Sets of Points Self-Conjugate with regard to a Quadric in n Dimensions

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§ 1. In space of three dimensions the properties of self-conjugate tetrads, pentads and hexads with regard to a quadric are well known (see Baker's *Principles of Geometry*, vol. iii). The general theorem in space of n dimensions S_n is *to establish the existence of a set of* $n + p + 1$ *points* A_0, A_1, ..., A_{n+p} $(0 \leqslant p \leqslant n - 1)$ *such that the pole, with respect to a given quadric, of the* $(n - 1)$*-flat determined by any set of* n *of the points lies in the* p*-flat determined by the remaining* $p + 1$ *points.*

§ 2. Consider a space S_{n+p} containing S_n, and $n + p + 1$ linearly independent points A'_0, A'_1, ..., A'_{n+p}. Then there is a quadric Q' in S_{n+p} with respect to which the points form a self-polar simplex. If $p = 0$ this simplex forms the self-conjugate set of $n + 1$ points whose existence is in question. If $p > 0$, let S_{p-1} be the polar $(p - 1)$-flat of S_n with respect to Q', and project the figure on to S_n with S_{p-1} as axis of projection. The process is as follows. To project a point A': determine the p-flat through A' and S_{p-1}; this cuts S_n in the corresponding point A. Generally, to project an r-flat R': determine the $(p + r)$-flat through R' and S_{p-1}; this cuts S_n in the corresponding r-flat R. To project the quadric Q': an $(n + p - 1)$-flat through S_{p-1} and touching the quadric has $n - 1$ degrees of freedom and cuts S_n in an $(n - 1)$-flat which envelopes the corresponding quadric Q in S_n. In the present case, since S_{p-1} is the polar of S_n, Q is actually the section of Q' by S_n. The assemblage of $(n + p - 1)$-flats through S_{p-1} envelopes a hypercone of species p having S_{p-1} as vertex-edge; this is a tangent hypercone to Q', and the points of contact form the quadric Q.

We proceed to show that the $n + p + 1$ points A_r obtained in this way form a self-conjugate set with respect to the quadric Q, *i.e.* that the pole, with respect to Q, of the $(n - 1)$-flat α determined by any set of n of the points A_0, ..., A_{n-1} lies in the p-flat β determined by the remaining $p + 1$ points A_n, ..., A_{n+p}.

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The corresponding n points $A'_0, \ldots, A'_{n-1}$ determine an $(n-1)$-flat α', and this determines with S_{p-1} an $(n+p-1)$-flat whose pole with respect to Q' lies in S_n and also in the p-flat $\beta' \equiv (A'_n, \ldots, A'_{n+p})$. P is therefore the point of intersection of S_n with β'. Now the corresponding p-flat $\beta \equiv (A_n, \ldots, A_{n+p})$ is the intersection of S_n with the $(2p)$-flat determined by S_{p-1} and β'; hence P lies in β. (It is necessary that $2p < n+p$, and therefore $p < n$). Also since P is conjugate, with respect to Q', to every point in the $(n+p-1)$-flat (α', S_{p-1}), it is conjugate to every point in the section of this by S_n. But this section is α, and the section of Q' by S_n is Q; hence P is the pole of α with respect to Q, and it has been proved also that P lies in β.

It follows further that *the polar r-flat* $(0 \leqslant r < n-p)$ *of the* $(n-r-1)$*-flat determined by any set of* $n-r$ *of the points* $(A_0, \ldots, A_{n-r-1})$ *lies in the* $(p+r)$*-flat determined by the remaining* $p+r+1$ *points* $(A_{n-r}, \ldots, A_{n+p})$.

§ 3. If the simplex $A'_0, \ldots, A'_{n+p}$ is taken as frame of reference, the tangential equation of the quadric Q' is of the form

$$\xi_0^2 + \ldots + \xi_{n+p}^2 = 0.$$

Any linear equation in (ξ_r) represents a point P' in S_{n+p}. The point A'_r is represented by the equation $\xi_r = 0$. S_{p-1} is represented by p linear equations $\Sigma_1 = 0, \ldots, \Sigma_p = 0$ in $\xi_0, \ldots, \xi_{n+p}$. Any linear equation in (ξ_r), together with the p equations $\Sigma_p = 0$, represents the assemblage of $(n+p-1)$-flats through S_{p-1} and the point P'; these cut S_n in an assemblage of $(n-1)$-flats all passing through the corresponding point P. The quadratic equation $\Sigma\, \xi_r^2 = 0$, together with the p equations $\Sigma_p = 0$, represents the assemblage of $(n+p-1)$-flats through S_{p-1} and touching Q'; these cut S_n in an assemblage of $(n-1)$-flats all touching Q.

Hence in S_n the equation

$$\xi_0^2 + \ldots + \xi_{n+p}^2 = 0$$

where ξ_r are connected by p linear equations $\Sigma_p = 0$, represents a quadric Q, and the equation $\xi_r = 0$ represents the point A_r. *The self-conjugate set of* $n+p+1$ *points* A_r *is thus related to the representation of the quadric by a tangential equation in terms of* $n+p+1$ *squares.*

§ 4. The reciprocal relations are at once deduced. When a quadric Q in S_n is represented by a point-equation

$$x_0^2 + \ldots + x_{n+p}^2 = 0$$

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in terms of $n + p + 1$ squares, the variables x_r being connected by p linear equations $S_p = 0$, the $n + p + 1$ primes, or $(n - 1)$-flats, $x_r = 0$ form a self-conjugate set such that the polar prime with respect to Q of the point common to any n of the primes passes through the $(n - p - 1)$-flat common to the remaining $p + 1$ primes; and so on.

§5. If S and Σ' are two quadrics such that a simplex exists which is inscribed in S and self-polar with respect to Σ' then an infinity of such simplexes exists, and S is said to be *outpolar* to Σ'.

If the point-equation of S is

$$S \equiv \Sigma \Sigma a_{rs} x_r x_s = 0 \tag{1}$$

and the tangential equation of Σ' is

$$\Sigma' \equiv \Sigma \Sigma A'_{rs} \xi_r \xi_s = 0 \tag{2}$$

the condition that S should be outpolar to Σ' is the vanishing of the bilinear invariant

$$\Theta' \equiv \Sigma \Sigma a_{rs} A'_{rs} = 0. \tag{3}$$

These relations still hold when S is a cone of any species. Let S be a cone in S_n whose vertex-edge S_{p-1} is the $(p-1)$-flat determined by the vertices $A_0, \ldots, A_{p-1}$ of the simplex of reference. Its equation is then a homogeneous quadratic containing the variables $x_p, \ldots, x_n$ alone, *i.e.*, in equation (1), $a_{rs} = 0$ if r or $s < p$.

Let S_{n-p} denote the polar $(n-p)$-flat of S_{p-1} with respect to Σ'. We may choose the frame of reference so that S_{n-p} is represented by the equations $x_0 = 0, \ldots, x_{p-1} = 0$. Then the point-equation of Σ' is

$$S' \equiv \Sigma \Sigma a'_{rs} x_r x_s = 0 \tag{4}$$

where $a'_{rs} = 0$ if $r < p$ and $s > p - 1$ or *vice versa*.

Consider the sections C and C' of S and S' by S_{n-p}. C is represented by

$$\Sigma \Sigma a_{rs} x_r x_s = 0, \quad x_0 = 0, \ldots, x_{p-1} = 0, \tag{5}$$

the first equation being precisely the same as (1).

C' is represented by

$$\Sigma \Sigma a'_{rs} x_r x_s = 0, \quad x_0 = 0, \ldots, x_{p-1} = 0, \tag{6}$$

and in the first equation we may now assume $a'_{rs} = 0$ if r or $s < p$. (5) and (8) then represent two quadrics C and C' in S_{n-p} in terms of the coordinates $x_p, \ldots, x_n$.

The determinant of S' is

$$\Delta' \equiv \begin{vmatrix} a'_{00} & \dots & a'_{0,p-1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a'_{p-1,0} & \dots & a'_{p-1,p-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & a'_{pp} & \dots & a'_{pn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a'_{np} & \dots & a'_{nn} \end{vmatrix}$$

and that of C' is

$$\delta' \equiv \begin{vmatrix} a'_{pp} & \dots & a'_{pn} \\ \dots & \dots & \dots \\ a'_{np} & \dots & a'_{nn} \end{vmatrix}.$$

A'_{rs} is the cofactor of a'_{rs} in Δ'; let α'_{rs} be the cofactor of a'_{rs} in δ'. Then if

$$D \equiv \begin{vmatrix} a'_{00} & \dots & a'_{0,p-1} \\ \dots & \dots & \dots \\ a'_{p-1,0} & \dots & a'_{p-1,p-1} \end{vmatrix}$$

we have

$$A'_{rs} = D\alpha'_{rs}, \tag{7}$$

when r and s are each greater than $p-1$.

When the cone S is outpolar to the quadric Σ', we have

$$\Sigma \Sigma a_{rs} A'_{rs} = 0,$$

the summations extending from p to n. Hence from (7)

$$\Sigma \Sigma a_{rs} \alpha'_{rs} = 0.$$

Therefore the quadric C is outpolar to the quadric C'.

Hence *if S is a cone with vertex-edge S_{p-1}, Σ' any quadric, and S_{n-p} the polar of S_{p-1} with respect to Σ', then if S is outpolar to Σ' the section of S by S_{n-p} is outpolar to the section of Σ'.*

§ 6. Returning now to the simplex $A'_0, \dots, A'_{n+p}$ in S_{n+p} and its projection $A_0, \dots, A_{n+p}$ on the n-flat S_n, and the quadric Q' for which $A'_0, \dots, A'_{n+p}$ is self-polar, S_{p-1} being the polar of S_n with respect to Q', let R' be a cone with vertex-edge S_{p-1} and passing through the $n+p+1$ points A'_r. R' is thus outpolar to Q'. Let R be the section of R' by S_n, and Q the section of Q'. Then R is outpolar to Q and contains the $n+p+1$ points $A_0, \dots, A_{n+p}$ which form a self-conjugate set with respect to Q.

Hence *if Σ' and S are two quadrics in S_n such that there exists a set of $n+p+1$ points $(0 \leqslant p \leqslant n-1)$ inscribed in S and self-conjugate with respect to Σ', S is outpolar to Σ'.* The existence of a simplex inscribed in S and self-polar with respect to Σ' thus implies also the existence of a self-conjugate r-ad $(n+1 \leqslant r \leqslant 2n)$ similarly inscribed.

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