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A FUNNEL SECTION PROPERTY FOR SYSTEMS WITH QUASIMONOTONE INCREASING RIGHT-HAND SIDE

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Abstract. Let $u' = f(t, u)$, $u(0) = u_0$ be an initial value problem with quasimonotone increasing right-hand side. We prove that if u, v are solutions such that $u(t_0) \ll v(t_0)$ then there is a solution w with $u(t_0) < w(t_0) < v(t_0)$.

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1. Introduction. Let $u_0 \in \mathbb{R}^n$, and $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous with $\|f(t, x)\| \leq c$, $(t, x) \in [0, T] \times \mathbb{R}^n$. We consider the initial value problem

$$u' = f(t, u), \quad u(0) = u_0. \tag{1}$$

Let L denote the Kneser funnel of problem (1); that is

$$L := \{u \in C([0, T], \mathbb{R}^n) : u \text{ solves } (1)\}.$$

According to Kneser's Theorem L is compact and connected; see for example [**1**, p. 24]. In particular for $t_0 \in [0, T]$ the funnel section

$$L_{t_0} = \{u(t_0) : u \in L\}$$

is a compact and connected subset of $\mathbb{R}^n$. For further investigations of the topological properties of funnels and funnel sections see [**3**], [**4**], [**6**] and references given there. In this paper we will consider the case that $\mathbb{R}^n$ is ordered by a cone and that f is in addition quasimonotone increasing.

Consider $\mathbb{R}^n$ together with a partial ordering $\leq$ induced by a cone K. A *cone* K is a closed convex subset of $\mathbb{R}^n$ with $\lambda K \subseteq K$, $\lambda \geq 0$, and $K \cap (-K) = \{0\}$. We will always assume that K is solid; that is Int $K \neq \emptyset$. As usual $x \leq y \iff y - x \in K$, and we use the notations $x < y$ if $x \leq y$ but $x \neq y$, and $x \ll y$ if $y - x \in \text{Int } K$. Let K^* denote the dual wedge; that is the set of all continuous linear functionals φ on $\mathbb{R}^n$ with $\varphi(x) \geq 0$, $x \geq 0$. For $x \leq y$ let $[x, y]$ be the order interval $\{z \in \mathbb{R}^n : x \leq z \leq y\}$. Since K is solid we can fix $p \in \text{Int } K$ and norm $\mathbb{R}^n$ by the Minkowski functional $\|\cdot\|$ of $[-p, p]$. For the sequel let $K_r(x)$ denote the closed ball $\{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. Finally let $C([0, T], \mathbb{R}^n)$ be endowed with the corresponding maximum norm $\|\cdot\|_\infty$.

Let $D \subset \mathbb{R}^n$. A function $f : [0, T] \times D \to \mathbb{R}^n$ is called *quasimonotone increasing* (in the sense of Volkmann [**7**]) if

$$t \in [0, T],\ x, y \in D,\ x \leq y,\ \varphi \in K^*,\ \varphi(x) = \varphi(y) \Rightarrow \varphi(f(t, x)) \leq \varphi(f(t, y)).$$

For quasimonotone increasing right-hand sides in problem (1) we prove the following additional structure of L_{t_0}.

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Theorem 1. *Let* $f: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *be quasimonotone increasing, let* $u, v \in L$ *and* $t_0 \in (0, T]$. *If* $u(t_0) \ll v(t_0)$, *then there exists* $w \in L$ *such that* $u(t_0) < w(t_0) < v(t_0)$.

Remarks. 1. Under the assumptions of Theorem 1 there is always a maximal and a minimal solution of problem (1); that is, there are functions $\underline{u}, \overline{u} \in L$ such that $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, $t \in [0, T]$, for each $u \in L$. See [**5**].

2. In dimension $n = 1$ every function is quasimonotone increasing ($K = [0, \infty)$; $<$ and $\ll$ means the same), L_{t_0} is a point or a compact interval, and if $u(t_0) < v(t_0)$ then Theorem 1 can be proved by starting at $\eta \in (u(t_0), v(t_0))$ and going left along a solution until one hits $\underline{u}$ or $\overline{u}$. This proof of course does not work for $n \geq 2$.

3. In Theorem 1 it is not supposed that $u(t)$ and $v(t)$ are comparable for all t. The assertion of Theorem 1 is equivalent to $x, y \in L_{t_0}$, $x \ll y$ implies that there exists $z \in L_{t_0}$ such that $x < z < y$. In fact the proof shows that $x < z \ll y$ is possible. By an analogous proof one can obtain $x \ll z < y$.

4. An analogous local version ($f: [0, T] \times K_r(u_0) \to \mathbb{R}^n$, $||f|| \leq c$ and $t_0 \in (0, \min\{T, r/c\}]$) of Theorem 1 holds. The proof is more technical but is essentially the same.

5. Consider dimension $n = 2$ and $K = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. Set $\alpha = 1/10$. For example the set defined by

$$([-1, 1]^2) \setminus \{(x, y) : -\alpha < y + x < \alpha, y < x\}$$

is compact, connected, contains a maximal and a minimal element, but cannot be a funnel section if f is quasimonotone increasing according to Theorem 1 (although it can be a funnel section if f is not quasimonotone increasing; see [**6**, Corollary (5.5)]).

Note added in proof. Dr. Roland Uhl showed me the following example. Consider $f: [0, 2] \times \mathbb{R}^2 \to \mathbb{R}^2$, $f(t, x, y) = (4((1 - t)_+)(x_+)^{1/2}, 2((x + y - 1)_+)^{1/2})$. This function is monotone in (x, y) (with respect to K as in 5.), and for problem (1) with $u_0 = (0, 0)$ we have $L_2 = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$.

$$f(t, x, y) = (4(1 - t)_+(x_+)^{1/2}, 2((x + y - 1)_+)^{1/2}).$$

2. Proof. To prove Theorem 1 we shall use the following Proposition.

Proposition 1. *Let* $\varepsilon > 0$ *and* $R > 0$. *Then there is a continuous function* $f_\varepsilon : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *such that:*

1 $||f(t, x) - f_\varepsilon(t, x)|| \leq \varepsilon, \quad (t, x) \in [0, T] \times K_R(u_0)$;

2 $||f_\varepsilon(t, x)|| \leq c, \quad (t, x) \in [0, T] \times \mathbb{R}^n$;

3 there exists $L_\varepsilon \geq 0$ *such that*

$$||f_\varepsilon(t, x) - f_\varepsilon(t, y)|| \leq L_\varepsilon ||x - y||, \quad (t, x), (t, y) \in [0, T] \times \mathbb{R}^n;$$

4 f_ε *is quasimonotone increasing.*

Proof. Let $0 < \delta < 1$ be such that $||f(t, x) - f(t, y)|| \leq \varepsilon$ if (t, x), $(t, y) \in [0, T] \times K_{R+1}(u_0)$ with $||x - y|| \leq \delta$. Let $h \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that

$$h(x) \geq 0, \ x \in \mathbb{R}^n, \quad \text{supp}\, h \subset K_\delta(0), \quad \int_{\mathbb{R}^n} h(x)\, dx = 1.$$

Now let $f_\varepsilon : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$f_\varepsilon(t,x) := \int_{\mathbb{R}^n} h(\xi - x) f(t,\xi)\, d\xi = \int_{\mathbb{R}^n} h(\xi) f(t, \xi + x)\, d\xi.$$

By standard reasoning (see [**2**, S.25]) *1*, *2* and *3* hold. To prove *4* let (t,x), $(t,y) \in [0,T] \times \mathbb{R}^n$ and $\varphi \in K^*$ with $x \le y$ and $\varphi(x) = \varphi(y)$. Then $\xi + x \le \xi + y$ and $\varphi(\xi + x) = \varphi(\xi + y)$, for each $\xi \in \mathbb{R}^n$, and therefore

$$\varphi(f_\varepsilon(t,x)) = \int_{\mathbb{R}^n} h(\xi)\varphi(f(t,\xi + x))\, d\xi \le$$

$$\int_{\mathbb{R}^n} h(\xi)\varphi(f(t,\xi + y))\, d\xi = \varphi(f_\varepsilon(t,y)).$$

□

Proof of Theorem 1. Let $u, v \in L$ with $u(t_0) \ll v(t_0)$. We set $R = T(c+3)$. Fix $\varepsilon \in (0,1)$. There is a function $f_\varepsilon : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, as in Proposition 1.

We set

$$q(\varepsilon) := ||f(\cdot, u) - f_\varepsilon(\cdot, u)||_\infty + ||f(\cdot, v) - f_\varepsilon(\cdot, v)||_\infty.$$

For $\lambda \in [0,1]$ we consider the function $F_\lambda : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $F_\lambda(t,x) =$

$$f_\varepsilon(t,x) + (1-\lambda)(f(t,u(t)) - f_\varepsilon(t,u(t))) + \lambda(f(t,v(t)) - f_\varepsilon(t,v(t))) + \lambda q(\varepsilon)p.$$

Now consider the initial value problems

$$w_\lambda' = F_\lambda(t, w_\lambda), \quad w_\lambda(0) = u_0, \qquad (\lambda \in [0,1]).$$

Since F_λ is Lipschitz the solutions $w_\lambda : [0,T] \to \mathbb{R}^n$ are unique and depend continuously on $\lambda \in [0,1]$, and we have $w_0(t) = u(t)$, $t \in [0,T]$. Moreover we have $v(t) \le w_1(t)$, $t \in [0,T]$, and $w_\lambda(t) \le w_\mu(t)$, $t \in [0,T]$ if $0 \le \lambda \le \mu \le 1$. For $t \in [0,T]$, we have

$$v'(t) - F_1(t, v(t)) = -q(\varepsilon)p \le 0 = w_1'(t) - F_1(t, w_1(t)), \quad v(0) = u_0 = w_1(0).$$

Since F_λ is Lipschitz and quasimonotone increasing we have $v(t) \le w_1(t)$ according to a classical theorem on differential inequalities; see [**7**]. To prove the second inequality consider

$$\frac{d}{d\lambda}\Big((1-\lambda)(f(t,u(t)) - f_\varepsilon(t,u(t))) + \lambda(f(t,v(t)) - f_\varepsilon(t,v(t))) + \lambda q(\varepsilon)p\Big) =$$

$$-(f(t,u(t)) - f_\varepsilon(t,u(t))) + (f(t,v(t)) - f_\varepsilon(t,v(t))) + q(\varepsilon)p \ge 0,$$

according to the property $||x||p - x \ge 0$ of the chosen norm. Therefore $F_\lambda(t,x) - f_\varepsilon(t,x)$ is monotone increasing in λ (and independent of x). Hence for $\lambda \le \mu$ and $t \in [0,T]$

$$w_\lambda'(t) - F_\lambda(t, w_\lambda(t)) = 0 \le w_\mu'(t) - F_\lambda(t, w_\mu(t)), \quad w_\lambda(0) = u_0 = w_\mu(0).$$

Again we conclude that $w_\lambda(t) \leq w_\mu(t)$, $t \in [0, T]$ and, in particular, we have $u(t) \leq w_\lambda(t)$, $t \in [0, T]$ and $\lambda \in [0, 1]$. Next, we have $||u(t) - u_0|| \leq Tc \leq R$ and $||v(t) - u_0|| \leq Tc \leq R$, $t \in [0, T]$. Therefore, for each $\lambda \in [0, 1]$, we have

$$||F_\lambda(t, x)|| \leq c + 3\varepsilon, \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$

Hence

$$||w_\lambda(t) - u_0|| \leq T(c + 3\varepsilon) \leq T(c + 3) = R, \quad t \in [0, T];$$

that is $w_\lambda(t) \in K_R(u_0)$, $t \in [0, T]$, which implies that

$$||F_\lambda(t, w_\lambda(t)) - f(t, w_\lambda(t))|| \leq 4\varepsilon, \quad t \in [0, T].$$

Since $u(t_0) \ll v(t_0)$ there exists $\delta > 0$ such that $x \ll v(t_0)$ for each $x \in K_\delta(u(t_0))$. We define $\Phi : C([0, T], \mathbb{R}^n) \to \mathbb{R}$ by

$$\Phi(h) = ||h(t_0) - u(t_0)|| - \delta.$$

The function Φ is continuous. We have $\Phi(w_0) = \Phi(u) = -\delta < 0$ and $\Phi(w_1) \geq \Phi(v) > 0$. Hence there exists $\lambda = \lambda(\varepsilon) \in (0, 1)$ with $\Phi(w_\lambda) = 0$.

According to the construction above and for $\varepsilon_k = 1/(k + 1)$ we find a sequence of functions $w_k : [0, T] \to \mathbb{R}^n$ with the following properties ($k \in \mathbb{N}$):

1. $u(t) \leq w_k(t)$, $t \in [0, T]$;
2. $\Phi(w_k) = 0$;
3. w_k is a solution of $w_k'(t) = f(t, w_k(t)) + g_k(t)$, $w_k(0) = u_0$, with $||g_k(t)|| \leq 4\varepsilon_k = 4/(k + 1)$.

According to the Arzelà-Ascoli Theorem there is a limit $w \in L$ of a subsequence of $(w_k)_{k=1}^\infty$. We have $u(t) \leq w(t)$, $t \in [0, T]$, and $u(t_0) < w(t_0)$ since $||w(t_0) - u(t_0)|| = \delta$. According to the choice of δ we have $w(t_0) \ll v(t_0)$, especially $w(t_0) < v(t_0)$.

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REFERENCES

1. K. Deimling, *Ordinary differential equations in Banach spaces*, Lecture Notes in Mathematics No. 596 (Springer-Verlag, 1977).

2. K. Deimling, *Nichtlineare Gleichungen und Abbildungsgrade* (Springer-Verlag, 1974).

3. P. Hartman, *Ordinary differential equations* (John Wiley & Sons, Inc., 1964).

4. E. Horst, Differential equations in Banach spaces: five examples, *Arch. Math. (Basel)* **46** (1986), 440–444.

5. R. Lemmert, S. Schmidt and P. Volkmann, Ein Existenzsatz für gewöhnliche Differentialgleichungen mit quasimonoton wachsender rechter Seite, *Math. Nachr*. **153** (1991), 349–352.

6. C. C. Pugh, Funnel sections, *J. Differential Equations* **19** (1975), 270–295.

7. P. Volkmann, Gewöhnliche Differentialungleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen, *Math. Z.* **127** (1972), 157–164.