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# Rationality and the Jordan–Gatti–Viniberghi Decomposition

Jason Levy

*Abstract.* We verify our earlier conjecture and use it to prove that the semisimple parts of the rational Jordan–Kac–Vinberg decompositions of a rational vector all lie in a single rational orbit.

## 1 Introduction

Let $k$ be a field of characteristic 0, and write $\bar{k}$ for its algebraic closure. Let $G$ be a reductive algebraic group (not necessarily connected), acting on a vector space $V$, with $G$, $V$, and the action all defined over $k$. Given a point $v \in V$, write $G_v$ for the stabilizer of $v$; it is an algebraic subgroup of $G$.

In [6], Kac and Vinberg made the following definitions:

***Definition 1.1***

(i) A vector $s \in V$ is *semisimple* if the orbit $G \cdot s$ is Zariski closed.

(ii) A vector $n \in V$ is *nilpotent* with respect to $G$ if the Zariski closure $\overline{G \cdot n}$ contains the vector 0.

(iii) A *Jordan decomposition* of a vector $\gamma \in V$ is a decomposition $\gamma = s + n$, with

- (a) $s$ semisimple,
- (b) $n$ nilpotent with respect to $G_s$,
- (c) $G_\gamma \subseteq G_s$.

This Jordan–Kac–Vinberg decomposition matches the standard Jordan decomposition when $V$ is the Lie algebra $\mathfrak{g}$ of $G$. In that case, every $\gamma \in \mathfrak{g}$ has a unique Jordan decomposition $\gamma = s + n$, and if $\gamma$ lies in $\mathfrak{g}(k)$ then so do $s$ and $n$. For general $V$, however, as noted in [7], an element $\gamma \in V$ may have multiple Jordan–Kac–Vinberg decompositions. For all of them, the element $s$ lies in a single $G$-orbit, namely the unique closed $G$-orbit in $\overline{G \cdot \gamma}$.

In [7], Kac showed that every vector $\gamma \in V$ has a Jordan–Gatti–Viniberghi decomposition (as he called it), as a simple application of the Luna slice theorem. A rational version of the Luna slice theorem has been proven by Bremigan [5], and it implies (Lemma 4.3) that every $k$-point $\gamma \in V(k)$ has a $k$-Jordan–Kac–Vinberg decomposition, that is a Jordan–Kac–Vinberg decompostion $\gamma = s + n$ with $s$ (and hence $n$) in $V(k)$. This fact has not previously appeared in the literature.

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In this paper we show that given $\gamma \in V(k)$, the semisimple parts $s$ of all $k$-Jordan–Kac–Vinberg decompositions $\gamma = s + n$ of $\gamma$ all lie in a single $G(k)$-orbit. In other words, even though $k$-Jordan–Kac–Vinberg decompositions are not unique, the $G(k)$-orbit of the semisimple parts is. This uniqueness is important in producing the fine geometric expansion in relative trace formulas (see the discussion in [9]).

We use two tools to prove this. One is the rational version of the Hilbert–Mumford Theorem, as proven by Kempf [8] and Rousseau. The Hilbert–Mumford Theorem allows us to restate the problem in terms of limits of the form $\lim_{t\to 0} \lambda(t) \cdot \gamma$, for $k$-cocharacters $\lambda$ in $G$.

The other is a recent rationality result of Bate–Martin–Röhrle–Tange [2] on such limits.

In fact we prove the somewhat more general result, Theorem 3.4, that given $\gamma \in V(k)$, the limit points $\lim_{t\to 0} \lambda(t) \cdot \gamma$ that are semisimple, ranging over all cocharacters $\lambda$ defined over $k$, all lie in a single $G(k)$-orbit. This solves the conjecture in [10].

## 2 Preliminaries

We begin with some notation. Let $k$ be a field (of any characteristic), and write $\overline{k}$ for its algebraic closure. Let $G$ be a reductive algebraic group (not necessarily connected) defined over $k$. Write $X^*(G)$ for the group of characters $\chi\colon G \to \mathrm{GL}(1)$, and $X_*(G)$ for the set of cocharacters $\lambda\colon \mathrm{GL}(1) \to G$. Similarly write $X^*(G)_k$ (resp. $X_*(G)_k$) for those characters (resp. cocharacters) defined over $k$. Define the map $\langle\ ,\ \rangle\colon X_*(G) \times X^*(G) \to \mathbb{Z}$ by requiring the identity $\chi\big(\lambda(t)\big) = t^{\langle\lambda,\chi\rangle}$. The group $G$ acts naturally on $X_*(G)$:

$$(g\cdot\lambda)(t) = g\lambda(t)g^{-1}, \quad \text{for } g \in G, \lambda \in X_*(G), t \in \mathrm{GL}(1).$$

Given $\lambda \in X_*(G)_k$ and $g \in G(k)$, the cocharacter $g \cdot \lambda$ is also in $X_*(G)_k$.

Suppose that $V$ is an affine $G$-variety. Given $\lambda \in X_*(G)$ and $v \in V$, we say that the *limit*

$$\lim_{t\to 0} \lambda(t) \cdot v \tag{2.1}$$

exists and equals $x$ if there is a morphism of varieties $\ell\colon \mathbb{A}^1 \to V$ with $\ell(t) = \lambda(t) \cdot v$ for $t \neq 0$, and $\lambda(0) = x$. Notice that if $\ell$ exists then it is unique; also if $V$ and $\lambda$ are defined over $k$, with $v \in V(k)$, then $\ell$ must also be defined over $k$, and so $x$ must lie in $V(k)$. Given $v \in V(k)$, write $\Lambda(v, k)$ for the set of $\lambda \in X_*(G)_k$ such that the limit (2.1) exists.

The group $G$ acts on itself via the action $y \mapsto xyx^{-1}$. Given $\lambda \in X_*(G)$, let $P(\lambda)$ be the subvariety

$$P(\lambda) = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}.$$

It is an algebraic group, defined over $k$ if $\lambda$ is. These groups $P(\lambda)$ were defined in [11] and are called the Richardson parabolic subgroups in [2]. The map

$$h_\lambda\colon P(\lambda) \to G, \quad h_\lambda(g) = \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$$

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is a homomorphism of algebraic groups, defined over $k$ if $\lambda$ is. The image and kernel are given by

$$\operatorname{Im} h_\lambda = G^\lambda = \{g \in G \mid \lambda(t) g \lambda(t)^{-1} = g, \text{ for all } t\}$$
$$\ker h_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1\} = R_u\big(P(\lambda)\big)$$

(see [11] for details).

Now suppose that $V$ is a $G$-module defined over $k$. Given any $\lambda \in X_*(G)$, we can then define the $G^\lambda$-modules

$$V_{\lambda,n} = \{v \in V \mid \lambda(t) \cdot v = t^n v \text{ for all } t\}, \quad n \in \mathbb{Z}$$
$$V_{\lambda,+} = \sum_{n>0} V_{\lambda,n}, \quad V_{\lambda,0+} = \sum_{n\geq 0} V_{\lambda,n} = V_{\lambda,0} \oplus V_{\lambda,+}.$$

Notice that $V_{\lambda,0+}$ consists of those vectors $v \in V$ such that the limit (2.1) exists, and is invariant under $P(\lambda)$; in fact for $g \in P(\lambda)$, and $v \in V_{\lambda,0+}$,

$$\lim_{t\to 0} \lambda(t) \cdot (g \cdot v) = h_\lambda(g) \cdot \big(\lim_{t\to 0} \lambda(t) \cdot v\big). \tag{2.2}$$

Further, for $v \in V_{\lambda,0+} = V_{\lambda,0} \oplus V_{\lambda,+}$, the limit (2.1) is just the projection of $v$ to $V_{\lambda,0}$.

Suppose next that $A$ is a $k$-defined torus in $G$. For each $\chi \in X^*(A)_k$ define $V^\chi$ by

$$V^\chi = \{v \in V \mid a \cdot v = \chi(a) v, \text{ for all } a \in A\}.$$

Then only finitely many $V^\chi$ are nonzero and $V$ is their direct sum. Given a vector $v \in V$, write $v_\chi$ for the component of $v$ in the space $V^\chi$, $\chi \in X^*(A)_k$, and set

$$\operatorname{supp} v = \operatorname{supp}_A v = \{\chi \in X^*(A)_k \mid v_\chi \neq 0\},$$

so that

$$v = \sum_{\chi \in \operatorname{supp} v} v_\chi.$$

For any $\lambda \in X_*(A)_k \subset X_*(G)_k$, each vector space $V_{\lambda,n}$, $n \in \mathbb{Z}$, is also a direct sum of weight spaces:

$$V_{\lambda,n} = \sum_{\substack{\chi \in X^*(A)_k \\ \langle \lambda,\chi\rangle = n}} V^\chi.$$

We record the following obvious statement for later use.

***Lemma 2.1*** *Suppose that $\lambda$ is in $X_*(A)_k$. For a vector $v \in V$, the limit (2.1) exists if and only if for every $\chi \in \operatorname{supp} v$ we have $\langle \lambda, \chi\rangle \geq 0$; in this case the limit equals*

$$\sum_{\substack{\chi \in X^*(A)_k \\ \langle \lambda,\chi\rangle = 0}} v_\chi.$$

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## 3 Limits

In this section we assume that $k$ is perfect. We summarize some results that we will later use. First is the rational version of the Hilbert–Mumford Theorem, [8, Corollary 4.3].

***Lemma 3.1*** *If $\gamma \in V(k)$, then there exists $\lambda \in X_*(G)_k$ so that the limit*

$$\lim_{t\to 0} \lambda(t) \cdot \gamma$$

*exists and is semisimple.*

***Remark*** Note that this limit point must necessarily lie in $V(k)$.

The following two results are also essential to our proof. The first is a restatement of [2, Lemma 2.15].

***Lemma 3.2*** *Suppose that $A \subset G$ is a $k$-defined torus and $\lambda, \lambda_0$ are in $X_*(A)_k$. Suppose that vectors $\gamma, v_0, v' \in V$ are related by*

$$v_0 = \lim_{t\to 0} \lambda_0(t) \cdot \gamma,$$
$$v' = \lim_{t\to 0} \lambda(t) \cdot v_0.$$

*Then there exists $\mu \in X_*(A)_k$ such that*

$$V_{\mu,0} = V_{\lambda_0,0} \cap V_{\lambda,0}$$
$$V_{\mu,+} \supseteq V_{\lambda_0,+}, \quad V_{\mu,0+} \subseteq V_{\lambda_0,0+},$$
$$v' = \lim_{t\to 0} \mu(t) \cdot \gamma.$$

***Remark*** The cocharacter $\mu$ can be of the form $n\lambda_0 + \lambda$ for any sufficiently large $n \in \mathbb{N}$.

The second result is [2, Cor. 3.7].

***Lemma 3.3*** *Let $v \in V(k)$ be semisimple. For every $\lambda \in X_*(G)_k$, if the limit $\lim_{t\to 0} \lambda(t) \cdot v$ exists, then it lies in $G(k) \cdot v$.*

***Remark*** In fact, [2, Cor. 3.7] shows that the limit must lie in $R_u\big(P(\lambda)\big)(k) \cdot v$.

Our main result in this section is the following.

***Theorem 3.4*** *Let $G$ be a reductive group and $V$ a $G$-module. Suppose that $k$ is perfect and let $\gamma \in V(k)$. Then for every $\lambda, \mu \in X_*(G)_k$ such that both vectors $v = \lim_{t\to 0} \lambda(t) \cdot \gamma$ and $v' = \lim_{t\to 0} \mu(t) \cdot \gamma$ exist and are semisimple, $v'$ lies in $G(k) \cdot v$.*

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***Remarks*** (a) This solves Conjecture 1.5 of [10].

(b) As is well-known (see for example [2, Remark 2.8] or [8, Lemma 1.1]), we can embed any affine $G$-variety over $k$ inside a $k$-defined rational $G$-module, and hence Theorem 3.4 is also valid for affine $G$-varieties.

***Definition 3.5*** Let $\Lambda(\gamma, k)_{\min}$ be the set of cocharacters that minimize $\dim V_{\lambda,0}$, among $\lambda \in X_*(G)_k$ such that $\lim_{t\to 0} \lambda(t) \cdot \gamma$ exists and is semisimple.

***Remark*** By the Kempf–Rousseau–Hilbert–Mumford theorem 3.1, and because $\dim V_{\lambda,0}$ is always a nonnegative integer, the set $\Lambda(\gamma, k)_{\min}$ is non-empty.

***Lemma 3.6*** *Given $\lambda \in \Lambda(\gamma, k)_{\min}$ and $p \in P(\lambda)(k)$, we have that $\lambda \in \Lambda(p\cdot\gamma, k)_{\min}$. Further, the limit points*

$$v = \lim_{t\to 0} \lambda(t)\cdot\gamma \quad \textit{and} \quad \lim_{t\to 0} \lambda(t)\cdot(p\cdot\gamma)$$

*lie in the same $G(k)$-orbit.*

**Proof** Let $\lambda \in \Lambda(\gamma, k)_{\min}$. By (2.2), $\lim_{t\to 0} \lambda(t)\cdot(p\cdot\gamma) = h_\lambda(p)\cdot v$, so the limit exists and lies in the $G(k)$-orbit of $v$; consequently its $G$-orbit is closed.

On the other hand, given any $\mu \in \Lambda(p\cdot\gamma, k)$, we have that $p^{-1}\cdot\mu \in \Lambda(\gamma, k)$ and $\dim V_{\mu,0} = \dim V_{p^{-1}\cdot\mu,0}$; since $\lambda \in \Lambda(\gamma, k)_{\min}$, this dimension is at least $\dim V_{\lambda,0}$; hence $\lambda$ lies in $\Lambda(p\cdot\gamma, k)_{\min}$. ■

***Lemma 3.7*** *Given $\lambda_0 \in \Lambda(\gamma, k)_{\min}$, write $v_0 = \lim_{t\to 0} \lambda_0(t)\cdot\gamma$. Suppose that $A \subset G$ is a $k$-defined torus with $\lambda_0 \in X_*(A)_k$. Suppose that for $\lambda \in X_*(A)_k$, the limit $v = \lim_{t\to 0} \lambda(t)\cdot\gamma$ exists and has a closed $G$-orbit. Then $v$ lies in $G(k)\cdot v_0$.*

**Proof** By Lemma 2.1 the existence of the limit $v$ implies that for every $\chi \in \operatorname{supp}(\gamma)$ we have $\langle\lambda, \chi\rangle \geq 0$, and the vector $v$ is the sum

$$\sum_{\substack{\chi\in\operatorname{supp}\gamma \\ \langle\lambda,\chi\rangle=0}} \gamma_\chi,$$

the projection of $\gamma$ to $V_{\lambda,0}$; in particular $\operatorname{supp} v \subseteq \operatorname{supp}\gamma$ and $\gamma - v \in V_{\lambda,+}$. Similarly $\operatorname{supp}(v_0)$ is contained in $\operatorname{supp}(\gamma)$, and so by Lemma 2.1 we may conclude that the limit $v' = \lim_{t\to 0} \lambda(t)\cdot v_0$ exists. Since $G\cdot v_0$ is closed, $v'$ lies in $G\cdot v_0$, so that $G\cdot v'$ is also closed.

We then obtain, from Lemma 3.2, a $\mu \in \Lambda(\gamma, k)$ with $v' = \lim_{t\to 0} \mu(t)\cdot\gamma$, having a closed $G$-orbit, and

$$V_{\mu,0} = V_{\lambda_0,0} \cap V_{\lambda,0} \tag{3.1}$$

$$V_{\mu,+} \supseteq V_{\lambda_0,+}, \quad V_{\mu,0+} \subseteq V_{\lambda_0,0+}. \tag{3.2}$$

Since $\lambda_0$ lies in $\Lambda(\gamma, k)_{min}$, we may conclude that $V_{\mu,0} = V_{\lambda_0,0}$, and hence by (3.1), (3.2), also that $V_{\mu,0+} = V_{\lambda_0,0+}$. The limit point $v'$ is the projection of $\gamma$ to $V_{\mu,0} = V_{\lambda_0,0}$, hence $v' = v_0$.

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Since $\lim_{t\to 0}\mu(t)\cdot\gamma$ exists, $\gamma$ and hence $v$ lie in $V_{\mu,0+}$. Now, the projection of $\gamma - v$ to

$$V_{\lambda,0} = \sum_{\substack{\chi\in X^*(A)_k \\ \langle\lambda,\chi\rangle=0}} V^\chi$$

is zero. By (3.1), $V_{\mu,0} \subseteq V_{\lambda,0}$, so the projection of $\gamma - v \in V_{\mu,0+}$ to $V_{\mu,0}$ is also zero, and hence

$$\lim_{t\to 0}\mu(t)\cdot v = \lim_{t\to 0}\mu(t)\cdot\gamma = v' = v_0.$$

By 3.3, we can finally conclude that $v$ lies in $G(k)\cdot v_0$. ■

**Proof of Theorem 3.4** First, note that a cocharacter in $G$ is necessarily a cocharacter in the connected component $G^0$ of the identity in $G$, and that it is sufficient to prove Theorem 3.4 for $G^0$. Without loss of generality, we therefore assume that $G$ is connected.

Pick $\lambda_0 \in \Lambda(\gamma,k)_{\min}$, set $v_0 = \lim_{t\to 0}\lambda_0(t)\cdot\gamma$. Since being in the same $G(k)$-orbit is an equivalence relation, it is clearly sufficient to prove the theorem for $\mu = \lambda_0$, $v' = v_0$.

The image of $\lambda_0$ lies in a maximal torus, and by [4, 1.4] must in fact lie in a maximal $k$-split torus $A$. Fix a minimal $k$-defined parabolic subgroup $P$ of $G$, with $C_G(A) \subseteq P \subseteq P(\lambda_0)$. The choice of $P$ corresponds to a choice of basis ${}_k\Delta$ of simple roots of $G$ with respect to $A$.

The image of $\lambda$ also lies in some maximal $k$-split torus, so since all maximal $k$-split tori are conjugate over $G(k)$ [3, Thm. 20.9(ii)], there exists $g \in G(k)$ so that the image of $g\cdot\lambda$ lies in $A$. Multiplying $g$ on the left by an element of $N_G(A)(k)$ if necessary, we can arrange that $\langle g\cdot\lambda,\alpha\rangle \geq 0$ for every $\alpha \in {}_k\Delta$, that is, $P \subseteq P(g\cdot\lambda)$. Let us write $\lambda_A$ for $g\cdot\lambda \in X_*(A)$.

We now apply the Bruhat decomposition: write

$$g = pwu, \quad p \in P(k) \subseteq P(\lambda_A)(k), \quad w \in N_G(A), \quad u \in R_u(P)(k).$$

Then

$$\begin{aligned} v &= \lim_{t\to 0}\lambda(t)\cdot\gamma = g^{-1}\cdot\lim_{t\to 0}\lambda_A(t)g\cdot\gamma \\ &= g^{-1}\cdot[\lim_{t\to 0}\lambda_A(t)p\lambda_A(t)^{-1}]\cdot\lim_{t\to 0}\lambda_A(t)wu\cdot\gamma \\ &= g^{-1}h_{\lambda_A}(p)\cdot\lim_{t\to 0}\lambda_A(t)wu\cdot\gamma \\ &= g^{-1}h_{\lambda_A}(p)w\cdot\lim_{t\to 0}(w^{-1}\cdot\lambda_A)(t)\cdot(u\cdot\gamma), \end{aligned} \tag{3.3}$$

with $gh_{\lambda_A}(p)w \in G(k)$. Note that the existence of the first limit in (3.3) implies the existence of the others.

Now, $u \in R_u(P)(k) \subseteq P(k) \subseteq P(\lambda_0)(k)$, so by Lemma 3.6, $\lambda_0 \in \Lambda(u\cdot\gamma,k)_{\min}$. Notice also that $\lambda_0$ and $w^{-1}\cdot\lambda_A$ both lie in $X_*(A)_k$. By Lemmas 3.7 and 3.6,

$$\lim_{t\to 0}(w^{-1}\cdot\lambda_A)\cdot(u\cdot\gamma) \in G(k)\cdot\lim_{a\to 0}\lambda_0(t)\cdot(u\cdot\gamma) = G(k)\cdot v_0$$

so $v$ is also in $G(k)\cdot v_0$. ■

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## 4 Application to Jordan Decompositions

In this section, we require $k$ to have characteristic 0.

***Definition 4.1*** (i) A *Jordan–Kac–Vinberg decomposition* of a vector $\gamma \in V$ is as in Definition 1.1(iii).

(ii) Given $\gamma \in V(k)$, a *k-Jordan–Kac–Vinberg decomposition* of $\gamma$ is a Jordan–Kac–Vinberg decomposition $\gamma = s + n$ with $s$ (and hence $n$) in $V(k)$.

Kac [7] used the Luna Slice theorem to prove that every vector has a Jordan–Kac–Vinberg decomposition. We now show that every vector in $V(k)$ has a $k$-Jordan–Kac–Vinberg decomposition.

Bremigan proved a rational version of the Luna Slice Theorem in [5]. The following is an immediate consequence of it.

***Lemma 4.2*** *Given $v \in V(k)$ semisimple, let $F$ be the subvariety of points $\gamma \in V$ with $G \cdot v \subseteq \overline{G \cdot \gamma}$. Then there is a $G$-invariant retraction $\psi\colon F \to G \cdot v$ that is defined over $k$ such that $\psi(\gamma) \in \overline{G_{\psi(\gamma)} \cdot \gamma}$ for every $\gamma \in F$.*

**Proof** A $G$-invariant retraction $\psi\colon F \to G \cdot v$, defined over $k$, is given in [5, Cor. 3.4]. A point $\gamma \in F$ is written as $\gamma = g \cdot x$ with $g \in G$ and $v \in \overline{G_v \cdot x}$ (and $x$ in the selected Luna slice), and $\psi(\gamma)$ is then set to be $g \cdot v$. But then

$$\psi(\gamma) = g \cdot v \in \overline{G_{g \cdot v} \cdot (g \cdot x)} = \overline{G_{\psi(\gamma)} \cdot \gamma},$$

as required. ∎

***Remark*** For fields of positive characteristic, the Luna Slice Theorem does not hold without additional assumptions. See [1] for further details.

***Corollary 4.3*** *Every $\gamma \in V(k)$ has a k-Jordan–Kac–Vinberg decomposition.*

**Proof** Let $\gamma \in V(k)$. By Lemma 3.1, there exists a semisimple $v \in \overline{G \cdot \gamma} \cap V(k)$. Lemma 4.2 provides a $G$-invariant map $\psi$, defined over $k$, from $F$ to $G \cdot v$. Setting $s = \psi(\gamma)$, we immediately see that $s \in V(k)$, that $s$ is semisimple, that $G_\gamma \subseteq G_s$, and that the unique closed $G_s$-orbit in $\overline{G_s \cdot \gamma}$ is $s$. Subtracting $s$, the unique closed $G_s$-orbit in $\overline{G_s \cdot (\gamma - s)}$ is 0. Therefore $\gamma = s + (\gamma - s)$ is a $k$-Jordan–Kac–Vinberg decomposition. ∎

We can use the Hilbert–Mumford theorem to provide an alternate description of a Jordan–Kac–Vinberg decomposition.

***Proposition 4.4*** *A decomposition $\gamma = s + n$, with $s$ semisimple, and $G_\gamma \subseteq G_s$, is a Jordan–Kac–Vinberg decomposition if and only if there exists $\lambda \in X_*(G_s)$ so that*

$$\lim_{t \to 0} \lambda(t) \cdot \gamma = s. \tag{4.1}$$

*If $\gamma \in V(k)$, then $\gamma = s + n$ is a k-Jordan–Kac–Vinberg decomposition if and only if $\lambda$ can be taken to be in $X_*(G_s)_k$.*

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**Proof** The first part of the proposition is just the second part over $\bar{k}$, so we need only consider the second part.

Given a $k$-Jordan–Kac–Vinberg decomposition $\gamma = s+n$, we know that $0 \in \overline{G_s \cdot n}$. The Hilbert–Mumford Theorem (Lemma 3.1) provides a $\lambda \in X_*(G_s)_k$ such that

$$\lim_{t\to 0} \lambda(t)\cdot n = 0. \tag{4.2}$$

However, since the image of $\lambda$ is in $G_s$, we can add $s$ and obtain (4.1).

In the other direction, given $\lambda \in X_*(G_s)_k$, subtracting $s$ from (4.1) gives (4.2), implying that $n$ is nilpotent with respect to $G_s$. Since $\gamma$ and $\lambda$ are defined over $k$, so are $s$ and $n$, hence $\gamma = s + n$ is a $k$-Jordan–Kac–Vinberg decomposition. ■

From Proposition 4.4 and Theorem 3.4, we immediately obtain the following.

***Corollary 4.5*** *For any two k-Jordan–Kac–Vinberg decompositions $\gamma = s + n$, $\gamma = s' + n'$ of $\gamma \in V(k)$, we have $s' \in G(k) \cdot s$.*

This means that although a vector $\gamma \in V(k)$ may have multiple $k$-Jordan–Kac–Vinberg decompositions, all such decompositions lie in a single $G(k)$-orbit.

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*Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa, ON K1N 6N5*
*e-mail*: jlevy@uottawa.ca

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