\chi \neq \mathbf{1}$. The unramified case was considered in [CN].

Given a ramified character χ_G of T_G, we may write it as $\chi_G = (\chi_1, \chi_2, \chi_3)$ for characters $\chi_i \colon \mathrm{k}^\times \to \mathbb{C}^\times$. Recall that the *conductor* of a character χ_i of $\mathrm{k}^\times$ is the least $m \geq 0$ such that $\mathcal{U}^m \subseteq \ker(\chi_i)$; thus we make the convention that $\operatorname{cond}(\chi_i) = 0$ if and only if $\chi_i|_{\mathcal{R}^\times} = \mathbf{1}$.

The use of normalized induction implies that $\operatorname{Ind}_{B_G}^G \chi \simeq \operatorname{Ind}_{B_G}^G \chi^w$ for any w in the Weyl group of G, so we may reorder the characters χ_i in a convenient way. Moreover, if ψ is a character of $\mathrm{k}^\times$ and $\psi \cdot \chi = (\psi\chi_1, \psi\chi_2, \psi\chi_3)$, then

$$\operatorname{Ind}_{B_G}^G \psi \cdot \chi = (\psi \circ \det) \operatorname{Ind}_{B_G}^G \chi.$$

It follows that we may assume that $\chi_1 = \mathbf{1}$ and that

$$0 \leq M = \operatorname{cond}(\chi_2) \leq \operatorname{cond}(\chi_3) = N.$$

Then $\operatorname{cond}(\chi_1\chi_2^{-1}) = M$ and we may furthermore assume that $\operatorname{cond}(\chi_2\chi_3^{-1}) = \operatorname{cond}(\chi_1\chi_3^{-1}) = N$. Define $\mathtt{m} = (M, N, N)$. We will assume throughout that $\chi \neq \mathbf{1}$, so in particular $\chi_3 \neq \mathbf{1}$ and $N > 0$.

Let $\mathtt{T} = \{\mathtt{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3 \mid 0 \leq c_1, c_2 \leq c_3 \leq c_1 + c_2\}$ and note that $\mathtt{m} = (M, N, N) \in \mathtt{T}$. Then $\mathtt{T}$ is a poset with $\mathtt{c} \preceq \mathtt{d}$ if $c_i \leq d_i$ for all i. Define the subposet $\mathtt{T}_\mathtt{m} = \{\mathtt{c} \in \mathtt{T} \mid \mathtt{c} \succeq \mathtt{m}\}$. Then for any $\mathtt{c}, \mathtt{d} \in \mathtt{T}_\mathtt{m}$, their greatest common descendant $\gcd\{\mathtt{c}, \mathtt{d}\}$ exists and is unique. It is defined as the maximal element of the set $\{\mathtt{e} \in \mathtt{T}_\mathtt{m} \mid \mathtt{e} \preceq \mathtt{c}, \mathtt{d}\}$ and is given explicitly by $e_i = \min\{c_i, d_i\}$ for $i = 1, 2$ and $e_3 = \min\{c_3, d_3, e_1 + e_2\}$. For example, if $\mathtt{c} = (1, 3, 4)$ and $\mathtt{d} = (2, 2, 4)$, then $\gcd\{\mathtt{c}, \mathtt{d}\} = (1, 2, 3) \neq \min\{\mathtt{c}, \mathtt{d}\}$.

Given $\mathtt{c} \in \mathtt{T}$, we define a subgroup $C_\mathtt{c}$ by

$$C_\mathtt{c} = \begin{bmatrix} \mathcal{R} & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^{c_1} & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^{c_3} & \mathcal{P}^{c_2} & \mathcal{R} \end{bmatrix} \cap K.$$

Then $C_\mathtt{c} \subseteq C_\mathtt{d}$ if and only if $\mathtt{c} \succeq \mathtt{d}$.

Let K_n denote the n-th principal congruence subgroup of K, that is, the normal subgroup of K consisting of all those matrices which are equivalent to the identity matrix modulo $\mathcal{P}^n$. Then for all $\mathtt{c} \in \mathtt{T}$ we have $C_\mathtt{c} \supset K_{c_3}$.

Subrepresentations Let χ be the restriction to T of a character of T_G, with the above conventions. Then, in particular, $\chi_1 = \mathbf{1}$. If $\mathtt{c} \in \mathtt{T}_\mathtt{m}$, then we can extend χ to a character of $C_\mathtt{c}$, denoted $\chi_\mathtt{c}$ or simply χ if there is no possibility of confusion. Namely, given $g = (g_{ij}) \in C_\mathtt{c}$, we define $\chi_\mathtt{c}(g) = \chi_2(g_{22})\chi_3(g_{33})$. One verifies directly that this is multiplicative exactly when $c_1 \geq M$ and $c_2, c_3 \geq N$.

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Definition 2.1 For each $\mathtt{c} \in \mathtt{T_m}$, set $U_{\mathtt{c}} = \operatorname{Ind}_{C_{\mathtt{c}}}^{K} \chi_{\mathtt{c}}$.

We have from [CN] that $\dim U_{\mathtt{c}} = (q+1)(q^2+q+1)q^{c_1+c_2+c_3-3}$ if $c_1c_2 > 0$ and $\dim U_{\mathtt{c}} = (q^2+q+1)q^{2(c_1+c_2-1)}$ if exactly one of c_1 or c_2 is zero. The representation $U_{\mathtt{c}}$ is naturally a subrepresentation of ϕ. In fact, it is contained in the subspace of K_{c_3}-fixed vectors of ϕ, a space which itself can be identified with $U_{\mathtt{d}}$, where $\mathtt{d} = (c_3, c_3, c_3)$. Consequently, one may also view $U_{\mathtt{c}}$ as a representation of the finite group K/K_{c_3}.

Note that $U_{\mathtt{d}} \subseteq U_{\mathtt{c}}$ if and only if $\mathtt{d} \preceq \mathtt{c}$. In fact, we can say slightly more.

Lemma 2.2 *Let* $\mathtt{c}, \mathtt{d} \in \mathtt{T_m}$ *and set* $\mathtt{e} = \gcd\{\mathtt{c}, \mathtt{d}\}$. *Then* $U_{\mathtt{c}} \cap U_{\mathtt{d}} = U_{\mathtt{e}}$.

Proof Since $\mathtt{e} \preceq \mathtt{c}, \mathtt{d}$, we immediately have $U_{\mathtt{e}} \subset U_{\mathtt{c}} \cap U_{\mathtt{d}}$. The opposite inclusion follows because $C_{\mathtt{e}}$ is the subgroup generated by $C_{\mathtt{c}}$ and $C_{\mathtt{d}}$. ■

Now consider the quotient

$$V_{\mathtt{c}} = U_{\mathtt{c}} / \sum_{\mathtt{d} \in \mathtt{T_m}, \mathtt{d} \prec \mathtt{c}} U_{\mathtt{d}}.$$

This quotient can be identified with a summand of ϕ. These summands are the building blocks of the decomposition of ϕ that we wish to study, so let us refine our description of $V_{\mathtt{c}}$.

Let $\mathtt{c} \in \mathtt{T_m}$. If $c_1c_2 \neq 0$, define for each $i \in \{1,2,3\}$ the triple

$$\mathtt{c}_{\{i\}} = (c_1 - \delta_{i1}, c_2 - \delta_{i2}, c_3 - \delta_{i3}) \in \mathbb{Z}^3.$$

If $c_1 = 0$, then only consider $\mathtt{c}_{\{3\}} = (0, c_2 - 1, c_2 - 1)$ and if $c_2 = 0$, then only consider $\mathtt{c}_{\{3\}} = (c_1 - 1, 0, c_1 - 1)$. Set $\mathcal{S}_{\mathtt{c}} = \{i \mid \mathtt{c}_{\{i\}} \in \mathtt{T_m}\}$. Then for all $\mathtt{d} \prec \mathtt{c}$ such that $\mathtt{d} \in \mathtt{T_m}$, there is some $i \in \mathcal{S}_{\mathtt{c}}$ such that $\mathtt{d} \preceq \mathtt{c}_{\{i\}}$.

Further, let $\mathtt{c}_{\varnothing} = \mathtt{c}$ and for each non-empty $I \subseteq \mathcal{S}_{\mathtt{c}}$ set $\mathtt{c}_I = \gcd\{\mathtt{c}_{\{i\}} \mid i \in I\}$. Thus $\mathcal{S}_{\mathtt{c}}$ defines a poset $\{U_{\mathtt{c}_I} \mid I \subseteq \mathcal{S}_{\mathtt{c}}\}$ of subrepresentations of $U_{\mathtt{c}}$ with the property that, for all $\mathtt{d} \in \mathtt{T_m}$ such that $U_{\mathtt{d}} \subseteq U_{\mathtt{c}}$, if $U_{\mathtt{d}} \neq U_{\mathtt{c}_I}$ for any $I \subseteq \mathcal{S}_{\mathtt{c}}$, then $U_{\mathtt{d}} \subsetneq U_{\mathtt{c}_{\mathcal{S}_{\mathtt{c}}}}$. The argument of [CN, Proposition 3.3] therefore applies to give the following result in the Grothendieck group of K.

Theorem 2.3 *For any* $\mathtt{c} \in \mathtt{T_m}$ *we have*

$$[V_{\mathtt{c}}] = \sum_{I \subseteq \mathcal{S}_{\mathtt{c}}} (-1)^{|I|} [U_{\mathtt{c}_I}],$$

where $[V]$ *denotes the equivalence class of* V *in the Grothendieck group of* K.

Since the $U_{\mathtt{c}}$ are essentially induced representations of finite groups, the dimension $\mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{d}})$ of the space of intertwining operators between $U_{\mathtt{c}}$ and $U_{\mathtt{d}}$ is equal to $\dim \mathcal{H}(\chi_{\mathtt{c}}, \chi_{\mathtt{d}})$, where

$$\mathcal{H}(\chi_{\mathtt{c}}, \chi_{\mathtt{d}}) = \{f \colon K \to \mathbb{C} \mid f(gkg') = \chi_{\mathtt{c}}(g) f(k) \chi_{\mathtt{d}}(g') \quad \forall g \in C_{\mathtt{c}}, g' \in C_{\mathtt{d}}\}.$$

As an immediate corollary of the above theorem we therefore have an effective means of determining the number of intertwining operators between the various quotients $V_{\mathtt{c}}$.

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Corollary 2.4 *Let $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$. Then the dimension of the space of intertwining operators between $V_{\mathtt{c}}$ and $V_{\mathtt{d}}$ is*

$$\mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{d}}) = \sum_{\substack{I \subseteq \mathcal{S}_{\mathtt{c}} \\ J \subseteq \mathcal{S}_{\mathtt{d}}}} (-1)^{|I|+|J|}\, \mathcal{I}(U_{\mathtt{c}_I}, U_{\mathtt{d}_J}).$$

2.2 Distinguished Double Coset Representatives of $C_{\mathtt{c}} \backslash K / C_{\mathtt{d}}$

We recall the parametrization of representatives for the double coset space $C_{\mathtt{c}} \backslash K / C_{\mathtt{d}}$ as given in [CN].

Let $\mathtt{T}^1 = \{\mathtt{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3 \mid 1 \le a_1, a_2 \le a_3\}$. Given $\mathtt{c} \in \mathtt{T}$, define $\underline{\mathtt{c}} \in \mathtt{T} \cap \mathtt{T}^1$ by $\underline{c}_i = \max\{c_i, 1\}$ for each i.

Definition 2.5 For any $\mathtt{c}, \mathtt{d} \in \mathtt{T}$, set

$$\mathtt{T}_{\mathtt{c},\mathtt{d}} = \left\{ \mathtt{a} \in \mathtt{T}^1 \mid \mathtt{a} \preceq \underline{\mathtt{c}}, \mathtt{a} \preceq \underline{\mathtt{d}} \text{ and } a_3 \le \min\{a_1 + \underline{c}_2, \underline{d}_1 + a_2\} \right\}$$

with the following exceptions:

$$\mathtt{T}_{\mathtt{c},\mathtt{d}} = \begin{cases} \{(1,1,1)\} & \text{if } \mathtt{c} \text{ or } \mathtt{d} \text{ equals } (0,0,0), \\ \{(1,a,a) \mid a \le \min\{c_2, d_2\}\} & \text{if } c_2 d_2 > 0 \text{ and } c_1 = d_1 = 0, \\ \{(a,1,a) \mid a \le \min\{c_1, d_1\}\} & \text{if } c_1 d_1 > 0 \text{ and } c_2 = d_2 = 0. \end{cases}$$

Next, for $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}}$ set $\min\{\mathtt{a}\} = \min\{a_1, a_2, a_3\}$. Then we define

$$\mathtt{a}(\mathtt{c},\mathtt{d}) = \max\Big\{ 0, \min\{a_1, a_2, a_3 - a_1, a_3 - a_2, \mathtt{c} - \mathtt{a}, \mathtt{d} - \mathtt{a}, \\ a_1 + c_2 - a_3, d_1 + a_2 - a_3\}\Big\}$$

and

$$\mathtt{a}(\mathtt{c},\mathtt{d})' = \max\{0, \min\{d_3 - a_3, c_3 - a_3, c_1 - a_1, d_2 - a_2\}\} \ge \mathtt{a}(\mathtt{c},\mathtt{d}).$$

Now identify $\mathcal{R}/\mathcal{P}^i$ with a set of representatives in $\mathcal{R}$ chosen so that they contain the representatives corresponding to $\mathcal{R}/\mathcal{P}^j$ for all $j < i$, and so that the representative of the zero element of $\mathcal{R}/\mathcal{P}^i$ has valuation i. Set $\mathcal{R}/\mathcal{P}^0 = \{1\}$, and for $i > 0$ set $(\mathcal{R}/\mathcal{P}^i)^\times = \{x \in \mathcal{R}/\mathcal{P}^i \mid \operatorname{val}(x) = 0\}$. With these conventions, define

$$\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}} = \begin{cases} (\mathcal{R}/\mathcal{P}^{\mathtt{a}(\mathtt{c},\mathtt{d})})^\times & \text{if } a_1 + a_2 \ne a_3, \\ \bigcup_{i=0}^{\mathtt{a}(\mathtt{c},\mathtt{d})'} (\mathcal{U}^i \setminus \mathcal{U}^{i+1}) \cap (\mathcal{R}/\mathcal{P}^{\mathtt{a}(\mathtt{c},\mathtt{d})+i})^\times \cap (\mathcal{R}/\mathcal{P}^{\mathtt{a}(\mathtt{c},\mathtt{d})'})^\times & \text{if } a_1 + a_2 = a_3. \end{cases}$$

In other words, in this latter case, any two elements $x, y \in \mathcal{U}^i \setminus \mathcal{U}^{i+1}$ represent the same element of $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}$ if and only if $\operatorname{val}(x - y) \ge \min\{i + \mathtt{a}(\mathtt{c},\mathtt{d}), \mathtt{a}(\mathtt{c},\mathtt{d})'\}$.

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Definition 2.6 Let $\mathtt{c}, \mathtt{d} \in \mathtt{T}$. Enumerate the elements of $W \simeq S^3$ as

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, w_0\},$$

where s_i is the transposition $(i \quad i+1)$ and w_0 is the longest element. Define a subset $W_{\mathtt{c},\mathtt{d}}$ of W as

$$W_{\mathtt{c},\mathtt{d}} = \begin{cases} W & \text{if } \mathtt{c}, \mathtt{d} \succeq (1,1,1), \\ \{1, s_1, w_0\} & \text{if } c_1d_1(c_2+d_2) > 0 \text{ and } c_2d_2 = 0, \\ \{1, s_2, w_0\} & \text{if } c_1d_1 = 0 \text{ and } (c_1+d_1)c_2d_2 > 0, \\ \{1, w_0\} & \text{if } c_1c_2 = 0 \text{ and } d_1d_2 = 0 \text{ but } (c_1+c_2)(d_1+d_2) > 0, \\ \{1\} & \text{if } \mathtt{c} = (0,0,0) \text{ or } \mathtt{d} = (0,0,0). \end{cases}$$

The following theorem is proved in [CN].

Proposition 2.7 *Let* $\mathtt{c}, \mathtt{d} \in \mathtt{T}$. *A complete set of distinct double coset representatives* $\mathtt{R}_{\mathtt{c},\mathtt{d}}$ *of* $C_{\mathtt{c}} \backslash K / C_{\mathtt{d}}$ *is* $\mathtt{R}_{\mathtt{c},\mathtt{d}} = \bigcup_{w \in W_{\mathtt{c},\mathtt{d}}} \mathtt{R}^w_{\mathtt{c},\mathtt{d}}$, *where for* $w \in W_{\mathtt{c},\mathtt{d}}$ *we define* $\mathtt{R}^w_{\mathtt{c},\mathtt{d}}$ *as follows.*

(i) $$\mathtt{R}^1_{\mathtt{c},\mathtt{d}} = \left\{ t_{\mathtt{a},x} = \begin{bmatrix} 1 & 0 & 0 \\ \pi^{a_1} & 1 & 0 \\ x\pi^{a_3} & \pi^{a_2} & 1 \end{bmatrix} \;\middle|\; \mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}}, x \in \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}} \right\};$$

(ii) $$\mathtt{R}^{s_1}_{\mathtt{c},\mathtt{d}} = \left\{ s_1^{(\alpha,\beta)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \pi^\beta & \pi^\alpha & 1 \end{bmatrix} \;\middle|\; \begin{matrix} 1 \le \alpha \le \min\{\underline{d}_2, c_3\} \\ 1 \le \beta \le \min\{\underline{c}_2, d_3\} \\ -c_1 \le \beta - \alpha \le d_1 \end{matrix} \right\};$$

(iii) $$\mathtt{R}^{s_2}_{\mathtt{c},\mathtt{d}} = \left\{ s_2^{(\alpha,\beta)} = \begin{bmatrix} 1 & 0 & 0 \\ \pi^\beta & 0 & 1 \\ \pi^\alpha & 1 & 0 \end{bmatrix} \;\middle|\; \begin{matrix} 1 \le \alpha \le \min\{\underline{d}_1, c_3\} \\ 1 \le \beta \le \min\{\underline{c}_1, d_3\} \\ -c_2 \le \beta - \alpha \le d_2 \end{matrix} \right\};$$

(iv) $$\mathtt{R}^{s_1s_2}_{\mathtt{c},\mathtt{d}} = \left\{ s_1s_2^{(\alpha)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \pi^\alpha & 1 & 0 \end{bmatrix} \;\middle|\; 1 \le \alpha \le \min\{d_1, c_2\} \right\};$$

(v) $$\mathtt{R}^{s_2s_1}_{\mathtt{c},\mathtt{d}} = \left\{ s_2s_1^{(\alpha)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \pi^\alpha & 1 \\ 1 & 0 & 0 \end{bmatrix} \;\middle|\; 1 \le \alpha \le \min\{c_1, d_2\} \right\};$$

(vi) $$\mathtt{R}^{w_0}_{\mathtt{c},\mathtt{d}} = \left\{ w_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

3 Determination of the Set of Double Cosets Supporting Intertwining Operators

To understand the space of intertwining operators of the finite-dimensional representations $U_{\mathtt{c}}$ given in Definition 2.1, we construct bases for the spaces $\mathcal{H}(\chi_{\mathtt{c}}, \chi_{\mathtt{d}})$. That is, for any $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$ we must identify among the double cosets enumerated in Proposition 2.7 those which support intertwining operators of $U_{\mathtt{c}}$ with $U_{\mathtt{d}}$. Denote the

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subset of these cosets by $\mathtt{S}_{\mathtt{c},\mathtt{d}} \subseteq \mathtt{R}_{\mathtt{c},\mathtt{d}}$, and write $\mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{d}}) = \dim \mathcal{H}(\chi_{\mathtt{c}}, \chi_{\mathtt{d}}) = |\mathtt{S}_{\mathtt{c},\mathtt{d}}|$. (Note that in the case that $\chi = \mathbf{1}$, which we continue to exclude, $\mathtt{R}_{\mathtt{c},\mathtt{d}} = \mathtt{S}_{\mathtt{c},\mathtt{d}}$ and there is nothing to show.)

If $\mathtt{a} \in \mathtt{T}^1$, then define $\mathtt{a}^{\mathrm{op}} = (a_3 - a_2, a_3 - a_1, (a_3 - a_1) + (a_3 - a_2))$.

Theorem 3.1 *Let $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$, with $\mathtt{m} = (M, N, N) \succ (0, 0, 0)$ as before. Then a set of representatives for double cosets in $C_{\mathtt{c}} \backslash K / C_{\mathtt{d}}$ supporting elements of $\mathcal{H}(\chi_{\mathtt{c}}, \chi_{\mathtt{d}})$ is*

$$\mathtt{S}_{\mathtt{c},\mathtt{d}} = \bigcup_{w \in W_{\mathtt{c},\mathtt{d}}} \mathtt{S}^{w}_{\mathtt{c},\mathtt{d}},$$

where the subsets $\mathtt{S}^w_{\mathtt{c},\mathtt{d}} \subseteq \mathtt{R}^w_{\mathtt{c},\mathtt{d}}$ are defined as follows.

(i) $\mathtt{S}^1_{\mathtt{c},\mathtt{d}}$ *is the set of all $t_{\mathtt{a},x}$ with $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}}$ and $x \in \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}$ such that one of the following holds:*

- (1) *$a_1 \geq M$ and $a_2 \geq N$;*
- (2) *$a_1 < M$, $a_2 \geq N$, and one of the following holds:*
  - (a) *$a_1 + a_2 < a_3$ and $M \leq \min\{\mathtt{c} - \mathtt{a}, \mathtt{d} - \mathtt{a}, c_2 + a_1 - a_3, d_1 + a_2 - a_3\}$,*
  - (b) *$a_1 + a_2 > a_3$ and $M \leq \min\{\mathtt{c} - \mathtt{a}^{\mathrm{op}}, \mathtt{d} - \mathtt{a}^{\mathrm{op}}\}$,*
  - (c) *$a_1 + a_2 = a_3$ and $M \leq \min\{\mathtt{c} - \mathtt{a}, \mathtt{d} - \mathtt{a}\}$ and $\mathrm{val}(x - 1) \leq \mathtt{a}(\mathtt{c}, \mathtt{d})' - M$;*
- (3) *$a_1 \geq N$ and $a_2 < N$ and the same conditions* (a), (b), (c) *with M replaced by N;*

(ii) $\mathtt{S}^{s_1}_{\mathtt{c},\mathtt{d}}$ *is the set of all $s_1^{(\alpha,\beta)}$ such that*

$$N \leq \alpha \leq \min\{d_2, c_3\} - M,$$
$$N \leq \beta \leq \min\{c_2, d_3\} - M,$$
$$M - c_1 \leq \beta - \alpha \leq d_1 - M;$$

(iii) $\mathtt{S}^{s_2}_{\mathtt{c},\mathtt{d}}$ *is the set of all $s_2^{(\alpha,\beta)}$ such that*

$$N \leq \alpha \leq \min\{d_1, c_3\} - N,$$
$$N \leq \beta \leq \min\{c_1, d_3\} - N,$$
$$N - c_2 \leq \beta - \alpha \leq d_2 - N;$$

(iv) $\mathtt{S}^w_{\mathtt{c},\mathtt{d}} = \varnothing$ *for all other $w \in W_{\mathtt{c},\mathtt{d}}$ and for any $w \notin W_{\mathtt{c},\mathtt{d}}$.*

Proof Let us first show that none of the cosets represented by elements of $\mathtt{R}^{s_1 s_2}_{\mathtt{c},\mathtt{d}} \cup \mathtt{R}^{s_2 s_1}_{\mathtt{c},\mathtt{d}} \cup \mathtt{R}^{w_0}_{\mathtt{c},\mathtt{d}}$ can support intertwining operators. Choose an element $b \in \mathcal{R}^\times$ such that $\chi_3(b) \neq 1$. Set $g = \mathrm{diag}(b, 1, 1)$ and $g' = \mathrm{diag}(1, 1, b)$. These are elements of $C_{\mathtt{c}}$ and $C_{\mathtt{d}}$ for any $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$. One verifies that $g s_1 s_2^{(\alpha)} = s_1 s_2^{(\alpha)} g'$, $g w_0 = w_0 g'$, and $s_2 s_1^{(\alpha)} g = g' s_2 s_1^{(\alpha)}$, but that $\chi(g) \neq \chi(g')$. Consequently none of these representatives are in $\mathtt{S}_{\mathtt{c},\mathtt{d}}$.

From now on, let us adopt the notational convention that if $g = (g_{ij}) \in C_{\mathtt{c}}$, then $g_{21} = \gamma_{21}\pi^{c_1}$, $g_{32} = \gamma_{32}\pi^{c_2}$, and $g_{31} = \gamma_{31}\pi^{c_3}$. So $g' \in C_{\mathtt{d}}$ would have $g'_{21} = \gamma'_{21}\pi^{d_1}$, and so forth. Moreover, given a coset representative $h \in \mathtt{R}_{\mathtt{c},\mathtt{d}}$ and a pair of elements $g \in C_{\mathtt{c}}$ and $g' \in C_{\mathtt{d}}$ such that $gh = hg'$, we will call (g, g') a *coset pair*.

Suppose now that $(g, g') \in C_{\mathtt{c}} \times C_{\mathtt{d}}$ are a coset pair for the representative $s_1^{(\alpha,\beta)}$. We determine directly that the matrix coefficients of g and g' satisfy

$$\begin{aligned} g_{22} &= g_{33} - g_{23}\pi^{\beta} - \gamma'_{21}\pi^{d_1+\alpha-\beta} + \gamma_{32}\pi^{c_2-\beta} - \gamma'_{31}\pi^{d_3-\beta}, \\ g'_{22} &= g_{33} - g_{23}\pi^{\beta} - \gamma_{21}\pi^{c_1+\beta-\alpha} - \gamma'_{32}\pi^{d_2-\alpha} + \gamma_{31}\pi^{c_3-\alpha}, \\ g'_{33} &= g_{33} - g_{23}\pi^{\beta} - g'_{23}\pi^{\alpha}, \end{aligned}$$

with the remaining coefficients given by

$$\begin{aligned} g_{11} &= g'_{22} - g'_{23}\pi^{\alpha}, & g'_{11} &= g_{22} + g_{23}\pi^{\beta}, \\ g_{12} &= \gamma'_{21}\pi^{d_1} - g'_{23}\pi^{\beta}, & g'_{12} &= g_{23}\pi^{\alpha} + \gamma_{21}\pi^{c_1}, \\ g_{13} &= g'_{23}, & g'_{13} &= g_{23}. \end{aligned}$$

This allows us to compare

$$\chi_{\mathtt{c}}(g) = \chi_2(g_{22})\chi_3(g_{33}) = \chi_2(g_{33} - g_{23}\pi^{\beta} - \gamma'_{21}\pi^{d_1+\alpha-\beta} + \gamma_{32}\pi^{c_2-\beta} - \gamma'_{31}\pi^{d_3-\beta})\chi_3(g_{33})$$

with

$$\begin{aligned} \chi_{\mathtt{d}}(g') &= \chi_2(g'_{22})\chi_3(g'_{33}) \\ &= \chi_2(g_{33} - g_{23}\pi^{\beta} - \gamma_{21}\pi^{c_1+\beta-\alpha} - \gamma'_{32}\pi^{d_2-\alpha} + \gamma_{31}\pi^{c_3-\alpha}) \cdot \\ &\qquad \chi_3(g_{33} - g_{23}\pi^{\beta} - g'_{23}\pi^{\alpha}). \end{aligned}$$

It follows that whenever $M \leq \min\{c_1 - \alpha + \beta, d_1 + \alpha - \beta, c_2 - \beta, d_2 - \alpha, c_3 - \alpha, d_3 - \beta\}$ and $N \leq \min\{\alpha, \beta\}$, then $\chi_{\mathtt{c}}(g) = \chi_{\mathtt{d}}(g')$, and so $s_1^{(\alpha,\beta)} \in \mathtt{S}_{\mathtt{c},\mathtt{d}}$. Conversely, when these inequalities are not satisfied, and additionally $\alpha, \beta \geq 1$, then we can use the relations above to construct a coset pair (g, g') on which the characters do not agree. This proves part (ii); the proof of part (iii) is analogous and is omitted.

To prove part (i) of the theorem, suppose $t_{\mathtt{a},x} \in \mathtt{R}^1_{\mathtt{c},\mathtt{d}}$ and let $(g, g') \in C_{\mathtt{c}} \times C_{\mathtt{d}}$ be a coset pair such that $gt_{\mathtt{a},x} = t_{\mathtt{a},x}g'$. To simplify notation, set $r_x = \pi^{a_1+a_2} - x\pi^{a_3}$. One calculates directly that the matrix coefficients of g and g' satisfy the relation

$$\begin{aligned} (g_{12}\pi^{a_1} + g_{13}\pi^{a_1+a_2} - g_{23}\pi^{a_2})xr_x = {}& -\gamma_{21}x\pi^{c_1+a_2} - \gamma'_{21}r_x\pi^{d_1+a_2-a_3} \\ & + \gamma_{32}r_x\pi^{a_1+c_2-a_3} + \gamma'_{32}x\pi^{a_1+d_2} \\ & + \gamma_{31}\pi^{c_3-a_3+a_1+a_2} - \gamma'_{31}\pi^{d_3-a_3+a_1+a_2}, \end{aligned} \tag{3.1}$$

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and all other matrix coefficients are determined by the equations

$$\begin{aligned} g'_{11} &= g_{22} + g_{23}x\pi^{a_3-a_1} + \gamma_{21}\pi^{c_1-a_1} - \gamma'_{21}\pi^{d_1-a_1}, \\ g_{11} &= g'_{11} - g_{12}\pi^{a_1} - g_{13}x\pi^{a_3}, \\ g'_{22} &= g_{22} - g_{12}\pi^{a_1} - g_{13}\pi^{a_1+a_2} + g_{23}\pi^{a_2}, \\ g'_{33} &= g_{22} - g_{12}r_x\pi^{-a_2} - \gamma_{32}\pi^{c_2-a_2} + \gamma'_{32}\pi^{d_2-a_2}, \\ g_{33} &= g'_{33} - g_{13}r_x + g_{23}\pi^{a_2}, \end{aligned}$$

together with $g'_{12} = g_{12} + g_{13}\pi^{a_2}$, $g'_{13} = g_{13}$ and $g'_{23} = g_{23} - g_{13}\pi^{a_1}$. Note that in this case, as opposed to the one for $s_1^{(\alpha,\beta)}$ above, although any solution (with coefficients in $\mathcal{R}$) of (3.1) gives a pair of matrices (g, g') satisfying the relation $gt_{\mathtt{a},x} = t_{\mathtt{a},x}g'$, it must be additionally verified that g and g' are invertible in K.

Now, given a coset pair (g, g'), we have

$$\chi_{\mathtt{c}}(g) = \chi_2(g_{22})\chi_3(g_{33}) = \chi_2(g_{22})\chi_3(g'_{33} - g_{13}r_x + g_{23}\pi^{a_2}), \tag{3.2}$$

while

$$\chi_{\mathtt{d}}(g') = \chi_2(g_{22} - g_{12}\pi^{a_1} - g_{13}\pi^{a_1+a_2} + g_{23}\pi^{a_2})\chi_3(g'_{33}). \tag{3.3}$$

Hence these characters agree whenever $a_1 \geq M$ and $a_2 \geq N$, proving part (i)(1).

Now suppose a_1 and a_2 are both less than N. Choose a pair $(g_{12}, g_{23}) \in \mathcal{R} \times \mathcal{R}$ of minimum valuation satisfying $g_{23}\pi^{a_2} = g_{12}\pi^{a_1}$. Set

$$g_{13} = \gamma_{21} = \gamma'_{21} = \gamma_{31} = \gamma'_{31} = \gamma_{32} = \gamma'_{32} = 0$$

and set $g_{22} = 1$. These are easily seen to define a coset pair $(g, g') \in C_{\mathtt{c}} \times C_{\mathtt{d}}$. Since $\mathrm{val}(g_{12}\pi^{a_1}) = \mathrm{val}(g_{23}\pi^{a_2}) = \max\{a_1, a_2\} < N = \mathrm{cond}(\chi_3)$, we have $\chi_{\mathtt{c}}(g) \neq \chi_{\mathtt{d}}(g')$ and it follows that $t_{\mathtt{a},x} \notin \mathtt{S}^1_{\mathtt{c},\mathtt{d}}$.

There are exactly two cases left to consider: when $a_1 < M$ and $a_2 \geq N$, or when $a_1 \geq N$ and $a_2 < N$. Comparing (3.2) and (3.3), and noting that $\max\{a_1, a_2\} \leq \min\{a_3, \mathrm{val}(r_x)\}$, we deduce the following.

(A) If $a_1 < M$ and $a_2 \geq N$, then $t_{\mathtt{a},x} \in \mathtt{S}^1_{\mathtt{c},\mathtt{d}}$ if and only if $\mathrm{val}(g_{12}\pi^{a_1}) \geq M$ for all coset pairs (g, g').

(B) If $a_1 \geq N$ and $a_2 < N$, then $t_{\mathtt{a},x} \in \mathtt{S}^1_{\mathtt{c},\mathtt{d}}$ if and only if $\mathrm{val}(g_{23}\pi^{a_2}) \geq N$ for all coset pairs (g, g').

Consider case (A), that is, assume that $a_1 < M$ and $a_2 \geq N$. If $\mathrm{val}(g_{12}\pi^{a_1}) \geq a_2 \geq N$, then we are done; otherwise, the term with least valuation on the left-hand side of (3.1) is $g_{12}\pi^{a_1}xr_x$. Comparing with the right-hand side, we deduce $\mathrm{val}(g_{12}\pi^{a_1}) + \mathrm{val}(r_x) \geq \alpha$ where

$$\begin{aligned} \alpha = \min\bigl\{ & c_1 + a_2, d_1 + a_2 - a_3 + \mathrm{val}(r_x), a_1 + c_2 - a_3 + \mathrm{val}(r_x), a_1 + d_2, \\ & c_3 - a_3 + a_1 + a_2, d_3 - a_3 + a_1 + a_2\bigr\}. \end{aligned}$$

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It follows that if $\alpha \geq M + \mathrm{val}(r_x)$, then $t_{\mathtt{a},x} \in \mathtt{S}_{\mathtt{c},\mathtt{d}}$ by (A) above. Restating this condition in the three cases $a_1 + a_2 < a_3$, $a_1 + a_2 > a_3$, and $a_1 + a_2 = a_3$ yields the conditions described in part (i)(2)(a,b,c) of the theorem.

Conversely, suppose $\alpha < M + \mathrm{val}(r_x)$ and set $g_{13} = g_{23} = 0$. Choose a term of least possible valuation on the right-hand side of (3.1); set its coefficient (either γ_{ij} or γ'_{ij}, for some $i > j$) to be $\pi^{a_1-\alpha}$ if $\alpha < a_1 + \mathrm{val}(r_x)$ and 1 otherwise. Then set the remaining coefficients of the right-hand side of (3.1) equal to zero and solve for g_{12}, which is now necessarily in $\mathcal{R}^\times$. Take $g_{22} = 1$ and solve for the remaining coefficients. This results in a coset pair $(g, g') \in C_\mathtt{c} \times C_\mathtt{d}$ such that $\mathrm{val}(g_{12}\pi^{a_1}) < M$, so by (A) we conclude $t_{\mathtt{a},x} \notin \mathtt{S}_{\mathtt{c},\mathtt{d}}$, as required.

A similar argument establishes condition (i)(3) of the Theorem, following case (B) above. ∎

Let us conclude this section by deriving some consequences of Theorem 3.1. The first, which is immediate, is a convenient restatement of the theorem in a special case. Note that when $\mathtt{c} = \mathtt{d} = (n, n, n)$, we have simply

$$\mathtt{a}(\mathtt{c},\mathtt{d}) = \min\{a_1, a_2, a_3 - a_1, a_3 - a_2, n - a_3\} \quad \text{and} \quad \mathtt{a}(\mathtt{c},\mathtt{d})' = n - a_3.$$

Corollary 3.2 *Set $\mathtt{c} = (n, n, n)$ for $n \geq N$. The space of intertwining operators of $U_\mathtt{c} = V_\chi^{K_n}$ with itself has a basis parametrized by $\mathtt{S}_n = \bigcup_{w\in W} \mathtt{S}_n^w$, where*

(i) *$\mathtt{S}_n^1$ is the set of all $t_{\mathtt{a},x}$ such that $1 \leq a_1, a_2 \leq a_3 \leq n$, $x \in \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{c}}$ and one of conditions* (a), (b), or (c) *is met:*
   - (1) $\mathtt{a} \succeq \mathtt{m}$*; or*
   - (2) *$a_1 < M$ and $a_2 \geq N$ and:*
     - (a) *$a_1 + a_2 < a_3 \leq n - M$, or*
     - (b) *$a_1 + a_2 > a_3$ and $a_1 + a_2 \geq M - n + 2a_3$, or*
     - (c) *$a_1 + a_2 = a_3 \leq n - M$ and $\mathrm{val}(r_x) \leq n - M$; or*
   - (3) *$a_1 \geq N$, $a_2 < N$ and the same conditions* (a), (b), (c), *with M replaced by N, are satisfied;*

(ii) $\mathtt{S}_n^{s_1} = \{s_1^{(\alpha,\beta)} \mid N \leq \alpha, \beta \leq n - M\}$*;*

(iii) $\mathtt{S}_n^{s_2} = \{s_2^{(\alpha,\beta)} \mid N \leq \alpha, \beta \leq n - N\}$*;*

(iv) $\mathtt{S}_n^{s_1s_2} = \mathtt{S}_n^{s_2s_1} = \mathtt{S}_n^{w_0} = \varnothing$.

Our second corollary will be relevant for the purposes of calculating $\mathcal{I}(V_\mathtt{c}, V_\mathtt{d})$ in Section 4.

Corollary 3.3 *Suppose that $\mathtt{c}, \mathtt{d}, \mathtt{c}', \mathtt{d}' \in \mathtt{T}_\mathtt{m}$ with $\mathtt{c} \preceq \mathtt{c}'$ and $\mathtt{d} \preceq \mathtt{d}'$. Then $\mathtt{S}_{\mathtt{c},\mathtt{d}} \subseteq \mathtt{S}_{\mathtt{c}',\mathtt{d}'}$.*

Proof Recall that we have identified elements of $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}$ with a set of representatives in $\mathcal{R}^\times$ in such a way that if $\mathtt{a}(\mathtt{c},\mathtt{d}) \leq \mathtt{a}(\mathtt{c}',\mathtt{d}')$ and $\mathtt{a}(\mathtt{c},\mathtt{d})' \leq \mathtt{a}(\mathtt{c}',\mathtt{d}')'$, then $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}} \subseteq \mathtt{X}^{\mathtt{a}}_{\mathtt{c}',\mathtt{d}'}$. It now easily follows that $\mathtt{R}_{\mathtt{c},\mathtt{d}} \subseteq \mathtt{R}_{\mathtt{c}',\mathtt{d}'}$. Furthermore, it is clear that the list of constraints on elements of $\mathtt{S}$ in Theorem 3.1 can only become less constrictive as $\mathtt{c}$ or $\mathtt{d}$ increases. ∎

In particular, it makes sense to ask, for a given distinguished double coset representative $g \in \bigcup_{\mathtt{c},\mathtt{d}} \mathtt{R}_{\mathtt{c},\mathtt{d}}$, whether there exist $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$ for which $g \in \mathtt{S}_{\mathtt{c},\mathtt{d}}$.

Theorem 3.4 *The double cosets which support self-intertwining operators of $U_{\mathtt{c}}$ for some $\mathtt{c} \in \mathtt{T}_{\mathtt{m}}$, are represented by $\mathtt{S} = \mathtt{S}_{\mathtt{c},\mathtt{d}} = \bigcup_{w\in W} \mathtt{S}^{w}$, where*

(i) $\mathtt{S}^{1} = \{t_{\mathtt{a},x} \mid a_3 \geq \max\{a_1, a_2\} \geq N, x \in \mathcal{R}^{\times}\}$;
(ii) $\mathtt{S}^{s_1} = \{s_1^{(\alpha,\beta)} \mid \alpha, \beta \geq N\}$;
(iii) $\mathtt{S}^{s_2} = \{s_2^{(\alpha,\beta)} \mid \alpha, \beta \geq N\}$;
(iv) $\mathtt{S}^{s_1 s_2} = \mathtt{S}^{s_2 s_1} = \mathtt{S}^{w_0} = \varnothing$.

Moreover, up to identifying $t_{\mathtt{a},x}$ and $t_{\mathtt{a},y}$ whenever x and y have the same image in $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}$ for $\mathtt{c}$, $\mathtt{d}$ sufficiently large, these elements all represent distinct cosets.

Proof This follows from Corollary 3.2 by allowing n to grow without bound. ■

4 Irreducibility

The results of the preceding section allow us to restate Corollary 2.4 in terms of the sets $\mathtt{S}_{\mathtt{c},\mathtt{d}}$. That is, for any $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$, we have

$$\mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{d}}) = \sum_{\substack{I \subseteq \mathcal{S}_{\mathtt{c}} \\ J \subseteq \mathcal{S}_{\mathtt{d}}}} (-1)^{|I|+|J|} \, |\mathtt{S}_{\mathtt{c}_I, \mathtt{d}_J}|.$$

The irreducibility of $U_{\mathtt{m}} = V_{\mathtt{m}}$ is known from Howe's work [H1, Theorem 1]. In this section, we demonstrate that this extends to many, but not all, of the quotients which are "extremal" in the sense that they have few immediate descendants in the poset $\mathtt{T}_{\mathtt{m}}$.

We retain the notation of the previous sections and begin with a lemma.

Lemma 4.1 *Let $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$. Then $\mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{d}}) = 0$ if $c_3 \neq d_3$.*

Proof Note that $U_{\mathtt{c}} \subseteq U_{(c_3,c_3,c_3)} = V_{\chi}^{K_{c_3}}$. Without loss of generality, assume $d_3 < c_3$. Then $\mathtt{e} = \gcd(\mathtt{c}, (d_3, d_3, d_3)) \prec \mathtt{c}$, so from Lemma 2.2 we deduce

$$V_{\mathtt{c}} \subset U_{\mathtt{c}} / (U_{\mathtt{c}} \cap V_{\chi}^{K_{d_3}}).$$

Hence although $V_{\mathtt{d}}$ consists of K_{d_3}-fixed vectors, no subrepresentation of $V_{\mathtt{c}}$ does, so they cannot intertwine. ■

Theorem 4.2 *For each $n \in \mathbb{Z}$ with $N \leq n \leq N + M$, the K-module $V_{(M,N,n)}$ is irreducible.*

Proof Set $\mathtt{c} = (M, N, n)$. If $n = N$, then $\mathcal{S}_{\mathtt{c}} = \varnothing$; otherwise, $\mathcal{S}_{\mathtt{c}}$ is a singleton corresponding to the triple $(M, N, n - 1)$. By Corollary 2.4 and induction, it thus suffices to show that $|\mathtt{S}_{\mathtt{c},\mathtt{c}}| = n - N + 1$.

If $M = 0$, then $W_{\mathtt{c},\mathtt{c}} = \{1, w_0\}$, so $\mathtt{S}^{s_1}_{\mathtt{c},\mathtt{c}} = \mathtt{S}^{s_2}_{\mathtt{c},\mathtt{c}} = \varnothing$. If $M > 0$, then $\min\{c_2, c_3\} - M = N - M < N$ and $\min\{c_1, c_3\} - N = M - N < N$, so again $\mathtt{S}^{s_1}_{\mathtt{c},\mathtt{c}} = \mathtt{S}^{s_2}_{\mathtt{c},\mathtt{c}} = \varnothing$, regardless of the value of n. Thus $\mathtt{S}_{\mathtt{c},\mathtt{c}} = \mathtt{S}^{1}_{\mathtt{c},\mathtt{c}}$.

Now let $t_{\mathtt{a},x} \in \mathtt{S}^1_{\mathtt{c},\mathtt{c}}$, so one of Theorem 3.1(i)(1), (2), or (3) applies. If it were (2), then $a_1 < M$ and $a_2 \geq N$ imply that $M > 0$ and $a_2 = N$, so neither case (a) nor case (c) could apply, since $M > c_2 - a_2 = 0$. Were case (b) to apply, then $M \leq \min\{\mathtt{c} - \mathtt{a}^{\mathrm{op}}\}$ would imply that $a_2 = a_3 = N$ and so $N - (a_3 - a_1) = a_1$, which is not greater than or equal to M, a contradiction. We similarly deduce that case (3) cannot apply. This leaves case (1), which consists of the elements $t_{\mathtt{a},x}$ with $\mathtt{a} = (M, N, m)$, $N \leq m \leq n$ and $x \in \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{c}}$, each of which support an intertwining operator. For each such $\mathtt{a}$, we have $\mathtt{a}(\mathtt{c},\mathtt{c}) = \mathtt{a}(\mathtt{c},\mathtt{c})' = 0$, so in fact $|\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{c}}| = 1$. The desired conclusion follows. ■

Theorem 4.3 *$V_{(m,n,n+m)}$ is irreducible for each $m \geq M$ and $n \geq N$.*

Proof We first consider the case that $\mathtt{c} = (m, n, m+n)$ with $m \geq 1$. Then $\mathcal{S}_{\mathtt{c}}$ is a singleton corresponding to $\mathtt{d} = \mathtt{c}_{\{3\}} = (m, n, m+n-1)$. Hence $\mathcal{I}(U_{\mathtt{c}}, V_{\mathtt{c}}) = \mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{c}}) - \mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{d}})$ and it suffices to show that $|\mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}}| = 1$.

First note that since $c_1 = d_1$ and $c_2 = d_2$, and that both are at most $d_3 < c_3$, we have $\mathtt{S}^w_{\mathtt{c},\mathtt{c}} = \mathtt{S}^w_{\mathtt{c},\mathtt{d}}$ for each $w \in W \setminus \{1\}$.

Next note that $\mathtt{T}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{T}_{\mathtt{c},\mathtt{d}}$ consists of the single element $(m, n, n+m) = \mathtt{c}$. Since $|\mathtt{X}^{\mathtt{c}}_{\mathtt{c},\mathtt{c}}| = 1$, there is a unique distinguished double coset of the form $t_{\mathtt{c},x} \in \mathtt{R}^1_{\mathtt{c},\mathtt{c}}$; it is clearly in $\mathtt{S}_{\mathtt{c},\mathtt{c}}$. We claim that this is the only element of $\mathtt{R}^1_{\mathtt{c},\mathtt{c}} \setminus \mathtt{R}^1_{\mathtt{c},\mathtt{d}}$. Namely, let $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}}$. Since $0 \leq d_3 - a_3 = (d_1 + d_2 - 1) - a_3 = (d_1 + a_2 - a_3) + (d_2 - a_2) - 1$, it must be true that $d_3 - a_3 \geq \min\{d_1 + a_2 - a_3, d_2 - a_2\}$, so necessarily $\mathtt{a}(\mathtt{c},\mathtt{c}) = \mathtt{a}(\mathtt{c},\mathtt{d})$. If furthermore $a_1 + a_2 = a_3$, then $\mathtt{a}(\mathtt{c},\mathtt{c})' = \mathtt{a}(\mathtt{c},\mathtt{d})'$ by the same reasoning. Hence $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{c}} = \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}$ for all such $\mathtt{a}$, as claimed.

We next claim that $\mathtt{S}^1_{\mathtt{c},\mathtt{c}} \cap \mathtt{R}^1_{\mathtt{c},\mathtt{d}} = \mathtt{S}^1_{\mathtt{c},\mathtt{d}}$. Namely, given $t_{\mathtt{a},x} \in \mathtt{S}^1_{\mathtt{c},\mathtt{c}} \cap \mathtt{R}^1_{\mathtt{c},\mathtt{d}}$, since $d_3 - a_3 \geq \min\{d_1 + a_2 - a_3, d_2 - a_2\} = \min\{c_1 + a_2 - a_3, c_2 - a_2\}$, we see that all of the conditions set out in Theorem 3.1(i) are unchanged in passing from the pair $(\mathtt{c},\mathtt{c})$ to the pair $(\mathtt{c},\mathtt{d})$. Hence $t_{\mathtt{a},x} \in \mathtt{S}^1_{\mathtt{c},\mathtt{d}}$.

This shows, for the case $m \geq 1$, that $\mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}}$ is a singleton, from which we deduce the irreducibility of $V_{\mathtt{c}}$.

When $\mathtt{c} = (0, n, n)$, we have instead $\mathcal{S}_{\mathtt{c}} = \{3\}$ corresponding to $\mathtt{d} = \mathtt{c}_{\{3\}} = (0, n-1, n-1)$. We have $\mathtt{S}_{\mathtt{c},\mathtt{d}} = \mathtt{S}^1_{\mathtt{c},\mathtt{d}}$, $\mathtt{S}_{\mathtt{c},\mathtt{c}} = \mathtt{S}^1_{\mathtt{c},\mathtt{c}}$ and neither of the cases (i)(2) nor (i)(3) of Theorem 3.1 can apply. It thus follows easily that $|\mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}}| = 1$ in this case as well. ■

Theorem 4.4 *$V_{(m,n,\max\{n,m\})}$ is irreducible for each $m > M$ and $n > N$.*

Proof Suppose first that $m > \max\{M, 1\}$ and $n > N$, and that $\max\{m, n\} = n$. Then $\mathtt{c} = (m, n, n)$ and $\mathcal{S}_{\mathtt{c}} = \{1, 2\}$ with the corresponding triples $\mathtt{c}_{\{1\}} = (m-1, n, n)$, $\mathtt{c}_{\{2\}} = (m, n-1, n)$, and $\mathtt{c}_{\{1,2\}} = (m-1, n-1, n)$. We compute the alternating sum

$$\mathcal{I}(U_{\mathtt{c}}, V_{\mathtt{c}}) = \mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{c}}) - \mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{c}_{\{1\}}}) - \mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{c}_{\{2\}}}) + \mathcal{I}(U_{\mathtt{c}}, U_{\mathtt{c}_{\{1,2\}}}) \tag{4.1}$$

as a sum of differences by defining

$$\mathcal{A}_0 = \mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{2\}}} \quad \text{and} \quad \mathcal{A}_1 = \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{1\}}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{1,2\}}}.$$

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Thus we have $\mathcal{I}(U_{\mathtt{c}}, V_{\mathtt{c}}) = |\mathcal{A}_0| - |\mathcal{A}_1|$. We use $(\mathtt{d}, \mathtt{d}')$ to denote either of the pairs $(\mathtt{c}, \mathtt{c}_{\{2\}})$ or $(\mathtt{c}_{\{1\}}, \mathtt{c}_{\{1,2\}})$, for ease of notation.

Suppose first that $s_1^{(\alpha,\beta)} \in \mathtt{S}_{\mathtt{c},\mathtt{d}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}'}$. Then comparing the constraints on α and β in Theorem 3.1(ii) for $\mathtt{d}$ and $\mathtt{d}'$, we see that necessarily $\alpha = d_2 - M = n - M$ and

$$\max\{N, M - c_1 + \alpha\} \leq \beta \leq \min\{d_1 - M + \alpha, c_2 - M\}.$$

Since $d_1 \geq M$ by hypothesis, $c_2 - M = n - M \leq \alpha + d_1 - M$, so these inequalities simplify to $\max\{N, n - m\} \leq \beta \leq n - M$. This constraint on the pair (α, β) is independent of the value of $d_1 \in \{m-1, m\}$, so $s_1^{(\alpha,\beta)} \in \mathcal{A}_0$ if and only if $s_1^{(\alpha,\beta)} \in \mathcal{A}_1$. Hence these cosets contribute nothing to the overall sum (4.1).

Now suppose that $s_2^{(\alpha,\beta)} \in \mathtt{S}_{\mathtt{c},\mathtt{d}}$. Then Theorem 3.1(iii) implies that $\beta - \alpha \leq (c_1 - N) - N$; but this bound is at most $d_2' - N$, since $c_1 - N = n - N \leq n - 1 = d_2'$. Similarly, $d_1 \leq c_1$ and $\beta > 0$ together imply that $\beta - \alpha \geq N - c_1$, regardless of the value of $d_1 \in \{m-1, m\}$. All other conditions on (α, β) being unchanged in passing from $(\mathtt{c}, \mathtt{d})$ to $(\mathtt{c}, \mathtt{d}')$, we deduce that $s_2^{(\alpha,\beta)} \in \mathtt{S}_{\mathtt{c},\mathtt{d}'}$. Hence none of these cosets appear in either $\mathcal{A}_0$ or $\mathcal{A}_1$.

Finally, consider distinguished coset representatives of the form $t_{\mathtt{a},x} \in \mathtt{R}^1_{\mathtt{c},\mathtt{d}}$. First note that $\mathtt{T}_{\mathtt{c},\mathtt{d}} \setminus \mathtt{T}_{\mathtt{c},\mathtt{d}'} = \{(a_1, n, n) \mid 1 \leq a_1 \leq \underline{d}_1\}$, and $|\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}| = |\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}'}| = 1$, since $a_3 - a_2 = 0$. Considering which of these are in $\mathtt{S}_{\mathtt{c},\mathtt{d}}$, we deduce that these triples give rise to $m - M + 1$ coset representatives in $\mathcal{A}_0$ and $m - M$ of them in $\mathcal{A}_1$.

Suppose now that $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}'}$. Since $0 \leq d_2' - a_2 = d_3 - 1 - a_2 = (d_3 - a_3) + (a_3 - a_2) - 1$, we have $d_2' - a_2 \geq \min\{d_3 - a_3, a_3 - a_2\}$ and so it follows that $\mathtt{a}(\mathtt{c}, \mathtt{d}) = \mathtt{a}(\mathtt{c}, \mathtt{d}')$. Similarly, if $a_1 + a_2 = a_3$, then $a_2 < a_3$ implies that $d_2' - a_2 \geq d_3 - a_3$, so $\mathtt{a}(\mathtt{c}, \mathtt{d})' = \mathtt{a}(\mathtt{c}, \mathtt{d}')'$. Hence for all $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}'}$ we have $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}} = \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}'}$.

So suppose $t_{\mathtt{a},x} \in \mathtt{S}^1_{\mathtt{c},\mathtt{d}} \cap \mathtt{R}^1_{\mathtt{c},\mathtt{d}'}$. We first note that if $t_{\mathtt{a},x}$ falls under any of the conditions (2)(a), (2)(c), (3)(a), or (3)(c) of Theorem 3.1, then the inequality $a_2 < a_3$ implies $d_2' - a_2 \geq d_3 - a_3$. Consequently, this condition is unchanged in passing from $\mathtt{d}$ to $\mathtt{d}'$ and so $t_{\mathtt{a},x} \in \mathtt{S}_{\mathtt{c},\mathtt{d}'}$. Similarly, if $t_{\mathtt{a},x}$ falls under condition (3)(b), then $a_2 < a_3$ so $d_2' - (a_3 - a_1) \geq d_3 - (a_3 - a_1) - (a_3 - a_2)$; again we deduce $t_{\mathtt{a},x} \in \mathtt{S}_{\mathtt{c},\mathtt{d}'}$. So none of these occur in either $\mathcal{A}_0$ or $\mathcal{A}_1$.

On the other hand, if $t_{\mathtt{a},x}$ falls under condition (2)(b) for the pair $(\mathtt{c}, \mathtt{d})$, then it fails (2)(b) for the pair $(\mathtt{c}, \mathtt{d}')$ exactly when $a_3 = a_2 \geq N$, $d_1 \geq M$, $d_2 - (a_3 - a_1) = M$, and $1 \leq a_1 < M$. Hence, noting also that this condition is independent of the choice of $d_1 \in \{m-1, m\}$, all such $t_{\mathtt{a},x}$ lie in both $\mathcal{A}_0$ and $\mathcal{A}_1$.

We deduce that $|\mathcal{A}_0| - |\mathcal{A}_1| = 1$, so the quotient $V_{\mathtt{c}}$ is indeed irreducible.

The case for $m \geq n$ follows by an analogous argument, where we interchange the roles of $\mathtt{c}_{\{1\}}$ and $\mathtt{c}_{\{2\}}$ throughout.

It only remains to show the case where $m = 1$ and $M = 0$. In this case, $\mathtt{c} = (1, n, n)$ with $n > N \geq 1$, so we have $\mathtt{c}_{\{1\}} = (0, n, n)$, $\mathtt{c}_{\{2\}} = (1, n-1, n)$, and $\mathtt{c}_{\{1,2\}} = (0, n-1, n-1)$. Define $\mathcal{A}_0$ and $\mathcal{A}_1$ as above. Since $c_1 - a_1 = 0$ for all $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}}$, for any $\mathtt{d}$, and since cases (i)(2) and (i)(3) cannot occur, the analysis is much simplified from the above. We readily see that $\mathcal{A}_0 = \{s_1^{(n,n-1)}, s_1^{(n,n)}, t_{(1,n,n),1}\}$, whereas $\mathcal{A}_1 = \{t_{(1,n,n),1}, t_{(1,n-1,n),1}\}$. Thus we conclude again in this case that $V_{\mathtt{c}}$ is irreducible. ■

Corollary 4.5 *If $n > N$, then the quotient $V_{(n,n,n)}$ is the unique irreducible module of maximal dimension in $V_\chi^{K_n}$.*

Proof We recall that $V_\chi^{K_n} \simeq U_{(n,n,n)}$, and that $V_{(n,n,n)}$ is irreducible by Theorem 4.4. Set $\alpha = (q+1)(q^2+q+1)$; then we have $\dim U_{\mathtt{c}} = \alpha q^{c_1+c_2+c_3-3}$ whenever $c_1c_2 \neq 0$. Thus, for $n > N \geq 1$, we have by Theorem 2.3 that

$$\begin{aligned}\dim V_{(n,n,n)} &= \dim U_{(n,n,n)} - \dim U_{(n-1,n,n)} - \dim U_{(n,n-1,n)} + \dim U_{(n-1,n-1,n)} \\ &= \alpha(q^{3n-3} - 2q^{3n-4} + q^{3n-5}) \\ &= \alpha q^{3n-5}(q-1)^2.\end{aligned}$$

However, it now follows that the complement of $V_{(n,n,n)}$ in $U_{(n,n,n)}$ has dimension $\alpha q^{3n-5}(2q-1)$, which is strictly less than $\dim V_{(n,n,n)}$ whenever $q > 3$. ■

The strict inequalities in Theorem 4.4 are necessary as the following proposition shows.

Proposition 4.6

(i) *Let $\mathtt{c} = (M,n,n)$ with $n > N$. Then*

$$\mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{c}}) = \begin{cases} n-N+1 & \text{if } n < M+N, \\ M+1 & \text{if } n \geq M+N. \end{cases}$$

(ii) *Let $\mathtt{c} = (n,N,n)$ with $n \geq N$. Then*

$$\mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{c}}) = \begin{cases} n-N+1 & \text{if } n < 2N, \\ N+1 & \text{if } n \geq 2N. \end{cases}$$

Proof To prove part (i), let $\mathtt{c} = (M,n,n)$ with $M > 0$ and $n > N$. Then $\mathcal{S}_{\mathtt{c}} = \{2\}$, corresponding to the triple $\mathtt{c}_{\{2\}} = (M, n-1, n)$. We first compute $\mathcal{I}(U_{\mathtt{c}}, V_{\mathtt{c}}) = |\mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{2\}}}|$. For ease of notation, set $(\mathtt{d}, \mathtt{d}') = (\mathtt{c}, \mathtt{c}_{\{2\}})$.

It is easy to see that $s_1^{(n-M,n-M)} \in \mathtt{S}_{\mathtt{c},\mathtt{d}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}'}$ if $n - M \geq N$, whereas $\mathtt{S}^{s_2}_{\mathtt{c},\mathtt{d}} = \mathtt{S}^{s_2}_{\mathtt{c},\mathtt{d}'} = \varnothing$.

Of the elements in $\mathtt{T}_{\mathtt{c},\mathtt{d}} \setminus \mathtt{T}_{\mathtt{c},\mathtt{d}'} = \{(a_1, n, n) \mid 1 \leq a_1 \leq M\}$, only (M,n,n) gives rise to a representative in $\mathtt{S}_{\mathtt{c},\mathtt{d}}$, and then exactly one, which we shall denote $t_{(M,n,n),1}$.

For each $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}'}$, we have $a_2 < n$ and $a_3 \leq n$. Since $\mathtt{a}(\mathtt{c},\mathtt{d}) \leq \min\{a_3 - a_2, n - a_3, d_2 - a_2\}$ and $a_3 \leq d_2$, we deduce that if $\mathtt{a}(\mathtt{c},\mathtt{d}) \neq \mathtt{a}(\mathtt{c},\mathtt{d}')$, then necessarily $a_3 = n$ and $a_3 = a_2$, a contradiction. Similarly, $\mathtt{a}(\mathtt{c},\mathtt{d})'$ does not depend on the value of d_2. Hence $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}} = \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}'}$.

So suppose $t_{\mathtt{a},x} \in (\mathtt{S}^1_{\mathtt{c},\mathtt{d}} \cap \mathtt{R}_{\mathtt{c},\mathtt{d}'}) \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}'}$. It does not fall under case (i)(1) of Theorem 3.1, since this case is independent of $\mathtt{d}'$; nor can case (i)(3) occur, since $a_1 \leq M$. In cases (i)(2)(a) and (c), the right-hand side can depend on the value of $d_2 \in \{n-1, n\}$ if and only if $a_2 = a_3$, contradicting the hypotheses. In case (i)(2)(b), which holds only if $a_2 \geq N$, we must have that $a_3 - a_2 = 0$ or else the right-hand side is less than M. It follows that the right-hand side depends on the value of d_2 exactly

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when $a_2 = a_3 \geq N$ and $n-(a_3-a_1)=M$; in each of these cases $|\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}| = |\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}'}| = 1$ and $t_{\mathtt{a},1} \in \mathtt{S}_{\mathtt{c},\mathtt{d}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}'}$.

We conclude that when $M > 0$

$$\mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{2\}}} = \{t_{(M-k,n-k,n-k),1} \mid 0 \leq k \leq \min\{M-1, n-N\}\} \cup \{s_1^{(n-M,n-M)} \mid n-M \geq N\}.$$

A simpler analysis, which we consequently omit, allows us to further deduce that $\mathtt{S}_{\mathtt{c}_{\{2\}},\mathtt{c}} = \mathtt{S}_{\mathtt{c}_{\{2\}},\mathtt{c}_{\{2\}}}$ and so $\mathcal{I}(U_{\mathtt{c}}, V_{\mathtt{c}}) = \mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{c}})$, and this has the value stated in the proposition.

When $M = 0$, we have instead $\mathtt{c} = (0, n, n)$ and $\mathtt{c}_{\{3\}} = (0, n-1, n-1)$, and $\mathtt{S}_{\mathtt{c},\mathtt{d}} = \mathtt{S}^1_{\mathtt{c},\mathtt{d}}$. Since neither (i)(2) nor (i)(3) of Theorem 3.1 can apply, and $\mathtt{a}(\mathtt{d},\mathtt{d}') = \mathtt{a}(\mathtt{d},\mathtt{d}')' = 0$ for all choices of $\mathtt{d},\mathtt{d}' \in \{\mathtt{c}, \mathtt{c}_{\{3\}}\}$ and for all $\mathtt{a} \in \mathtt{T}_{\mathtt{d},\mathtt{d}'}$, we readily conclude that $\mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{3\}}} = \{t_{(1,n,n),1}\}$. Hence the quotient $V_{\mathtt{c}}$ is irreducible in this case.

To prove part (ii), let $\mathtt{c} = (n, N, n)$ with $n \geq N$. Then $\mathcal{S}_{\mathtt{c}} = \{1\}$ with corresponding triple $(n-1, N, n)$. Reasoning as above, we deduce readily that $s_2^{(n-N,n-N)} \in \mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{1\}}}$ whenever $n \geq 2N$ and that $\mathtt{S}^{s_1}_{\mathtt{c},\mathtt{c}} = \mathtt{S}^{s_1}_{\mathtt{c},\mathtt{c}_{\{1\}}} = \varnothing$.

Set $(\mathtt{d},\mathtt{d}') = (\mathtt{c}, \mathtt{c}_{\{1\}})$. Note that $\mathtt{T}_{\mathtt{c},\mathtt{d}} \setminus \mathtt{T}_{\mathtt{c},\mathtt{d}'} = \{(n, a_2, n) \mid 1 \leq a_2 \leq N\}$ and each of these has $|\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}| = 1$. These triples thus give rise to only one coset in $\mathtt{S}^1_{\mathtt{c},\mathtt{d}}$, namely that represented by $t_{(n,N,n),1}$.

Of those $\mathtt{a} \in \mathtt{T}_{\mathtt{c},\mathtt{d}'}$, one sees as above that $\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}} = \mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}'}$. For such a triple $\mathtt{a}$, if $t_{\mathtt{a},x} \in \mathtt{S}_{\mathtt{c},\mathtt{d}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{d}'}$, then it falls under case (i)(3)(b) of Theorem 3.1 and we deduce as above that $N \leq a_1 = a_3 \leq n$ and $n - a_3 = N - a_2$. These conditions further imply that $|\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{d}}| = 1$ meaning each such triple gives rise to a unique double coset.

We conclude that

$$\mathtt{S}_{\mathtt{c},\mathtt{c}} \setminus \mathtt{S}_{\mathtt{c},\mathtt{c}_{\{1\}}} = \{t_{(n-k,N-k,n-k),1} \mid 0 \leq k \leq \min\{n-N, N-1\}\} \cup \{s_2^{(n-N,n-N)} \mid \text{if } n-N \geq N\}.$$

It is readily verified that $\mathtt{S}_{\mathtt{c}_{\{1\}},\mathtt{c}} = \mathtt{S}_{\mathtt{c}_{\{1\}},\mathtt{c}_{\{1\}}}$, and so $\mathcal{I}(U_{\mathtt{c}}, V_{\mathtt{c}}) = \mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{c}})$. Counting the double cosets in the expression above yields part (ii) of the proposition. ■

5 Examples

We conclude the paper with two examples meant to illustrate the results in Section 4.

Example 5.1 Suppose that $M = N = 2$ and let us consider the decomposition of $V_\chi^{K_4}$ under K. The values of $\mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{d}})$ are calculated using Corollary 2.4 and Theorem 3.1, with several values identified by Theorems 4.2, 4.3, and 4.4 and Proposition 4.6. The remaining computations were implemented in GAP [GAP] and the results are represented schematically in Figure 5.1, as follows.

Each triple $\mathtt{c}$ in Figure 5.1 corresponds to the induced representation $U_{\mathtt{c}}$, and the number beneath it is the value of $\mathcal{I}(V_{\mathtt{c}}, V_{\mathtt{c}})$. The arrows imply the partial order $\succeq$

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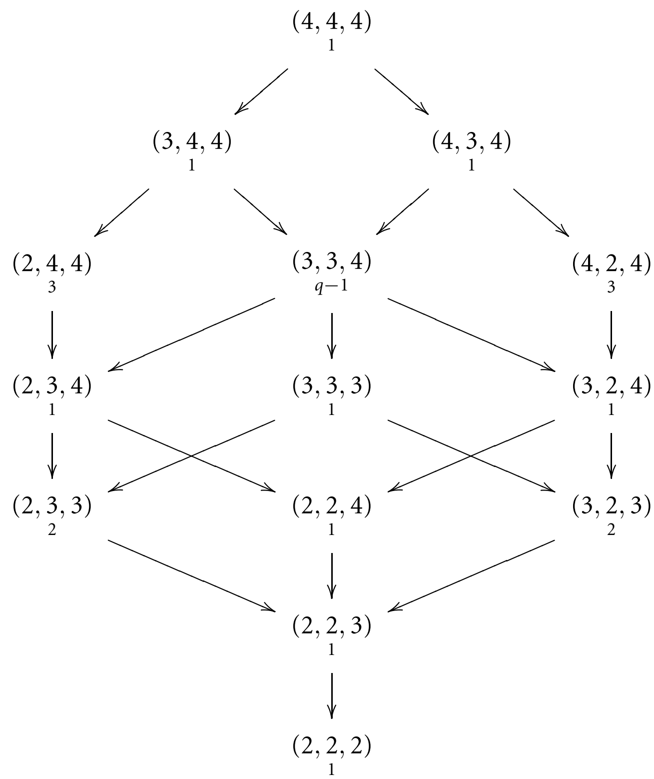

Figure 5.1: Reducibility of $V_{\mathtt{c}}$ for V_χ with $M = 2$ and $N = 2$.

on $\mathtt{T}$; hence the set of all components of the diagram below and including $\mathtt{c}$ may be identified with the whole of $U_{\mathtt{c}}$. For reference, we list in Table 5.1 the dimensions of the quotients $V_{\mathtt{c}}$ occuring in Figure 5.1. These are calculated using Theorem 2.3. We abbreviate $\alpha = (q+1)(q^2+q+1)$.

Figure 5.1 reveals several typical features of the K-representations $V_{\mathtt{c}}$. For example, we note that while many $V_{\mathtt{c}}$ are irreducible, several are not. Besides those identified by Proposition 4.6, for which the number of intertwining operators grows at most linearly with c_3, there exist components such as $V_{(3,3,4)}$, for which the number of intertwining operators is a polynomial function of q. Such components occur more frequently in $V_\chi^{K_n}$ as n increases, since they come into existence only when $|\mathtt{X}^{\mathtt{a}}_{\mathtt{c},\mathtt{c}}|$ is a polynomial in q, that is, when $\mathtt{a}(\mathtt{c},\mathtt{c}) > 0$.

Example 5.2 Consider a character χ for which $M = 1$ and $N = 2$. Figure 5.2 describes a portion of the restriction to K of V_χ, namely, the subrepresentations $U_{\mathtt{c}}$ for which $\dim U_{\mathtt{c}} \le \alpha q^9$. In terms of triples, this implies that we consider the elements $\mathtt{c} \in \mathtt{T_m}$ for which $c_1 + c_2 + c_3 \le 9$. Again, the number of intertwining operators between each pair of quotients is determined by Corollary 2.4.

Quotient	Dimension
$V_{(4,4,4)}$	$q^7(q-1)^2\alpha$
$V_{(3,4,4)}, V_{(4,3,4)}$	$q^6(q-1)^2\alpha$
$V_{(3,3,4)}$	$q^4(q-1)^3\alpha$
$V_{(2,4,4)}, V_{(4,2,4)}$	$q^6(q-1)\alpha$
$V_{(2,3,4)}, V_{(3,3,3)}, V_{(3,2,4)}$	$q^4(q-1)^2\alpha$
$V_{(2,3,3)}, V_{(2,2,4)}, V_{(3,2,3)}$	$q^4(q-1)\alpha$
$V_{(2,2,3)}$	$q^3(q-1)\alpha$
$V_{(2,2,2)}$	$q^3\alpha$

Table 5.1: Dimensions of $V_{\mathtt{c}}$ for V_χ with $M = 2$ and $N = 2$.

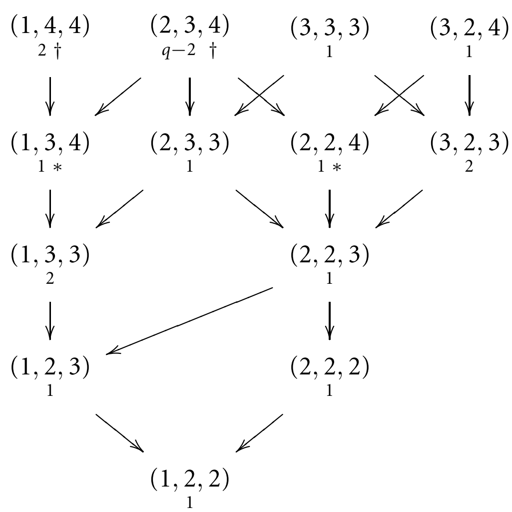

Figure 5.2: Reducibility of and equivalences between $V_{\mathtt{c}}$ for V_χ with $M = 1$ and $N = 2$.

This example illustrates a phenomenon not present in Example 5.1. There are two pairs of isomorphic irreducible representations: $V_{(1,3,4)} \simeq V_{(2,2,4)}$ (indicated by $*$ in Figure 5.2) and one of the two inequivalent irreducible summands of $V_{(1,4,4)}$ is isomorphic to exactly one of the irreducible summands of $V_{(2,3,4)}$ (indicated by † in Figure 5.2). By Lemma 4.1, such pairs of isomorphic irreducible summands, for distinct triples $\mathtt{c}, \mathtt{d} \in \mathtt{T}_{\mathtt{m}}$, can occur only when $c_3 = d_3$.

The dimensions of the representations in Figure 5.2 are given in Table 5.2. We have again abbreviated $\alpha = (q+1)(q^2+q+1)$. We conjecture that $V_{(2,3,4)}$ in fact decomposes as a sum of $q-2$ distinct irreducible summands, each of dimension equal to that of $V_{(1,3,4)}$, $V_{(2,2,4)}$, and $V_{(3,2,3)}$. This would be consistent with the remaining irreducible summand in $V_{(1,4,4)}$ having dimension equal to that of $V_{(3,3,3)}$ and $V_{(3,2,4)}$.

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Quotient	Dimension
$V_{(1,4,4)}$	$q^5(q-1)\alpha$
$V_{(3,3,3)}, V_{(3,2,4)}$	$q^4(q-1)^2\alpha$
$V_{(2,3,4)}$	$q^4(q-1)(q-2)\alpha$
$V_{(1,3,4)}, V_{(2,2,4)}, V_{(3,2,3)}$	$q^4(q-1)\alpha$
$V_{(2,3,3)}$	$q^3(q-1)^2\alpha$
$V_{(1,3,3)}$	$q^3(q-1)\alpha$
$V_{(2,2,3)}$	$q^2(q-1)^2\alpha$
$V_{(1,2,3)}, V_{(2,2,2)}$	$q^2(q-1)\alpha$
$V_{(1,2,2)}$	$q^2\alpha$

Table 5.2: Dimensions of $V_{\mathfrak{c}}$ for V_χ with $M = 1$ and $N = 2$.

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