Can. J. Math., Vol. XXIX, No. 5, 1977, pp. 1010–1030

RELATIONS BETWEEN GENERALIZED GROWTH CONDITIONS AND SEVERAL CLASSES OF CONVEXOID OPERATORS

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1. Introduction. In this paper we shall discuss some classes of bounded linear operators on a complex Hilbert space. If T is a bounded linear operator T acting on the complex Hilbert space H, then the following two inequalities always hold:

$$(1.1)\quad \frac{1}{d(\mu, \sigma(T))} \leqq ||(T-\mu)^{-1}|| \quad \text{for all } \mu \notin \sigma(T),$$

$$(1.2)\quad ||(T-\mu)^{-1}|| \leqq \frac{1}{d(\mu, W(T))} \quad \text{for all } \mu \notin \overline{W(T)},$$

where $\sigma(T)$ indicates the spectrum of T, $W(T)$ denotes the numerical range of T defined by $W(T) = \{(Tx, x) : ||x|| = 1 \text{ and } x \in H\}$ and $\overline{W(T)}$ means the closure of $W(T)$ respectively.

As some generalizations of ordinary growth conditions, we shall define generalized growth conditions as follows: an operator T is said to satisfy *the condition* (G_1) *for* (M, N) (in symbols, $T \in (G_1)$ for (M, N)) if

$$(1.3)\quad ||(T-\mu)^{-1}|| \leqq \frac{1}{d(\mu, M)} \quad \text{for all } \mu \notin N$$

where M and N are two closed sets satisfying $N \supset M \supset \sigma(T)$. Similarly an operator T is said to satisfy *the condiiton* $(w - G_1)$ *for* (M, N) (in symbols, $T \in (w - G_1)$ for (M, N)) if

$$(1.4)\quad w((T-\mu)^{-1}) \leqq \frac{1}{d(\mu, M)} \quad \text{for all } \mu \notin N$$

where M and N are two closed sets satisfying $N \supset M \supset \sigma(T)$ and $w(T)$ denotes the numerical radius of T defined by $w(T) = \sup \{|\lambda| : \lambda \in W(T)\}$. $T \in (G_1)$ for M [**22**] coincides with $T \in (G_1)$ for (M, M) and similarly $T \in (w - G_1)$ for M means $T \in (w - G_1)$ for (M, M) respectively.

An operator T is said to satisfy the condition (G_1) (in symbols, $T \in (G_1)$) if $T \in (G_1)$ for $\sigma(T)$, that is, $T \in (G_1)$ if T satisfies the equality in (1.1). An operator T is said to be *convexoid* [**11**] if $\overline{W(T)} = \text{co}\ \sigma(T)$ where co M means the convex hull of a set M in the complex plane. It is well known [**17**]

Received October 4, 1976.

that T is convexoid if and only if $T \in (G_1)$ for co $\sigma(T)$. Moreover, an operator T is said to be *spectraloid* (resp. *normnloid*) if $w(T) = r(T)$ (resp. $\|T\| = r(T)$) [**11**], where $r(T)$ denotes the spectral radius of T.

Luecke [**15**] introduced a new class of operators such that $T \in R$ if

$$(1.5)\quad \|(T - \mu)^{-1}\| = \frac{1}{d(\mu, W(T))} \quad \text{for all } \mu \notin \overline{W(T)}.$$

Namely, $T \in R$ if T satisfies the equality in (1.2). Moreover he showed the following characterization of this class:

THEOREM A [**15**]. *$T \in R$ if and only if $\partial W(T) \subset \sigma(T)$ where ∂M is the boundary of M.*

Luecke's class R consists of the multiples of the identity even in the finite dimensional Hilbert space and this class does not contain general normal operators.

Fujii [**4**, II] has defined the hen-spectrum $\tilde{\sigma}(T)$ of an operator T in order to construct a new class of operators which includes both (G_1) and R; that is, hen-spectrum $\tilde{\sigma}(T)$ is defined by $\tilde{\sigma}(T) = [[\sigma(T)^c]_\infty]^c$ where M^c is the complement of M and $[M]_\infty$ the unbounded component of M and $\tilde{\sigma}(T)$ is a compact set containing $\sigma(T)$ in the complex plane. Using this notion of $\tilde{\sigma}(T)$, Fujii [**4**, II] has introduced a new class of operators (G_1) for $\tilde{\sigma}(T)$ (in symbols, (H_1)) which properly contains both (G_1) and R, that is, $T \in (H_1)$ if

$$(1.6)\quad \|(T - \mu)^{-1}\| \leqq \frac{1}{d(\mu, \tilde{\sigma}(T))} \quad \text{for all } \mu \notin \tilde{\sigma}(T).$$

Moreover, he has given another interesting characterization of operators belonging to R as follows:

THEOREM B [**4**, II]. *$T \in R$ if and only if $\overline{W(T)} = \tilde{\sigma}(T)$.*

On the other hand C. R. Putnam considered conditions on an operator T implying

$$(*)\quad \operatorname{Re} \sigma(T) = \sigma(\operatorname{Re} T).$$

The equation (*) holds for normal and also seminormal operators [**19**]. Moreover, (*) has considerable significance for non-normal operators; namely, (*) plays a role in the proof of C. R. Putnam [**19**; **20**] which states that a seminormal operator whose spectrum has zero area is normal. S. K. Berberian has not only given a simple proof of Putnam's result, but he has also proved the following theorems.

THEOREM C [**3**]. *If T satisfies the growth condition (G_1) and $\sigma(T)$ is connected, then (*) holds.*

THEOREM D[**3**]. *If T is an operator such that $\sigma(T)$ is a spectral set for T, then (*) holds.*

Related to Theorem C, Patel [**18**] has established that the equation (*) also holds for operators in the class R, defined by Luecke without any restriction on the spectrum, as follows:

THEOREM E [**18**]. *If $T \in R$, then (*) holds.*

In this paper we shall give slightly different characterizations of convexoid operators (Proposition 2.1 in Section 2). As an application of this result we shall introduce a class S of convexoid operators (G_1) for $(\sigma(T), \overline{W(T)})$ which includes (H_1) in connection with generalized growth conditions and we shall construct non-trivial examples belonging to S in Section 3. Moreover, we shall show the slightly different characterizations of operators belonging to the class R in Section 4.

In Section 5, as an extension of an already known unified formulation, we shall give a construction of operators satisfying generalized growth conditions.

In Section 6 we shall introduce a more narrow class P which is properly contained in both classes (G_1) and R. We shall show a method to construct operators belonging to P and we shall discuss relations among (G_1), R, the class C of the set of all convexoid operators and the class P. Moreover,we shall discuss some related results.

In Section 7 we shall give some extensions of Theorem C and Theorem E, and as an application of this result, we shall show some extensions of Theorem D. And we shall give a method of construction of operators satisfying (*) and some counter examples.

In Section 8 we shall show another characterization of operators in R which is considered as both a converse of Theorem E and a precise estimation of a result of [**4**, II, Theorem 12], and we shall show a parallel result to this characterization and some related results.

Finally, in Section 9 we shall make some comments to Lin's results [**14**] related to generalized convexoid operators.

2. Characterizations of convexoid operators. First we show the following theorem.

THEOREM 2.1. *If $T \in (w - G_1)$ for (M, N), then $\overline{W(T)} \subset \text{co}\, M$.*

For the sake of convenience, we state the following obvious lemma in connection with subsequent discussion.

LEMMA 2.1. *If X is any bounded closed set in the complex plane, then*

(i) $\text{co}\, X = \{$*the intersection of all the closed half planes which contain the set* $X\}$

$$= \bigcap_{\theta} \left\{ \lambda : \operatorname{Re} \lambda e^{i\theta} \geqq \inf_{s \in X} \operatorname{Re} s e^{i\theta} \right\}$$

(ii) co X = {*the intersection of all the circles which contain the set* X}

$$= \bigcap_{\mu} \left\{ \lambda : |\lambda - \mu| \leqq \sup_{x \in X} |x - \mu| \right\}$$

(iii) co X = {*the intersection of all the circles with sufficiently large radii which contain the set* X}

$$= \bigcap_{\mu} \left\{ \lambda : |\lambda - \mu| \leqq \sup_{x \in X} |x - \mu| \text{ for all } \mu \text{ whose absolute values are sufficiently large} \right\}.$$

Proof of Theorem 2.1. By the hypothesis, we have

$$d(\mu, M)|(x, (T - \mu)x)| \leqq ||(T - \mu)x||^2$$

for all $\mu \notin N \supset M$ and $||x|| = 1$, so that

$$\inf_{s \in M} (|s|^2 - 2 \operatorname{Re} s\bar{\mu} + |\mu|^2)|(x, (T - \mu)x/|\mu|)|^2$$
$$\leqq ||(T - \mu)x/|\mu|\,||^2(||Tx||^2 - 2 \operatorname{Re} (Tx, x)\bar{\mu} + |\mu|^2).$$

Taking $\mu = -|\mu|e^{-i\theta}$ and dividing by $|\mu|$ and transfering $|\mu|$ to ∞, we obtain

$$\operatorname{Re} (Tx, x)e^{i\theta} \geqq \inf_{s \in M} \operatorname{Re} se^{i\theta} \quad \text{for } ||x|| = 1.$$

This implies $\overline{W(T)} \subset \text{co } M$ by (i) of Lemma 2.1 which is the desired relation and so the proof is complete.

Here we shall sum up several characterizations of convexoid operators for the convenience of subsequent discussion.

PROPOSITION 2.1. *Any one of the following conditions is necessary and sufficient in order that* T *is convexoid:*

(i) $T - \mu$ *is spectraloid for all complex* μ;

(ii) $T - \mu$ *is spectraloid for all complex* μ *whose absolute values are sufficiently large;*

(iii) $||(T - \mu)^{-1}|| \leqq \dfrac{1}{d(\mu, \text{co } \sigma(T))}$ *for all* $\mu \notin \text{co } \sigma(T)$,

(iv) $||(T - \mu)^{-1}|| \leqq \dfrac{1}{d(\mu, \text{co } \sigma(T))}$ *for all complex* μ *whose absolute values are sufficiently large,*

(v) $w((T - \mu)^{-1}) \leqq \dfrac{1}{d(\mu, \text{co } \sigma(T))}$ *for all* $\mu \notin \text{co } \sigma(T)$,

(vi) $w((T - \mu)^{-1}) \leqq \dfrac{1}{d(\mu, \text{co } \sigma(T))}$ *for all complex* μ *whose absolute values are sufficiently large.*

Other characterizations of convexoid operators were given in [**5**] and [**8**].

Proof of Proposition 2.1. (i) was shown in [**10**] and thereafter (i) was alternatively obtained in [**8**] by using (ii) of Lemma 2.1. (iii) was obtained in [**17**] and (v) was shown in [**14**], so that we have only to show the sufficiency of (ii), (iv) and (vi). Taking $X = \overline{W(T)}$ and $\sigma(T)$ in (iii) of Lemma 2.1 respectively, we have the following formulas, since $\overline{W(T)}$ is convex [**11**]:

$$(2.1)\quad \overline{W(T)} = \bigcap_{\mu} \{\lambda : |\lambda - \mu| \leqq w(T - \mu) \text{ for all complex } \mu \text{ whose absolute values are sufficiently large}\},$$

$$(2.2)\quad \text{co}\,\sigma(T) = \bigcap_{\mu} \{\lambda : |\lambda - \mu| \leqq r(T - \mu) \text{ for all complex } \mu \text{ whose absolute values are sufficiently large}\}$$

Hence the sufficiency of (ii) follows from (2.1) and (2.2). Take $M = \text{co}\,\sigma(T)$ and N the set of all complex numbers whose absolute values are sufficiently large in Theorem 2.1, so that $\overline{W(T)} \subset \text{co}\,\sigma(T)$. Hence the sufficiency of (vi) follows from the fact the opposite inclusion relation automatically holds [**11**]. The sufficiency of (iv) follows from that of (vi) since $||T|| \geqq w(T)$ is always valid. This completes the proof.

By (2.1) we have the following corollary.

COROLLARY 2.1. $0 \in \overline{W(T)}$ *if and only if* $|\mu| \leqq w(T - \mu)$ *for all complex* μ *whose absolute values are sufficiently large.*

The result that $0 \in \overline{W(T)}$ if and only if $|\mu| \leqq w(T - \mu)$ for all complex μ was shown in [**8**] and [**10**].

3. Construction of a class S. In this section we shall introduce a new class S of convexoid operators which properly includes the class (H_1) containing both (G_1) and R in connection with generalized growth conditions as an application of characterizations of convexoid operators (Proposition 2.1). We shall also construct non-trivial examples belonging to S and discuss some related results.

Definition 3.1. An operator T is said to *belong* to S if T satisfies the condition (G_1) for $(\sigma(T), \overline{W(T)})$, that is, T satisfies the following equality:

$$||(T - \mu)^{-1}|| = \frac{1}{d(\mu, \sigma(T))} \quad \text{for all } \mu \notin \overline{W(T)}.$$

Hence we remark that $T \in S$ if and only if $(T - \mu)^{-1}$ is normaloid for all $\mu \notin \overline{W(T)}$. First we shall show the following theorem.

THEOREM 3.1. *If T belongs to (H_1), then T also belongs to S.*

Proof. For all $\mu \notin \tilde{\sigma}(T)$, we have $d(\mu, \tilde{\sigma}(T)) = d(\mu, \sigma(T))$ since $\partial\tilde{\sigma}(T) = \overline{\tilde{\sigma}(T)} \cap [\overline{\tilde{\sigma}(T)^c}] \subset \sigma(T)$, so that $T \in (H_1)$ if and only if $T \in (G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$; namely, $(T - \mu)^{-1}$ is normaloid for all $\mu \notin \tilde{\sigma}(T)$. Consequently, if $T \in (H_1)$, then $T \in (G_1)$ for $(\sigma(T), \overline{W(T)})$ since $\sigma(T) \subset \tilde{\sigma}(T) \subset \text{co}\,\sigma(T) \subset \overline{W(T)}$ always holds [**4**, II]; that is, $T \in S$, so the proof is complete.

As we stated in the proof of Theorem 3.1, $T \in (H_1)$ if and only if $T \in (G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$. We remark that $T \in (H_1)$ and $\sigma(T)^c$ (the complement of $\sigma(T)$) is a connected set if and only if $T \in (G_1)$ for $\sigma(T)$. Next we show a construction of operators belonging to S.

THEOREM 3.2. *If A is an operator and B satisfies (G_1) for $(\sigma(B), \overline{W(B)})$ such that*

(3.1) $\quad d(\mu, \sigma(B)) \leqq d(\mu, W(A)) \quad$ *for all* $\mu \notin \overline{W(B)}$,

then $T = A \oplus B$ also satisfies (G_1) for $(\sigma(T), \overline{W(T)})$.

Proof. By the hypothesis we have the inequality $d(\mu, W(B)) \leqq d(\mu, \sigma(B)) \leqq d(\mu, W(A))$ for all $\mu \notin \overline{W(B)}$, so that $\overline{W(A)} \subset \overline{W(B)}$ and $\overline{W(T)} = \text{co}\,\{\overline{W(A)} \cup \overline{W(B)}\} = \overline{W(B)}$. Hence for all $\mu \notin \overline{W(T)}$,

$$||(T - \mu)^{-1}|| = \max\{||(A - \mu)^{-1}||, ||(B - \mu)^{-1}||\}$$
$$\leqq \max\left\{\frac{1}{d(\mu, W(A))}, \frac{1}{d(\mu, \sigma(B))}\right\} = \frac{1}{d(\mu, \sigma(B))} \leqq \frac{1}{d(\mu, \sigma(T))}$$

since $\sigma(T) = \sigma(A) \cup \sigma(B)$, so that $T \in (G_1)$ for $(\sigma(T), \overline{W(T)})$; namely, $T \in S$.

Example 3.1. We shall construct a non-trivial example T such that $T \in S$ but $T \notin (H_1)$ by using Theorem 3.2. Put A as follows:

(3.2) $$A = \begin{bmatrix} -3 & 2 \\ 0 & -3 \end{bmatrix}.$$

Then we have $\sigma(A) = \{-3\}$ and $\overline{W(A)} = \{\lambda: |\lambda + 3| \leqq 1\}$. Moreover let B be the normal operator with the following spectrum

$$\sigma(B) = \{\lambda: |\lambda| = 4, \text{Re}\,\lambda \leqq 0\} \cup \{\lambda: |\lambda| = 2, \text{Re}\,\lambda \leqq 0\},$$

and put $T = A \oplus B$. Then we have $\overline{W(B)} = \{\lambda: |\lambda| \leqq 4, \text{Re}\,\lambda \leqq 0\}$. It is easily verified that $d(\mu, \sigma(B)) \leqq d(\mu, W(A))$ for all $\mu \notin \overline{W(B)}$, so that $T = A \oplus B$ belongs to S by Theorem 3.2. Next we shall show that this operator T does not belong to (H_1). We put $\mu_0 = -3 + \frac{1}{2}i$; then $\mu_0 \notin \tilde{\sigma}(T) = \{-3\} \cup \sigma(B)$. By a simple calculation we have

$$||(A - \mu_0)^{-1}|| = \left\|\begin{bmatrix} -\frac{1}{2}i & 2 \\ 0 & -\frac{1}{2}i \end{bmatrix}^{-1}\right\| = \left\|\begin{bmatrix} 2i & 8 \\ 0 & 2i \end{bmatrix}\right\| > 2.$$

If $T \in (H_1)$, then we have the following contradiction:

$$2 < ||(A - \mu_0)^{-1}|| \leqq ||(T - \mu_0)^{-1}|| = \frac{1}{d(\mu_0, \hat{\sigma}(T))} = 2.$$

This implies $T \notin (H_1)$.

THEOREM 3.3. *The class S properly contains the class (H_1).*

Proof. The proof follows from Theorem 3.1 and Example 3.1.

Example 3.2. The spectrum $\sigma(B)$ in Example 3.1 is disconnected. But even if $\sigma(B)$ is connected, there exists an example $T = A \oplus B$ which does not always belong to (H_1) as follows: let A be the same in (3.2) and B be the normal operator with the connected spectrum

$$\sigma(B) = \{\lambda: \lambda = te^{i\theta}, 3 \leqq t \leqq 4, 0 \leqq \theta \leqq \tfrac{3}{2}\pi\}.$$

It is easily verified that $A \oplus B \in S$, but $A \oplus B \notin (H_1)$ by the method analogous to Example 3.1, so that we shall omit its calculation.

As an application of Proposition 2.1 we shall prove the following theorem.

THEOREM 3.4. *If T belongs to S, then T is convexoid.*

Proof. If $T \in S$, that is, $T \in (G_1)$ for $(\sigma(T), \overline{W(T)})$, then $T \in (G_1)$ for $(\text{co}\,\sigma(T), N)$ where N is an arbitrary closed set containing $\overline{W(T)}$, hence T is convexoid by (iv) of Proposition 2.1, so the proof is complete.

Remark 3.1. By Theorem 3.4 we remark that T satisfies (G_1) for $(\sigma(T), \text{co}\,\sigma(T))$ if and only if T belongs to S.

Here we shall show a construction of convexoid operators for the sake of convenience.

THEOREM F [**4**, I]. *If A is an operator and B is convexoid operator such that $\overline{W(B)} \supset \overline{W(A)}$, then $T = A \oplus B$ is also convexoid.*

The following example shows that the class S is properly contained in the class C, the set of all convexoid operators.

Example 3.3. Put A as follows:

$$(3.3) \quad A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Then $\sigma(A) = \{0\}$ and $\overline{W(A)} = D$ where D is the unit disk in the complex plane. Moreover let $B = \text{diag}\,\{\sqrt{2}, -\sqrt{2}, \sqrt{2}i, -\sqrt{2}i\}$ and put $T = A \oplus B$. Then we have $\overline{W(B)} \supset \overline{W(A)}$, so that $T = A \oplus B$ is convexoid by Theorem F and $\overline{W(T)}$ consists of the interior and the boundary of the square whose vertices are $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{2}i$ and $-\sqrt{2}i$ respectively. Then we shall show that this convexoid operator T does not belong to S as follows. Next we put

$\mu_0 = 1 + i$. Then for $\mu_0 \notin \overline{W(T)}$, we have

$$||(A - \mu_0)^{-1}|| = \left\|\begin{bmatrix} \dfrac{-1}{1+i} & \dfrac{-2}{(1+i)^2} \\ 0 & \dfrac{-1}{1+i} \end{bmatrix}\right\|$$

$$\geqq \sqrt{\frac{1}{|1+i|^2} + \left(\frac{2}{|1+i|^2}\right)^2} > 1.$$

If $T \in S$, then

$$1 < ||(A - \mu_0)^{-1}|| \leqq ||(T - \mu_0)^{-1}|| = \frac{1}{d(\mu_0, \sigma(T))} < 1,$$

and this contradiction implies $T \notin S$.

A closed set X in the complex plane is said to be a *spectral set* for an operator T if $\sigma(T) \subset X$ and

$$||f(T)|| \leqq ||f||_X = \sup_{z \in X} |f(z)|$$

where f is a rational function with poles off X [**13**].

Definition 3.2 [**4**, I-II]. An operator T is said to be *numeroid* if $\overline{W(T)}$ is a spectral set for T.

The remainder of this section is devoted to discussing the relation among S, the class of normaloids and the class of numeroids. It is known that if T is a numeroid, then $T - \mu$ is normaloid for all complex μ; hence a numeroid operator is normaloid.

Example 3.4. We shall construct a numeroid operator T which does not belong to S as follows. Put $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; then D is the spectral set for A and $\overline{W(A)} = \frac{1}{2} D$ which consists of the closed disk with center 0 and radius $\frac{1}{2}$. Let B be diag $\{2, -1 + 2i, -1 - 2i\}$ and put $T = A \oplus B$. Then $D \cup \overline{W(A)} \subset \overline{W(B)}$, so that $T = A \oplus B$ is a numeroid by [**4**, III, Remark]. Next we have $\sigma(T) = \sigma(A) \cup \sigma(B) = \{0, 2, -1 + 2i, -1 - 2i\}$ and $\overline{W(T)}$ consists of the interior and the boundary of the triangle whose vertices are 2, $-1 + 2i$ and $-1 - 2i$ respectively. Next we put $\mu_0 = -2$, then for $\mu_0 \notin \overline{W(T)}$ we have $||(A - \mu_0)^{-1}|| > \frac{1}{2}$. If $T \in S$, then

$$\frac{1}{2} < ||(A - \mu_0)^{-1}|| \leqq ||(T - \mu_0)^{-1}|| = \frac{1}{d(\mu_0, \sigma(T))} = \frac{1}{2}.$$

This contradiction implies $T \notin S$.

PROPOSITION 3.1. *There exists a numeroid which does not belong to S and vice versa.*

Proof. By Example 3.4 there exists a numeroid which does not belong to S. Conversely there exists an operator in (G_1) which is not a normaloid [**16**, Theorem 1.3], so that $(G_1) \subset (H_1) \subset S$ by Theorem 3.3 and this implies the desired assertion since non-normaloid is certainly non-numeroid, so the proof is complete.

By Proposition 3.1 we remark that there exists a normaloid which does not belong to S and vice versa.

4. Equivalent representations of the class R. As another application of Proposition 2.1 we shall give the equivalent representations of "characterizations in formula" which exactly indicate that R forms the subclass of the class C, the set of all convexoid operators.

PROPOSITION 4.1. *The following six conditions are mutually equivalent:*

(i) $||(T-\mu)^{-1}|| = \dfrac{1}{d(\mu, W(T))}$ *for all* $\mu \notin \overline{W(T)}$,

(ii) $||(T-\mu)^{-1}|| = \dfrac{1}{d(\mu, \operatorname{co}\sigma(T))}$ *for all* $\mu \notin \overline{W(T)}$,

(iii) $||(T-\mu)^{-1}|| = \dfrac{1}{d(\mu, \operatorname{co}\sigma(T))}$ *for all* $\mu \notin \operatorname{co}\sigma(T)$,

(iv) $w((T-\mu)^{-1}) = \dfrac{1}{d(\mu, W(T))}$ *for all* $\mu \notin \overline{W(T)}$,

(v) $w((T-\mu)^{-1}) = \dfrac{1}{d(\mu, \operatorname{co}\sigma(T))}$ *for all* $\mu \notin \overline{W(T)}$,

(vi) $w((T-\mu)^{-1}) = \dfrac{1}{d(\mu, \operatorname{co}\sigma(T))}$ *for all* $\mu \notin \operatorname{co}\sigma(T)$.

Proof. If T satisfies (ii) or (iii), then T is convexoid by Proposition 2.1, thus (ii) or (iii) implies (i). Conversely if T satisfies (i), then T is convexoid by Theorem A or Theorem B, namely, (i) implies (ii) and (iii) too. If T satisfies (iv), then T also satisfies (i) since the following inequality holds for any operator T:

$$w((T-\mu)^{-1}) \leqq ||(T-\mu)^{-1}|| \leqq \frac{1}{d(\mu, W(T))} \quad \text{for all } \mu \notin \overline{W(T)}.$$

Conversely, if T satisfies (i), then T belongs to (H_1) by [**4**, II]. Moreover, T also belongs to S by Theorem 3.1, namely, $(T-\mu)^{-1}$ is normaloid for all $\mu \notin \overline{W(T)}$, thus (i) implies (iv). The proof of the equivalence relation among (iv), (v) and (vi) is similar to one among (i), (ii) and (iii), so that we shall omit its proof.

Remark 4.1. Luecke characterized the operators in R in terms of the boundary of the numerical range and the spectrum of the operator (Theorem A) and Fujii also characterized the operators in R in terms of the numerical range and the hen-spectrum of the operator (Theorem B). Namely, related to the numerical range and the spectrum, both theorems also indicate the geometrical significations which imply that the operators in R are convexoid. On the other hand, it is somewhat interesting to note Proposition 4.1 indicates that the class R is the subclass of the class C, the set of all convexoid operators; that is, if $T \in R$, then T satisfies exactly the "equality" in the inequality named by the growth condition (G_1) for co $\sigma(T)$ (or equivalently, (G_1) for (co $\sigma(T)$, $\overline{W(T)}$)) which is valid for general convexoid operators by Proposition 2.1. Consequently we come to the conclusion that both theorems of Theorem A and Theorem B can be considered as "geometrical characterizations". On the other hand, Proposition 4.1 can be considered as "Characterizations in formula" in connection with Proposition 2.1.

5. Construction of operators satisfying generalized growth conditions. Fujii [**4**, II] has given the following two theorems as unified formulations of already known results.

THEOREM G. *If A is an operator, X a closed set in the plane with $\overline{W(A)} \subset X$ and B a normal operator with $\sigma(B) \subset X$, then $T = A \oplus B$ satisfies (G_1) for X.*

THEOREM H. *If A does not satisfy (G_1) for X which is a closed set with $\sigma(A) \subset X \subset \overline{W(A)}$ and $X \neq \overline{W(A)}$, then $T = A \oplus B$ does not satisfy (G_1) for X whenever B is a normal operator with $\sigma(B) \subset X$.*

As an extension of both the above theorems and Theorem 3.2, we shall give a method to construct operators satisfying generalized growth conditions as follows.

THEOREM 5.1. *If A is an operator, X and Y both closed sets in the complex plane and B satisfies (G_1) for $(\sigma(B), Y)$ such that for all $\mu \notin Y \supset X$, $d(\mu, X) \leqq \min\{d(\mu, W(A)), d(\mu, \sigma(B))\}$, then $T = A \oplus B$ satisfies (G_1) for (X, Y) in the generalized growth condition.*

Proof. By the hypothesis, for all $\mu \notin Y \supset X$, we have

$$||(T-\mu)^{-1}|| = \max\{||(A-\mu)^{-1}||, ||(B-\mu)^{-1}||\}$$
$$\leqq \max\left\{\frac{1}{d(\mu, W(A))}, \frac{1}{d(\mu, \sigma(B))}\right\} \leqq \frac{1}{d(\mu, X)},$$

so the proof is complete.

In Theorem 5.1, for all $\mu \notin Y \supset X$, $d(\mu, X) \leqq d(\mu, W(A))$ is essential; we can not replace this condition by $d(\mu, X) \leqq d(\mu, \sigma(A))$ for all $\mu \notin Y \supset X$ as follows.

PROPOSITION 5.1. *If X and Y are both closed sets in the complex plane and A does not satisfy (G_1) for (X, Y) such that for all $\mu \notin Y \supset X$, $d(\mu, X) \leqq d(\mu, \sigma(A))$ and there exists a $\lambda \notin Y$ such that $d(\lambda, X) > d(\lambda, W(A))$, then $T = A \oplus B$ does not satisfy (G_1) for (X, Y) whenever B satisfies (G_1) for $(\sigma(B), Y)$ such that the following property holds: $d(\mu, X) \leqq d(\mu, \sigma(B))$ for all $\mu \notin Y \supset X$.*

Proof. The proof is similar to that of Theorem H. By the hypothesis we have a $\mu_0 \notin Y \supset X$ such that

$$||(A - \mu_0)^{-1}|| > \frac{1}{d(\mu_0, X)}.$$

Then

$$||(T - \mu_0)^{-1}|| = \max \{||(A - \mu_0)^{-1}||, ||(B - \mu_0)^{-1}||\}$$
$$= \max \left\{ ||(A - \mu_0)^{-1}||, \frac{1}{d(\mu_0, \sigma(B))} \right\} > \frac{1}{d(\mu_0, X)}.$$

This is the desired relation.

Example 5.1. Related to Proposition 5.1 we shall give an example as follows: put $A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ and moreover let B be the unilateral shift operator and put $T = A \oplus B$. Then we have $\sigma(B) = D$ where D is the unit disk and $\sigma(T) = \sigma(A) \cup \sigma(B) = \{0\} \cup D = \sigma(B)$ and $\overline{W(T)} = \text{co}\{\overline{W(A)} \cup \overline{W(B)}\} = \overline{W(A)} = 2D$. Clearly A does not satisfy (G_1) for $(\sigma(T), \overline{W(T)})$ and for all $\mu \notin \overline{W(T)}$, we have $d(\mu, W(A)) < d(\mu, \sigma(B)) < d(\mu, \sigma(A))$. Then $T = A \oplus B$ does not satisfy (G_1) for $(\sigma(T), \overline{W(T)})$ by Proposition 5.1. In fact this example is not convexoid, so that this cannot belong to S.

6. Relations among C, (G_1), R and a class P. In Section 3 we have introduced a class S containing the class (H_1) which contains both (G_1) and R related to generalized growth conditions. In this section we shall introduce a new narrow class P which is properly contained in the both classes (G_1) and R, we shall show a method to construct operators belonging to this class P and we shall discuss relations among C, (G_1), R and this narrow class P.

Definition 6.1. An operator T is said to *belong* to P if T satisfies $\overline{W(T)} = \sigma(T)$.

An example of an operator belonging to P is the unilateral shift operator. Here we shall give a method to construct an operator belonging to a new class P.

THEOREM 6.1. *If A is an operator and B belongs to P such that $\sigma(B) \supset \overline{W(A)}$, then $T = A \oplus B$ also belongs to P.*

Proof. By the hypothesis $\overline{W(B)} = \sigma(B) \supset \overline{W(A)} \supset \sigma(A)$ and $\overline{W(T)} = \text{co}\{\overline{W(A)} \cup \overline{W(B)}\} = \overline{W(B)} = \sigma(B) = \sigma(T)$. This is the desired relation since $\sigma(T) = \sigma(A) \cup \sigma(B) = \sigma(B)$.

By replacing $\sigma(B)$ by $\tilde{\sigma}(B)$ and P by R in Theorem 6.1, we have the following parallel result to Theorem 6.1 which is implicitly contained in the proof of [**4**, II, Theorem 4].

THEOREM 6.2. *If A is an operator and B belongs to R such that $\tilde{\sigma}(B) \supset \overline{W(A)}$, then $T = A \oplus B$ also belongs to R.*

The typical examples in R are both the unilateral shift and bilateral shift operators.

Remark 6.1. In [**15**] the following proposition is cited: *If A is an operator on H, then $A \oplus N \in R$ on $H \oplus K$ whenever N is a normal operator on K with $\sigma(N) \supset \partial W(A)$.* But it seems to us that this statement is insufficient, that is, there exists a counter example as follows: put $A = 1$, $N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and define $T = A \oplus N$. Then N is normal and $\sigma(N) = \{1, 2\} \supset \partial W(A) = \{1\}$. But this normal operator T cannot belong to R since $\partial W(T) \not\subset \sigma(T)$, that is, $\overline{W(T)} \neq \tilde{\sigma}(T)$ by Theorem A or Theorem B.

Next we shall show that this narrow class P is properly contained in both the classes (G_1) and R.

THEOREM 6.3. *$P \subset (G_1) \cap R$ and this inclusion relation is proper.*

Proof. Since the inequality (1.2) in Section 1 holds for any operator T, if $T \in P$, namely, $\overline{W(T)} = \sigma(T)$, then $T \in (G_1)$ for $\sigma(T)$ by (1.1).

On the other hand if $T \in P$, then $\overline{W(T)} = \tilde{\sigma}(T)$ since $\sigma(T) \subset \tilde{\sigma}(T) \subset \text{co}\, \sigma(T) \subset \overline{W(T)}$ holds for any operator T, so that $T \in R$ by Theorem B. Consequently $P \subset (G_1) \cap R$. The strict inclusion relation follows from the fact that the bilateral shift operator T belongs to both (G_1) and R but this operator cannot belong to P since $\overline{W(T)} = D$ and $\sigma(T)$ is the unit circle. This completes the proof.

In the proof of Theorem 1.2 in [**16**], Luecke shows a method to construct an operator T belonging to (G_1); that is, if A is an operator and B a normal operator such that $\sigma(B) = \overline{W(A)}$, then $T = A \oplus B$ belongs to (G_1), but this operator T exactly belongs to P which is properly included in (G_1) by Theorem 6.1. Moreover Luecke shows a common concrete example which belongs to both (G_1) and R in the proofs of Theorem 1.3 in [**16**] and Theorem 4 in [**15**], but this common example also turns out to belong to P so that we have the following proposition along the lines of his results.

PROPOSITION 6.1. *There exists an invertible operator belonging to P such that* (i) *$T^2 \notin (G_1)$ and $T^2 \notin R$,* (ii) *$r(T) < ||T||$ (non-normaloid),* (iii) *$T^{-1} \notin (G_1)$ and $T^{-1} \notin R$.*

Definition 6.2 [**4**, II]. The class Q is the set of all operators T such that $\tilde{\sigma}(T) = \text{co}\, \sigma(T)$, that is, $T \in Q$ if $\partial\tilde{\sigma}(T)$ is a convex curve.

Fujii [**4**, II] showed that $R = C \cap Q$ where C is the set of all convexoids. Related to his result we shall introduce the following two new classes associated with the class P.

Definition 6.3. The classes U and V are the sets of all operators satisfying $\sigma(T) = \mathrm{co}\,\sigma(T)$ and $\tilde{\sigma}(T) = \sigma(T)$, respectively.

We shall show a characterization of operators belonging to P related to other known classes.

THEOREM 6.4. $P = V \cap R = C \cap U = V \cap Q \cap C$, *where C is the set of all convexoid operators.*

Proof. If $T \in P$, then we have $T \in V$ and $T \in R$ since $\sigma(T) \subset \tilde{\sigma}(T) \subset \mathrm{co}\,\sigma(T) \subset \overline{W(T)}$ holds for any operator T and Theorem B, hence $P \subset V \cap R$. Conversely, if $T \in V \cap R$, then $\sigma(T) = \tilde{\sigma}(T) = \overline{W(T)}$ so that $V \cap R \subset P$. Similarly we have $P = C \cap U = V \cap Q \cap C$, so the proof is complete.

As an extension of Corollary 2 in [**14**], we have the following result which indicates the relation among the classes C, S, (H_1), (G_1) R and P.

THEOREM 6.5. *If T has a convex spectrum, i.e. $\sigma(T) = \mathrm{co}\,\sigma(T)$, then the following statements are mutually equivalent:* (i) *T is convexoid,* (ii) *T belongs to P,* (iii) *T belongs to R,* (iv) *T belongs to (H_1),* (v) *T belongs to S,* (vi) *$(T - \mu)^{-1}$ is spectraloid for all $\mu \notin \overline{W(T)}$,* (vii) *$(T - \mu)^{-1}$ is spectraloid for all $\mu \notin \tilde{\sigma}(T)$.*

Proof. The proof follows from Proposition 2.1 and Theorem 6.3.

Definition 6.4 [**4**, I-II]. An operator T is said to be *hen-spectroid* (resp. spectroid) if $\tilde{\sigma}(T)$ (resp. $\sigma(T)$) is a spectral set for T.

THEOREM I [**4**, III]. *$T \in R$ is a hen-spectroid if and only if there is a sirong normal dilation N of T with $\overline{W(N)} = \tilde{\sigma}(T)$.*

As a parallel result to Theorem I, we assume a stronger hypothesis and give a stronger conclusion as follows.

PROPOSITION 6.2. *$T \in P$ is a spectroid if and only if there is a strong normal dilation N of T with $\overline{W(N)} = \sigma(T)$.*

The proof of Proposition 6.2 is similar to that of Theorem I which is based on Schreiber's result [**24**] and we shall omit it.

7. Some extensions of theorems of Berberian and Patel for operators implying $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$**.** At first we remark that the class (H_1) properly contains both (G_1) and R, and if $T \in R$, then $\mathrm{Re}\,\sigma(T)$ (this equals $\mathrm{Re}\,\overline{W(T)}$) is a closed interval because $T \in R$ if and only if $\partial W(T) \subset \sigma(T)$ by Theorem A (that is, $\overline{W(T)} = \tilde{\sigma}(T)$ by Theorem B) and $\overline{W(T)}$ is the convex set containing $\sigma(T)$ [**11**], so that the following Theorem 7.1 covers Theorem C and Theorem E.

THEOREM 7.1. *If T satisfies (H_1) and* Re $\sigma(T)$ *is a closed interval, then* (*) *holds.*

We shall show the following Lemma 7.1 and Lemma 7.2 along Berberian's idea [**3**] to prove Theorem 7.1.

LEMMA 7.1. *If T satisfies (H_1) and λ is a semibare point of hen-spectrum $\tilde{\sigma}(T)$, then*

(i) $A_\lambda(T) = A_{\lambda *}(T^*)$; *$\lambda$ is a normal approximate point spectrum*

(ii) $N_\lambda(T) = N_{\lambda *}(T^*)$; *$\lambda$ is a normal point spectrum, where*

$A_\lambda(T) = \{\{x_n\}; ||x_n|| = 1; ||Tx_n - \lambda x_n|| \to 0$ *as* $n \to \infty\}$

and $N_\lambda(T)$ denotes the kernel of $T - \lambda$.

Proof. If T satisfies (H_1), then $T - \lambda$ also satisfies (H_1) since $\tilde{\sigma}(T + \lambda I) = \tilde{\sigma}(T) + \lambda$ holds for every complex λ, so that we can assume $\lambda = 0$. As $\lambda = 0$ is a semibare point of $\tilde{\sigma}(T)$, we can choose a non-zero complex number $\lambda_0 \notin \tilde{\sigma}(T)$ such that $\{\lambda: |\lambda - \lambda_0| \leqq |\lambda_0|\}$ meets $\tilde{\sigma}(T)$ only at 0. As seen in the proof of Theorem 3.1, $d(\lambda_0, \tilde{\sigma}(T)) = d(\lambda_0, \sigma(T)) = |\lambda_0|$ holds since $\partial\tilde{\sigma}(T) = \overline{\tilde{\sigma}(T)} \cap \overline{[\tilde{\sigma}(T)^c]} \subset \sigma(T)$ and T satisfies (G_1) for $(\sigma(T), \tilde{\sigma}(T))$. We have the following equality:

$$(7.1) \quad ||(T - \lambda_0)^{-1}|| = \frac{1}{|\lambda_0|}$$

Suppose that $\{x_n\}$ is a sequence of unit vectors such that $Tx_n \to 0$. Then

$$\left\|(T - \lambda_0)^{-1}x_n + \frac{1}{\lambda_0}x_n\right\| \leqq ||(T - \lambda_0)^{-1}|| \left\|x_n + (T - \lambda_0)\frac{1}{\lambda_0}x_n\right\|$$
$$= ||(T - \lambda_0)^{-1}|| \left\|\frac{1}{\lambda_0}Tx_n\right\| \to 0,$$

so that $(T - \lambda_0)^{-1}x_n + (1/\lambda_0)x_n \to 0$. This convergence implies $(T^* - \lambda_0^*)^{-1}x_n + (1/\lambda_0^*)x_n \to 0$ by Schreiber's result [**23**] since (7.1) holds. Whence $T^*x_n \to 0$ by an easy calculation and when we replace T by T^* and λ by λ^*, the above argument is reversible, so we have (i). If we replace x_n by a vector x in the proof of (i), then we have (ii) so the proof is complete.

Remark. 7.1. Lemma 7.1 is derived from [**22**] which is shown by an application of the contraction and unitary dilation theorem of Sz.-Nagy [**26**]. Here we have given a simple proof along the idea due to Berberian [**3**].

LEMMA 7.2. *If T satisfies (H_1), then* Re $\sigma(T) \subset \sigma($Re $T)$ *holds.*

Proof. Let $\alpha_0 \in$ Re $\sigma(T)$. Then there exists $\lambda_0 \in \partial\tilde{\sigma}(T)$ such that Re $\lambda_0 = \alpha_0$ and λ_0 is an approximate point spectrum of T by the definition of hen-spectrum $\tilde{\sigma}(T)$. Let $D_n = \{\lambda: |\lambda - \lambda_0| \leqq 1/n\}$ for $n = 1, 2, \ldots$. Then D_n contains a point $\mu_n \notin \tilde{\sigma}(T)$ such that $|\mu_n - \lambda_0| < 1/2n$. If $\lambda_n \in \tilde{\sigma}(T)$ with $d(\mu_n, \tilde{\sigma}(T)) = d(\mu_n, \sigma(T)) = |\mu_n - \lambda_n|$, then $\lambda_n \in \partial\tilde{\sigma}(T)$ lies on the circumference of a closed disc centered at μ_n whose interior contains no point of $\tilde{\sigma}(T)$, whence λ_n is a semibare point of $\tilde{\sigma}(T)$. Since T satisfies (H_1), it follows that λ_n is a normal approximate point spectrum of T by Lemma 7.1; that is, there exists a sequence of unit vectors $\{x_n\}$ such that $Tx_n - \lambda_n x_n \to 0$ and $T^*x_n - \lambda_n^* x_n \to 0$ as $n \to \infty$. Then we have $Tx_n - \lambda_0 x_n \to 0$ as $n \to \infty$ because

$$||Tx_n - \lambda_0 x_n|| \leqq ||Tx_n - \lambda_n x_n|| + ||(\lambda_n - \lambda_0)x_n|| \to 0$$

as $n \to \infty$. Similarly $T^*x_n - \lambda_0^* x_n \to 0$ as $n \to \infty$, so that

$$||(\mathrm{Re}\, T - \mathrm{Re}\, \lambda_0)x_n|| \leqq \tfrac{1}{2}||Tx_n - \lambda_0 x_n|| + \tfrac{1}{2}||T^*x_n - \lambda_0^* x_n|| \to 0$$

as $n \to \infty$, whence $\mathrm{Re}\, \lambda_0 \in \sigma(\mathrm{Re}\, T)$ and this is the desired relation.

Remark 7.2. It is shown in [**1**] that by changing Hilbert space, we can suppose that the approximate point spectrum coincides with the point spectrum of an operator. By applying this technique and [**2**, Lemma 2], S. K. Berberian has shown Lemma 7.2 in the case that T satisfies (G_1) for $\sigma(T)$ [**3**]. Hence we have given a simple proof of Lemma 7.2 which is based on (i) of Lemma 7.1, without using the changing Hilbert space technique.

LEMMA 7.3. *If T is convexoid, then:*
(i) *if* $\mathrm{Re}\, \sigma(T) \subset \sigma(\mathrm{Re}\, T)$ *and* $\mathrm{Re}\, \sigma(T)$ *is connected, then* (*) *holds;*
(ii) *if* $\sigma(\mathrm{Re}\, T) \subset \mathrm{Re}\, \sigma(T)$ *and* $\sigma(\mathrm{Re}\, T)$ *is connected, then* (*) *holds; and*
(iii) *if both* $\mathrm{Re}\, \sigma(T)$ *and* $\sigma(\mathrm{Re}\, T)$ *are connected, then* (*) *holds* [**3**; **8**].

Proof. It is known that T is convexoid if and only if

$$(\Sigma\text{-}\theta) \quad \mathrm{Re}\, \Sigma(e^{i\theta}T) = \Sigma(\mathrm{Re}\, e^{i\theta}T) \quad \text{for all } 0 \leqq \theta \leqq 2\pi$$

where $\Sigma(T)$ denotes co $\sigma(T)$, and $(\Sigma - \theta)$ is equivalent to co $\mathrm{Re}\, \sigma(e^{i\theta}T) =$ co $\sigma(\mathrm{Re}\, e^{i\theta}T)$ for all $0 \leqq \theta \leqq 2\pi$ [**8**]. If T is convexoid, then we have the following property by $(\Sigma - \theta)$

$$(7.2) \quad \mathrm{co}\, \mathrm{Re}\, \sigma(T) = \mathrm{co}\, \sigma(\mathrm{Re}\, T).$$

On the other hand, by hypothesis of (i) we have

$$(7.3) \quad \mathrm{co}\, \mathrm{Re}\, \sigma(T) = \mathrm{Re}\, \sigma(T) \subset \sigma(\mathrm{Re}\, T) \subset \mathrm{co}\, \sigma(\mathrm{Re}\, T)$$

hence we have (*) by (7.2) and (7.3). Similarly we have (ii). By (7.2) and hypothesis of (iii), we have (iii).

Remark 7.3. As an immediate consequence of $(\Sigma - \theta)$ in [**8**], we have given (iii) of Lemma 7.3 in order to give an elementary and direct proof of Berberian's Lemma which implies that if T is a Toeplitz operator, then (*) holds. We shall use only (i) of Lemma 7.3 in order to prove Theorem 7.1, but here we state (ii) and (iii) for the sake of convenience and some related results.

Proof of Theorem 7.1. If T satisfies (H_1), then $\mathrm{Re}\, \sigma(T) \subset \sigma(\mathrm{Re}\, T)$ holds by Lemma 7.2 and T is convexoid [**4**, II], so that we have (*) by hypothesis and (i) of Lemma 7.3. This completes the proof.

It is known that the class of spectroid operators is properly contained in the class of hen-spectroids and the latter is properly contained in the class of numeroids [,**4** I-III]. Thus the following Theorem 7.2 covers Theorem B.

THEOREM 7.2. *If T is hen-spectroid, then* (*) *holds.*

Proof. If T is hen-spectroid, then T satisfies (H_1) [**4**, III, Proposition 3] and T turns out to be convexoid [**4**, II]. As in the proof of Lemma 4 in [**3**] (when we replace $\sigma(T)$ by $\tilde{\sigma}(T)$, as easily seen, the corresponding orthogonal decomposition of Williams [**27**, Theorem 4] also holds), we have $\sigma(\text{Re } T) \subset \text{Re } \sigma(T)$ and the reverse inequality is already obtained by Lemma 7.2, whence (*)holds and the proof is complete.

The remainder of this section is devoted to showing a construction of operators satisfying (*) and to discuss some related results. In [**3**] there is given an example which shows that a convexoid operator need not satisfy (*) and related to this result there is also given a non-convexoid operator satisfying (*) in [**8**]. It is already known in Section 3 that the class S is properly contained in the class C, the set of all convexoid operators, and this class S properly contains the class (H_1).

Example 7.1. We shall give an example belonging to S which does not satisfy (*) as follows. Put $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$; then $\sigma(A) = \{0\}$ and $\overline{W(A)} = D$ the closed unit disk. Let B be the normal operator with the spectrum

$$\sigma(B) = \{4e^{i\theta}: -\pi/3 \leqq \theta \leqq \pi/3 \text{ and } 2\pi/3 \leqq \theta \leqq 4\pi/3\},$$

and put $T = A \oplus B$. It is easily verified that $d(\mu, \sigma(B)) \leqq d(\mu, W(A))$ for all $\mu \notin \overline{W(B)}$. Then T belongs to S by Theorem 3.2. Next we have

$$\sigma(\text{Re } T) = \sigma(\text{Re } A) \cup \sigma(\text{Re } B) = \{-1, 1\} \cup \text{Re } \sigma(B)$$
$$= \{-1, 1\} \cup [-4, -2] \cup [2, 4],$$

where $[a, b]$ means the closed interval $\{x: a \leqq x \leqq b\}$. On the other hand $\text{Re } \sigma(T) = \{0\} \cup \text{Re } \sigma(B) = \{0\} \cup [-4, -2] \cup [2, 4]$; that is, T does not satisfy (*). Hence we can verify that the class S does not contain the class of operators satisfying (*) and vice versa. Incidentally, it also turns out that this operator T in Example 7.1 is numeroid since $2D \cup \overline{W(A)} \subset \overline{W(B)}$ by [**4**, III, Remark] so that this implies that Theorem 7.2 cannot be generalized for numeroid operator. We find that Example 3.4 in Section 3 is another counter example for numeroid in Theorem 7.2.

PROPOSITION 7.1 (Construction). *If A is an operator and B satisfies* (*) *such that*

$$(7.4) \quad \text{Re } \overline{W(A)} \subset \text{Re } \sigma(B),$$

then $T = A \oplus B$ also satisfies (*).

https://doi.org/10.4153/CJM-1977-099-0 Published online by Cambridge University Press

As stated in Section 1, the typical examples satisfying (*) are normal and seminormal operators.

Proof of Proposition 7.1. Since the closure of a numerical range contains its spectrum of an operator and hypothesis (7.4) holds, we have

(7.5) $\operatorname{Re} \sigma(A) \subset \operatorname{Re} \overline{W(A)} \subset \operatorname{Re} \sigma(B)$

(7.6) $\sigma(\operatorname{Re} A) \subset \overline{W(\operatorname{Re} A)} = \operatorname{Re} \overline{W(A)} \subset \operatorname{Re} \sigma(B)$

so that we obtain the following by (7.5), (7.6) and hypothesis:

$$\sigma(\operatorname{Re} T) = \sigma(\operatorname{Re} A) \cup \sigma(\operatorname{Re} B) = \sigma(\operatorname{Re} A) \cup \operatorname{Re} \sigma(B) = \operatorname{Re} \sigma(B)$$

and

$$\operatorname{Re} \sigma(T) = \operatorname{Re} \sigma(A) \cup \operatorname{Re} \sigma(B) = \operatorname{Re} \sigma(B).$$

Hence T satisfies (*), and the proof is complete.

We remark that both the operators belonging to (H_1) constructed by Fujii [**4**, Theorem 4] and the operators belonging to (G_1) for $\overline{\sigma(T)}$ by Luecke [**16**, Theorem 1.2] satisfy (*) since both $\overline{W(A)} \subset \tilde{\sigma}(B)$ and $\overline{W(A)} \subset \sigma(B)$ also satisfy (7.4) in Proposition 7.1.

8. Another characterization of operators in R**.** In this section we shall characterize operators in R (Theorem 8.1). This Theorem 8.1 can be considered as a converse of Theorem E and a precise estimation of [**4**, II, Theorem 12]. We shall also show a parallel result of Theorem 8.1. Finally we shall show some related results of operators belonging to R.

THEOREM 8.1. *An operator T belongs to R if and only if $\partial\tilde{\sigma}(T)$ is a convex curve and*

(σ-θ) $\sigma(\operatorname{Re} e^{i\theta}T) = \operatorname{Re} \sigma(e^{i\theta}T)$ (*in symbols*, $T \in (\sigma - \theta)$)

holds for all $0 \leqq \theta \leqq 2\pi$, *that is*, $R = (\sigma - \theta) \cap Q$ *holds*.

Proof. It T belongs to R, then $e^{i\theta}T$ also belongs to R and $\partial\tilde{\sigma}(T)$ is a convex curve since $\overline{W(T)} = \tilde{\sigma}(T)$ holds, so that $(\sigma - \theta)$ holds by Theorem E.

Conversely, if $(\sigma - \theta)$ holds, then $\operatorname{co} \sigma(\operatorname{Re} e^{i\theta}T) = \operatorname{co} \operatorname{Re} \sigma(e^{i\theta}T)$ for all $0 \leqq \theta \leqq 2\pi$, that is, $(\Sigma - \theta)$ holds. Thus T is convexoid. In addition, if $\partial\tilde{\sigma}(T)$ is a convex curve, then T belongs to R by [**4**, II, Theorem 12], and this completes the proof.

Remark 8.1. We remark that Theorem 8.1 is a precise estimation of $R = C \cap Q$ [**4**, II, Theorem 12], where C is the set of all convexoids, because the class $(\sigma - \theta)$ is properly contained in the class C.

PROPOSITION 8.1. *A operator T belongs to P if and only if T has convex spectrum and $(\sigma - \theta)$ holds.*

The proof of this is similar to that of Theorem 8.1 so we shall omit it.

If $S, T \in C$, then $S \oplus T \in C$ without any restriction and we shall show the corresponding relation for operators in R with some moderate restriction as follows.

PROPOSITION 8.2. *If $S, T \in R$, then $S \oplus T \in R$ if and only if $S \oplus T \in Q$.*

Proof. The proof of necessity follows from the relation

$$\overline{W(S \oplus T)} = \text{co}\,\{\overline{W(S)} \cup \overline{W(T)}\} = \text{co}\,\{\tilde{\sigma}(S) \cup \tilde{\sigma}(T)\}$$
$$= \text{co}\,\tilde{\sigma}(S \oplus T) = \text{co}\,\sigma(S \oplus T) = \tilde{\sigma}(S \oplus T).$$

The proof of sufficiency is reversible, so the proof is complete.

PROPOSITION 8.3. *If $S \otimes T \in R$, then $\overline{W(S \otimes T)} = \overline{\text{co}}\,\{\overline{W(S)} \cdot \overline{W(T)}\}$ and $S \otimes T \in Q$.*

Proof. This proof easily follows from [**21**] and [**4**, II], but here we state this for the sake of completeness. For arbitrary operators S and T, we have

$$\overline{W(S \otimes T)} \supseteq \overline{\text{co}}\,\{\overline{W(S)} \cdot \overline{W(T)}\} \supseteq \text{co}\,\{\text{co}\,\sigma(S) \cdot \text{co}\,\sigma(T)\}$$
$$= \text{co}\,\{\sigma(S) \cdot \sigma(T)\} = \text{co}\,\sigma(S \otimes T) \supseteq \tilde{\sigma}(S \otimes T),$$

so that the hypothesis $\overline{W(S \otimes T)} = \tilde{\sigma}(S \otimes T)$ completes the proof.

PROPOSITION 8.4. *If $S, T \in C$, then $S \otimes T \in R$ if and only if both $\overline{W(S \otimes T)} = \overline{\text{co}}\,\{\overline{W(S)} \cdot \overline{W(T)}\}$ and $S \otimes T \in Q$ hold.*

Proof. By the preceding proposition, we have only to show the proof of necessity. By hypothesis we have

$$\overline{W(S \otimes T)} = \overline{\text{co}}\,\{\overline{W(S)} \cdot \overline{W(T)}\} = \text{co}\,\{\text{co}\,\sigma(S) \cdot \text{co}\,\sigma(T)\}$$
$$= \text{co}\,\{\sigma(S) \cdot \sigma(T)\} = \text{co}\,\sigma(S \otimes T) = \tilde{\sigma}(S \otimes T),$$

so the proof is complete.

Proposition 8.4 is similar to the following: If $S, T \in C$, then $S \otimes T \in C$ if and only if $\overline{W(S \otimes T)} = \overline{\text{co}}\,\{\overline{W(S)} \cdot \overline{W(T)}\}$ [**9**, Corollary 3].

9. Generalized convexoid operators. In [**25**] Sz.-Nagy and Foias introduced, for each $\rho > 0$, the class C_ρ of operators T on a given complex Hilbert space H for which there exists a Hilbert space K containing H as a subspace and a unitary operator U on K satisfying the following relation: $T^n = \rho P U^n$ $(n = 1, 2, \ldots)$ where P is the orthogonal projection of K onto H. This unitary operator U is called a unitary ρ-dilation of T. In [**12**] Holbrook introduced the *operator radii* as follows: $w_\rho(T) = \inf\,\{u : u > 0,\ T/u \in C_\rho\}$. In particular, $w_1(T) = ||T||$, $w_2(T) = w(T)$, $w_\infty(T) = r(T)$ and $C_\rho = \{T;\ w_\rho(T) \leqq 1\}$. In [**14**] Lin defined a generalized numerical range of an operator T as follows:

$$(9.1) \quad W_\rho(T) = \bigcap_\mu \{\lambda : |\lambda - \mu| \leqq w_\rho(T - \mu)\}, \quad 1 \leqq \rho \leqq \infty.$$

It is known [**14**] that $W_\rho(T)$ turns out to be a compact convex subset containing

https://doi.org/10.4153/CJM-1977-099-0 Published online by Cambridge University Press

co $\sigma(T)$ and $W_\rho(T)$ $(1 \leqq \rho \leqq 2)$ coincides with the closure of the usual numerical range of T. It is also known [**14**] that $W_\rho(T)$ can be expressed by means of the following formula:

$$(9.2) \quad W_\rho(T) = \bigcap_\mu \{\lambda: |\lambda - \mu| \leqq w_\rho^{\,0}(T - \mu)\}$$

where $w_\rho^0(T)$ is defined by $w_\rho^0(T) = \sup\{|\lambda|: \lambda \in W_\rho(T)\}$, $1 \leqq \rho \leqq \infty$. Moreover, $w_\rho^0(T)$ satisfies the following properties [**14**]: $r(T) \leqq w_\rho^0(T) \leqq w_\rho(T)$, $w_\infty^0(T) = r(T)$, $w_\rho^0(cT) = |c|w_\rho^0(T)$ for all complex c and $w_2(T) = w_\rho^0(T)$, $1 \leqq \rho \leqq 2$.

Definition 9.1 [**14**]. An operator T is said to be ρ-convexoid if $W_\rho(T) =$ co $\sigma(T)$ $(2 \leqq \rho < \infty)$ and an operator T is said to be ρ-normaloid if $w_\rho(T) = r(T)$ $(1 \leqq \rho < \infty)$ respectively. 1-normaloid and 2-normaloid turn out to be normaloid and spectraloid respectively.

At first we remark that this ρ-normaloid was considered somewhat earlier in [**6**] and [**7**] by the name ρ-oid and it is known [**6**] that for each $\rho \geqq 1$, ρ-normaloid if and only if "power equality" $w_\rho(T^k) = (w_\rho(T))^k$ $(k = 1, 2, \ldots)$ holds in the "power inequality" [**12**] $w_\rho(T^k) \leqq (w_\rho(T))^k$ $(k = 1, 2, \ldots)$ which is always valid for any operator T. It is also known that for each $0 < \rho < 1$, there exists no non-zero ρ-normaloid operator which is included in the class of normaloids [**6**] and an idempotent ρ-normaloid is a projection. Moreover, periodic $(T^k = T, k \geqq 2)$ ρ-normaloid is normal and partial isometric, that is to say, the direct sum of zero and a unitary operator [**7**]. Next we remark that the relation $W_\infty(T) =$ co $\sigma(T)$ was shown in Theorem 2 of [**14**], but this result had been already obtained in (3) in [**10**] or (2) in [**8**] by using (ii) of Lemma 2.1 in Section 2. It is also known [**14**] that a convexoid is ρ-convexoid.

Here, by using (iii) of Lemma 2.1 in Section 2 we shall show the following formula in the same way as in the proof of Proposition 2.1:

$$(9.3) \quad W_\rho(T) = \bigcap_\mu \{\lambda: |\lambda - \mu| \leqq w_\rho^{\,0}(T - \mu) \quad \text{for all complex } \mu \text{ whose absolute values are sufficiently large}\},\ 1 \leqq \rho \leqq \infty.$$

By (9.2) and (9.3) we have the following result as an extension of Corollary 2.1.

COROLLARY 9.1. *The following conditions are mutually equivalent:*

(i) $0 \in W_\rho(T)$,

(ii) $|\mu| \leqq w_\rho^0(T - \mu)$ *for all complex* μ, *and*

(iii) $|\mu| \leqq w_\rho^0(T - \mu)$ *for all complex* μ *whose absolute values are sufficiently large.*

By using (2.2), (9.3) and the same techniques as in Section 2 and moreover scrutinizing Lin's paper [**14**], we can remark that the following propositions are some extensions of Proposition 2.1.

PROPOSITION 9.1. *The following conditions are mutually equivalent:*

(i) *T is ρ-convexoid*,

(ii) $w_\rho^0(T - \mu) = r(T - \mu)$ *for all complex* μ,

(iii) $w_\rho^0(T - \mu) = r(T - \mu)$ *for all complex* μ *whose absolute values are sufficiently large.*

PROPOSITION 9.2. *Any one of the following conditions is sufficient in order that T is ρ-convexoid:*

(i) $T - \mu$ *is ρ-normaloid for all complex* μ, $2 \leqq \rho < \infty$,

(ii) $T - \mu$ *is ρ-normaloid for all complex* μ *whose absolute values are sufficiently large*, $2 \leqq \rho < \infty$,

(iii) $w_\rho^0((T - \mu)^{-1}) \leqq \dfrac{1}{d(\mu, \operatorname{co} \sigma(T))}$ *for all* $\mu \notin \operatorname{co} \sigma(T)$,

(iv) $w_\rho^0((T - \mu)^{-1}) \leqq \dfrac{1}{d(\mu, \operatorname{co} \sigma(T))}$ *for all complex* μ *whose absolute values are sufficiently large.*

Remark 9.1. The equivalence between (i) and (ii) of Proposition 9.1 and (i) and (iii) of Proposition 9.2 were shown in [**14**]. Moreover, some results in [**14**] can be improved in this direction, that is, "for all μ" in sufficient conditions of some results in [**14**] can be readily replaced by "for all complex μ whose absolute values are sufficiently large" in the same way as Proposition 9.1 and Proposition 9.2, so that we shall omit describing them. On the other hand (iv) of Proposition 9.2 can be considered as "ρ-generalized growth conditions". Strictly speaking, $(w_\rho^0 - G_1)$ for $(\operatorname{co} \sigma(T), N)$ as some generalizations of both (G_1) for $(\operatorname{co} \sigma(T), N)$ and $(w - G_1)$ for $(\operatorname{co} \sigma(T), N)$; namely, both (iv) and (vi) of Proposition 2.1 in Section 2.

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