Can. J. Math., Vol. XXVIII, No. 1, 1976, pp. 19–23

OPERATORS OF RANK ONE IN REFLEXIVE ALGEBRAS

W. E. LONGSTAFF

1. Introduction. If H is a (complex) Hilbert space and $\mathscr{F}$ is a collection of (closed linear) subspaces of H it is easily shown that the set of all (bounded linear) operators acting on H which leave every member of $\mathscr{F}$ invariant is a weakly closed operator algebra containing the identity operator. This algebra is denoted by Alg $\mathscr{F}$. In the study of such algebras it may be supposed [**4**] that $\mathscr{F}$ is a *subspace lattice* i.e. that $\mathscr{F}$ is closed under the formation of arbitrary intersections and arbitrary (closed linear) spans and contains both the zero subspace (0) and H. The class of such algebras is precisely the class of reflexive algebras [**3**]. In [**2**] it is shown that if $\mathscr{F}$ is totally ordered then Alg$\mathscr{F}$ is the strongly closed algebra generated by the operators of rank one it contains. We consider the problem of which subspace lattices have this density property. Totally ordered complete lattices are completely distributive in the sense of G. N. Raney [**6**]. It is shown that a subspace lattice with this density property is completely distributive and the converse is established in the case where the underlying space is finite dimensional.

2. Notation and preliminaries. Most of the notation is taken from [**5**]. An abstract lattice L is called *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad (a, b, c \in L)$$

and its dual statement hold identically in L. In the following let L be a complete lattice. We adopt the conventions that $\vee \emptyset = 0$ and $\wedge \emptyset = 1$ where 0 and 1 are the zero and unit element of L respectively. The following notation and definition is taken from [**6**]. If Λ is a non-empty index set and $\phi = \{\phi_\alpha : \alpha \in \Lambda\}$ is a family of non-empty subsets of L let $S(\phi)$ denote the collection of mappings $s : \Lambda \rightarrow L$ with the property that $s(\alpha) \in \phi_\alpha (\alpha \in \Lambda)$. For $s \in S(\phi)$ let $s(\Lambda)$ denote the image of Λ under s. The complete lattice L is called *completely distributive* if for every such family ϕ both

$$\wedge \{ \vee \phi_\alpha : \alpha \in \Lambda\} = \vee \{ \wedge s(\Lambda) : s \in S(\phi)\}$$

and its dual statement are valid. This condition is stronger than distributivity. If $a \in L$ the elements a_- and a_* of L are defined by $a_- = \vee \{b \in L : a \nleqq b\}$ and $a_* = \wedge \{b_- : b \in L \text{ and } b \nleqq a\}$. Then a_-, $a_* \in L$ and $a \leqq a_*$. It is shown in [**5**] that L is completely distributive if and only if $a = a_*$ for every element a of L.

Received May 13, 1974 and in revised form, April 25, 1975.
The author is a Flinders University Research Fellow.

https://doi.org/10.4153/CJM-1976-003-1 Published online by Cambridge University Press

If e and f are non-zero vectors of the Hilbert space H denote by $e \otimes f$ the operator of rank one defined by $x \to (x|e)f \quad (x \in H)$. If $\mathscr{F}$ is a subspace lattice denote by $\mathscr{R}$ the set of operators of rank one belonging Alg $\mathscr{F}$. Denote by $\mathfrak{A}$ the algebra generated by $\mathscr{R}$ and by $\overline{\mathfrak{A}}$ the closure of $\mathfrak{A}$ in the strong operator topology. Then clearly if $\mathscr{R} = \emptyset$ we have $\mathfrak{A} = \{0\}$ and otherwise $\mathfrak{A}$ is the set of finite sums of operators in $\mathscr{R}$. It is also clear that $\overline{\mathfrak{A}} \subseteq \text{Alg}\,\mathscr{F}$. For any subspace N of H denote by P_N the (orthogonal) projection whose range is N.

3. A necessary condition. Let $\mathscr{F}$ be a subspace lattice on H and let $\mathscr{R}$, $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ be as described above.

THEOREM 3.1. *If* $\overline{\mathfrak{A}} = \text{Alg}\,\mathscr{F}$ *then* $\mathscr{F}$ *is completely distributive.*

Proof. Since $\mathscr{F}$ is a complete lattice we need only show that $M = M_*$ for every element M of $\mathscr{F}$. We may suppose that $\dim H \geqq 1$. Then clearly $\mathscr{R} \neq \emptyset$. Fix $M \in \mathscr{F}$. We first show that $(1 - P_M)(e \otimes f)P_{M_*} = 0$ for every operator $e \otimes f \in \mathscr{R}$. This is equivalent to showing that $e \otimes f$ maps M_* into M. By Lemma 3.1 of [**5**] there is a subspace $K \in \mathscr{F}$ such that $f \in K$ and $e \in H \ominus K_-$. If $K \subseteq M$ then $(e \otimes f)M_* \subseteq K \subseteq M$. If $K \nsubseteq M$ then $M_* \subseteq K_-$ and so $(e \otimes f)M_* = (0) \subseteq M$. Thus $(1 - P_M)(e \otimes f)P_{M_*} = 0$ for every operator $e \otimes f \in \mathscr{R}$. It follows that $(1 - P_M)AP_{M_*} = 0$ whenever $A \in \mathfrak{A}$. Since the mapping $T \to XTY$ (X, Y fixed operators) is strongly continuous it follows that $(1 - P_M)AP_{M_*} = 0$ for every operator $A \in \overline{\mathfrak{A}}$. Since $I \in \overline{\mathfrak{A}}$ we have $0 = (1 - P_M)P_{M_*} = P_{M_*} - P_M$ and so $M = M_*$. The proof of the theorem is complete.

As noted earlier a partial converse has been obtained by Erdos [**2**].

4. Finite-dimensional case. We now prove the converse of Theorem 3.1 in the case where the underlying space is finite-dimensional. Every finite distributive lattice is completely distributive. Every completely distributive lattice of finite length is distributive and so, by Theorem 5 of [**1**, p. 139], is finite. Thus if $\dim H < \infty$ the class of completely distributive subspace lattices on H is precisely the class of finite distributive subspace lattices. An element a of an abstract lattice L is called *join-irreducible* if $a = b \vee c\,(b, c \in L)$ implies that either $a = b$ or $a = c$. If L is distributive and finite, by Theorem 9 of [**1**, p. 142], every non-zero element of L is the join of all the non-zero join-irreducible elements it contains. Also in this case, if $a \in L$ is non-zero and join-irreducible then a covers $a \wedge a_-$, i.e. $a \wedge a_- < a$ and there is no element b of L satisfying $a \wedge a_- < b < a$. To see this notice that $a \wedge a_- = a$ would imply that $a = \vee\{a \wedge c : c \in L \text{ and } a \nleq c\}$ and this contradicts the join-irreducibility of a. So $a \wedge a_- < a$. Now notice that $b < a$, $b \in L$ implies $b \leq a \wedge a_-$. Another elementary fact we need is that if $\mathscr{P}$ is any partially ordered set and $\mathscr{S}$ is any finite subset of $\mathscr{P}$ we can enumerate $\mathscr{S}$, say $\mathscr{S} = \{x_1 x_2 \ldots x_n\}$, in such a way that the partial order induced on $\mathscr{S}$ is not violated. More precisely, such that $x_i < x_j$ implies $i < j$. This is easily proved by induction on the number of elements in $\mathscr{S}$.

The author wishes to thank Dr. K. J. Harrison for pointing out some important simplifications to the original proof of the following theorem.

THEOREM 4.1. *If $\mathscr{F}$ is a completely distributive subspace lattice on a finite-dimensional Hilbert space H then $\mathfrak{A} = \mathrm{Alg}\,\mathscr{F}$ where $\mathfrak{A}$ is the algebra generated by the set of operators of rank one belonging to* Alg $\mathscr{F}$.

Proof. Clearly we may suppose that $\dim H \geqq 1$. By our earlier remarks $\mathscr{F}$ is distributive and finite. Let $\mathscr{J}$ be the set of non-zero join-irreducible elements of $\mathscr{F}$. Then $\mathscr{J} \neq \emptyset$. For every $K \in \mathscr{J}$ select a basis $\mathscr{B}(K)$ for $K \ominus (K \cap K_-)$ and let $X = \bigcup \{\mathscr{B}(K) : K \in \mathscr{J}\}$.

First we show that X is a linearly independent set of vectors. Enumerate the elements of $\mathscr{J}$, say $K_1K_2 \ldots K_n$ in such a way that $K_i \subset K_j$ implies $i < j$. Consider

$$[K_j \ominus (K_j \cap K_{j-})] \cap \bigvee_{k=1}^{j-1} [K_k \ominus (K_k \cap K_{k-})] \quad \text{for } 2 \leqq j \leqq n.$$

Let $1 \leqq k \leqq j - 1$. If K_k and K_j are not comparable then $K_k \subseteq K_{j-}$. If they are comparable then $K_k \subset K_j$ by the method of enumeration and so again we have $K_k \subseteq K_{j-}$. Thus

$$[K_j \ominus (K_j \cap K_{j-})] \cap \bigvee_{k=1}^{j-1} [K_k \ominus (K_k \cap K_{k-})] \subseteq [K_j \ominus (K_j \cap K_{j-})] \cap K_{j-} = (0).$$

Hence

$$[K_j \ominus (K_j \cap K_{j-})] \cap \bigvee_{k=1}^{j-1} [K_k \ominus (K_k \cap K_{k-})] = (0) \quad \text{for } 2 \leqq j \leqq n.$$

It follows that X is a linearly independent set of vectors.

Next we show that $X \cap M$ is a basis for M, for every non-zero element M of $\mathscr{F}$. Notice that this result is true for every atom. For if K is an atom then $K \in \mathscr{J}$ and $K = K \ominus (K \cap K_-)$ so $X \cap K = \mathscr{B}(K)$. Suppose that $M \in \mathscr{F}$ is non-zero and is not an atom. Then M strictly contains an element of $\mathscr{J}$. If the result is true for every element of $\mathscr{J}$ strictly contained in M then it is true for M itself. For either $M \in \mathscr{J}$ or $M \notin \mathscr{J}$. In the former case $M \cap M_- = \vee\{K \in \mathscr{J} : K \subset M\}$ and $X \cap M$ contains $\mathscr{B}(M)$ and $X \cap K$ for every element $K \in \mathscr{J}$ satisfying $K \subset M$. In the latter case $M = \bigvee\{K \in \mathscr{J} : K \subset M\}$ and $X \cap M$ contains $X \cap K$ for every element $K \in \mathscr{J}$ satisfying $K \subset M$. Suppose then that $M \in \mathscr{F}$ is non-zero and is not an atom and that $X \cap M$ is not a basis for M. Then there is a non-atomic element $K^{(1)} \in \mathscr{J}$ with $K^{(1)} \subset M$ such that $X \cap K^{(1)}$ is not a basis for $K^{(1)}$. There is a non-atomic element $K^{(2)} \in \mathscr{J}$ with $K^{(2)} \subset K^{(1)}$ such that $X \cap K^{(2)}$ is not a basis for $K^{(2)}$. This process continues indefinitely and this contradicts the finiteness of $\mathscr{J}$. So $X \cap M$ is a basis for M for every non-zero element M of $\mathscr{F}$.

Let $X = \{f_1 f_2 \ldots f_m\}$. Then X is a basis for H. Let $\{e_1 e_2 \ldots e_m\}$ be the corresponding dual basis uniquely defined by the requirement that $(f_i|e_j) = \delta_{ij}$ $(1 \leqq i, j \leqq m)$. Let $R \in \text{Alg}\,\mathscr{F}$ be arbitrary. Then $R = \sum_{i=1}^m e_i \otimes Rf_i$. To complete the proof of the theorem we show that $e_i \otimes Rf_i \in \text{Alg}\,\mathscr{F}$ $(1 \leqq i \leqq m)$. Let $M \in \mathscr{F}$ be non-zero and let $x \in M$. If $f_i \in M$ then $Rf_i \in M$ and so $(e_i \otimes Rf_i)x = (x|e_i)Rf_i \in M$. If $f_i \notin M$ then $e_i \in H \ominus M$ since $X \cap M$ is a basis for M. Then $(e_i \otimes Rf_i)x = 0 \in M$. Thus $e_i \otimes Rf_i \in \text{Alg}\,\mathscr{F}$ $(1 \leqq i \leqq m)$ and the proof is complete.

5. Operators of finite rank on atomic Boolean algebras. The following theorem, due to J. R. Ringrose, is proved in [**2**].

THEOREM 5.1. *If $\mathscr{F}$ is a totally ordered subspace lattice every operator of finite rank belonging to* Alg $\mathscr{F}$ *can be written as a finite sum of operators of rank one, each belonging to* Alg $\mathscr{F}$.

Totally ordered complete lattices are completely distributive. Can 'totally ordered' be replaced by 'completely distributive' in the above theorem? If the underlying space is finite-dimensional the answer is affirmative by Theorem 4.1. We show that 'totally ordered' can be replaced by 'atomic Boolean algebra'.

THEOREM 5.2. *If the subspace lattice $\mathscr{F}$ is an atomic Boolean algebra, every operator of finite rank belonging to* Alg $\mathscr{F}$ *can be written as a finite sum of operators of rank one, each belonging to* Alg $\mathscr{F}$.

Proof. By Proposition 7.1 and Lemma 3.1 of [**5**] the operator of rank one $e \otimes f$ belongs to Alg$\mathscr{F}$ if and only if there is an atom $K \in \mathscr{F}$ such that $f \in K$ and $e \in H \ominus K'$ where K' denotes the complement of K in $\mathscr{F}$. Let $R \in \text{Alg}\,\mathscr{F}$ have rank n. Then R has the form $R = \sum_{i=1}^n e_i \otimes f_i$ where $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are each linearly independent sets of vectors. Since H is the span of all the atoms of $\mathscr{F}$, $P_K e_1 \neq 0$ for some atom K. Since

$$0 = (1 - P_K)RP_K = \sum_{i=1}^n (P_K e_i) \otimes (1 - P_K)f_i$$

we have

$$(1 - P_K)f_1 = \sum_{i=2}^n \lambda_i (1 - P_K)f_i \quad \text{where } \lambda_i = -\frac{(P_K e_1|P_K e_i)}{||P_K e_1||^2} \ (2 \leqq i \leqq n).$$

Thus, $e_1 \otimes f_1 = e_1 \otimes f_1' + \sum_{i=2}^n (\bar{\lambda}_i e_1) \otimes f_i$ where $f_1' = P_K[f_1 - \sum_{i=2}^n \lambda_i f_i]$. Hence, $R = e_1 \otimes f_1' + \sum_{i=2}^n e_i' \otimes f_i$ where $e_i' = e_i + \bar{\lambda}_i e_1 (2 \leqq i \leqq n)$. Now $f_1' \in K$ and so $(1 - P_{K'})f_1' \neq 0$. Since $(1 - P_{K'})RP_{K'} = 0$ we have $0 = P_{K'}R^*(1 - P_{K'}) = [(1 - P_{K'})f_1'] \otimes P_{K'}e_1 + \sum_{i=2}^n [(1 - P_{K'})f_i] \otimes P_{K'}e_i'$. Thus

$$P_{K'}e_1 = \sum_{i=2}^n \mu_i P_{K'}e_i' \text{ where } \mu_i = -\frac{((1 - P_{K'})f_1'|(1 - P_{K'})f_i)}{||(1 - P_{K'})f_1'||^2} \ (2 \leqq i \leqq n).$$

Hence $e_1 \otimes f_1' = e_1' \otimes f_1' + \sum_{i=2}^{n} e_i' \otimes \bar{\mu}_i f_1'$ where $e_1' = (1 - P_{K'})[e_1 - \sum_{i=2}^{n} \mu_i e_i']$. Thus $R = e_1' \otimes f_1' + \sum_{i=2}^{n} e_i' \otimes f_i'$ where $f_i' = f_i + \bar{\mu}_i f_1' (2 \leqq i \leqq n)$. Now $f_1' \in K$ and $e_1' \in H \ominus K'$ and K is an atom. Thus $e_1' \otimes f_1' \in \text{Alg } \mathscr{F}$. The proof is completed by a simple induction argument.

References

1. G. Birkhoff, *Lattice theory* (revised edition, A.M.S., New York 1948).
2. J. A. Erdos, *Operators of finite rank in nest algebras*, J. London Math. Soc. *43* (1968), 391–397.
3. P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. *76* (1970), 887–933.
4. K. J. Harrison, *Transitive atomic lattices of subspaces*, Indiana Univ. Math. J., *21* (1972), 621–642.
5. W. E. Longstaff, *Strongly reflexive lattices*, to appear, J. London Math. Soc.
6. G. N. Raney, *Completely distributive complete lattices*, Proc. Amer. Math. Soc. *3* (1952), 677–680.

The Flinders University of South Australia,
Bedford Park, S.A. 5042