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Restricting Fourier Transforms of Measures to Curves in $\mathbb{R}^2$

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Abstract. We establish estimates for restrictions to certain curves in $\mathbb{R}^2$ of the Fourier transforms of some fractal measures.

1 Introduction

The starting point for this note was the following observation: if μ is a compactly supported nonnegative Borel measure on $\mathbb{R}^2$ that, for some $\alpha > 3/2$, is α-dimensional in the sense that

$$\mu\big(B(y,r)\big) \lesssim r^{\alpha} \tag{1.1}$$

for $y \in \mathbb{R}^2$ and $r > 0$, then

$$\int_0^\infty |\widehat{\mu}(t,t^2)|^2\,dt < \infty. \tag{1.2}$$

The proof is easy: writing $d\lambda$ for the measure given by dt on the curve (t,t^2), we see that

$$\begin{aligned}\int_0^\infty |\widehat{\mu}(t,t^2)|^2\,dt &= \iiint e^{-2\pi i(t,t^2)\cdot(x-y)}\,d\mu(x)\,d\mu(y)\,dt \\ &= \iint \widehat{\lambda}(x-y)\,d\mu(x)\,d\mu(y) \\ &\lesssim \iint |x_2-y_2|^{-1/2}\,d\mu(x)\,d\mu(y),\end{aligned} \tag{1.3}$$

where we put $x = (x_1,x_2)$ if $x \in \mathbb{R}^2$ and the inequality comes from the van der Corput estimate $|\widehat{\lambda}(x)| \lesssim |x_2|^{-1/2}$. For fixed y, the compact support of μ implies that

$$\int |x_2-y_2|^{-1/2}\,d\mu(x) \lesssim \sum_{j=0}^\infty 2^{j/2}\mu(\{x : |x_2-y_2| \le 2^{-j}\}) \lesssim \sum_{j=0}^\infty 2^{j/2}2^{j}2^{-j\alpha}$$

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since $\{x : |x_2 - y_2| \leq 2^{-j}\}$ can be covered by $\lesssim 2^j$ balls of radius 2^{-j}. Clearly the last sum is finite if $\alpha > 3/2$, and then (1.3) is finite since μ is a finite measure.

The simplemindedness of this argument made it seem unlikely that the index $3/2$ is best possible, and the search for that best index was the motivation for this work. Our results here are the following theorems.

Theorem 1.1 *Suppose* $\phi \in C^2([1,2])$ *satisfies the estimates*

$$\phi' \approx m, \quad \phi'' \approx m \tag{1.4}$$

for some $m \geq 1$*, and let* $\gamma(t) = (t, \phi(t))$*. Suppose* μ *is a nonnegative and compactly supported m measure on* $\mathbb{R}^2$ *that is* α*-dimensional in the sense that* (1.1) *holds. Then for* $\epsilon > 0$,

$$\int_1^2 |\widehat{\mu}(R\gamma(t))|^2 \, dt \lesssim R^{-\alpha/2+\epsilon} m^{1-\alpha}, \tag{1.5}$$

when $R \geq 2$*. Here the implied constant in* (1.5) *depends only on* α, ϵ*, the implied constants in* (1.1) *and* (1.4)*, and the diameter of the support of* μ.

Theorem 1.2 *Suppose* μ *is as in Theorem* 1.1, $p > 1$*, and*

(i) $-1 < \gamma < \alpha p - \alpha/2 - p$ *if* $1 < \alpha < 2$,
(ii) $-1 < \gamma < -1/2$ *if* $1/2 < \alpha \leq 1$,
(iii) $-1 < \gamma < \alpha - 1$ *if* $0 < \alpha \leq 1/2$.

Then

$$\int_0^\infty |\widehat{\mu}(t, t^p)|^2 \, t^\gamma \, dt \leq C < \infty, \tag{1.6}$$

where C depends only on p, the implied constant in (1.1)*, and the diameter of the support of* μ.

Theorem 1.3 *If* (1.6) *holds for* $p > 1$ *and* $\alpha \in (0,2)$ *with C as stated in Theorem* 1.2*, then*

(i) $-1 < \gamma \leq \alpha p - \alpha/2 - p$ *if* $1 < \alpha < 2$,
(ii) $-1 < \gamma \leq -1/2$ *if* $1/2 < \alpha \leq 1$,
(iii) $-1 < \gamma \leq \alpha - 1$ *if* $0 < \alpha \leq 1/2$.

Remarks (i) Theorem 1.1 is a generalization of Theorem 1 in [7], which was reproved with a simpler argument in [1]. As described in §2, the proof of Theorem 1.1 is just an adaptation of ideas from [1,7].

(ii) The examples which comprise the proof of Theorem 1.3 are similar in spirit to those in the proof of Proposition 3.2 in [7].

(iii) If α_0 is the infimum of the α's for which (1.1) implies (1.2) whenever μ is compactly supported, it follows from Theorem 1.2 that $\alpha_0 \leq 4/3$. Then the proof of Theorem 1.3 and a uniform boundedness argument together imply that $\alpha_0 = 4/3$.

(iv) Analogues of Theorem 1.1 have been studied for hypersurfaces in $\mathbb{R}^d$ and, particularly, for the sphere S^{d-1}. See, for example, [1–6].

The remainder of this note is organized as follows: the proof of Theorem 1.1 is in §2 and the proofs of Theorems 1.2 and 1.3 are in §3.

2 Proof of Theorem 1.1

As mentioned above, the proof is an adaptation of ideas from [1,7]. Specifically, with μ as in Theorem 1.1 and

$$\Gamma_R = \{R\gamma(t) : 1 \le t \le 2\}, \quad \Gamma_{R,\delta} = \Gamma_R + B(0, R^\delta)$$

for $R \ge 2$ and $\delta > 0$, we will modify an uncertainty principle argument from [7] to show that (1.5) follows from the estimate

$$\int_{\Gamma_{R,\delta}} |\widehat{\mu}(y)|^2 \, dy \lesssim R^{1-\alpha/2+2\delta} \, m^{2-\alpha}. \tag{2.1}$$

We will then adapt a bilinear argument from [1] to prove (2.1).

So, arguing as in [7], if $\kappa \in C_c^\infty(\mathbb{R}^2)$ is equal to 1 on the support of μ, then

$$\begin{aligned} \int_1^2 |\widehat{\mu}\big(R\gamma(t)\big)|^2 \, dt &= \int_1^2 \Big| \int \widehat{\kappa}(R\gamma(t) - y) \, \widehat{\mu}(y) \, dy \Big|^2 \, dt \\ &\lesssim \int \int_1^2 |\widehat{\kappa}(R\gamma(t) - y)| dt \, |\widehat{\mu}(y)|^2 \, dy. \end{aligned} \tag{2.2}$$

If $y = (y_1, y_2)$, then

$$\begin{aligned} \int_1^2 |\widehat{\kappa}(R\gamma(t) - y)| \, dt &\lesssim \int_1^2 \frac{1}{(1 + |R\gamma(t) - y|)^{10}} \, dt \\ &\lesssim \frac{1}{(1 + \operatorname{dist}(\Gamma_R, y)\,)^8} \int_1^2 \frac{1}{(1 + |R\phi(t) - y_2|)^2} \, dt. \end{aligned}$$

Estimating the last integral using the hypothesized lower bound on ϕ', we see from (2.2) that

$$\int_1^2 |\widehat{\mu}(R\gamma(t))|^2 dt \lesssim \frac{1}{Rm} \int \frac{|\widehat{\mu}(y)|^2}{(1 + \operatorname{dist}(\Gamma_R, y)\,)^8} \, dy. \tag{2.3}$$

Now

$$\begin{aligned} \int \frac{|\widehat{\mu}(y)|^2}{(1 + \operatorname{dist}(\Gamma_R, y))^8} \, dy &= \int_{\Gamma_{R,\epsilon/2}} + \sum_{j=2}^\infty \int_{\Gamma_{R,j\epsilon/2} \sim \Gamma_{R,(j-1)\epsilon/2}} \\ &\lesssim \int_{\Gamma_{R,\epsilon/2}} |\widehat{\mu}(y)|^2 \, dy + \sum_{j=2}^\infty R^{-8(j-1)\epsilon/2} \int_{\Gamma_{R,j\epsilon/2}} |\widehat{\mu}(y)|^2 \, dy. \end{aligned}$$

Thus (1.5) follows from (2.1) and (2.3).

Turning to the proof of (2.1), we note that by duality (and the fact that μ is finite) it is enough to suppose that f, satisfying $\|f\|_2 = 1$, is supported on $\Gamma_{R,\delta}$ and then to establish the estimate

$$\int |\widehat{f}(y)|^2 \, d\mu(y) \lesssim R^{1-\alpha/2+2\delta} \, m^{2-\alpha}. \tag{2.4}$$

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The argument we will give differs from the proof of Theorem 3 in [1] only in certain technical details. But because those details are not always obvious, and for the convenience of any reader, we will give the complete proof.

For $y \in \mathbb{R}^2$, write y' for the point on the curve Γ_R that is closest to y (if there are multiple candidates for y', choose the one with least first coordinate). Then $y' = R\gamma(t')$ for some $t' \in [1,2]$. For a dyadic interval $I \subset [1,2]$, define

$$\Gamma_{R,\delta,I} = \{y \in \Gamma_{R,\delta} : t' \in I\}, \quad f_I = f \cdot \chi_{\Gamma_{R,\delta,I}}.$$

For dyadic intervals $I, J \subset [1,2]$, we write $I \sim J$ if I and J have the same length and are not adjacent, but have adjacent parent intervals. The decomposition

$$[1,2] \times [1,2] = \bigcup_{n \geq 2} \Big(\bigcup_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} (I \times J) \Big) \tag{2.5}$$

leads to

$$\int |\widehat{f}(y)|^2 \, d\mu(y) \leq \sum_{n \geq 2} \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \int |\widehat{f_I}(y) \widehat{f_J}(y)| \, d\mu(y). \tag{2.6}$$

Truncating (2.5) and (2.6) gives

$$\begin{aligned} &\int |\widehat{f}(y)|^2 \, d\mu(y) \\ &\qquad \leq \sum_{4 \leq 2^n \leq R^{1/2}} \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \int |\widehat{f_I}(y) \widehat{f_J}(y)| \, d\mu(y) + \sum_{I \in \mathcal{I}} \int |\widehat{f_I}(y)|^2 \, d\mu(y), \end{aligned} \tag{2.7}$$

where $\mathcal{I}$ is a finitely overlapping set of dyadic intervals I with $|I| \approx R^{-1/2}$.

To estimate the integrals on the right-hand side of (2.7), we begin with two geometric observations. The first of these is that if $I \subset [1,2]$ is an interval with length ℓ, then $\Gamma_{R,I} \doteq \{R\big(t, \phi(t)\big) : t \in I\}$ is contained in a rectangle D with side lengths $\lesssim R\ell m, R\ell^2$, which we will abbreviate by saying that D is an $(R\ell m) \times (R\ell^2)$ rectangle. (To see this, note that the since the sine of the angle between vectors $(1, M)$ and $(1, M+\kappa)$ is

$$\frac{\kappa}{\sqrt{1+M^2}\sqrt{1+(M+\kappa)^2}},$$

it follows from (1.4) that the angle between tangent vectors at the beginning and ending points of the curve $\Gamma_{R,I}$ is $\lesssim \ell/m$. Since the distance between these two points is $\lesssim R\ell m$, it is clear that $\Gamma_{R,I}$ is contained in a rectangle D of the stated dimensions.)

Secondly, we observe that if $\ell \gtrsim R^{-1/2}$, then an R^δ neighborhood of an $(R\ell m) \times (R\ell^2)$ rectangle is contained in an $(R^{1+\delta}\ell m) \times (R^{1+\delta}\ell^2)$ rectangle. It follows that if I has length $2^{-n} \gtrsim R^{-1/2}$, then the support of f_I is contained in a rectangle D with dimensions $(R^{1+\delta}2^{-n}m) \times (R^{1+\delta}2^{-2n})$.

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The next lemma is part of Lemma 3.1 in [1] (the hypothesis $1 \leq \alpha \leq 2$ there is not necessary for the conclusion of that lemma). To state it, we introduce some notation: ϕ is a nonnegative Schwartz function such that $\phi(x) = 1$ for x in the unit cube Q; $\phi(x) = 0$ if $x \notin 2Q$, and, for each $M > 0$,

$$|\widehat{\phi}| \leq C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}.$$

For a rectangle $D \subset \mathbb{R}^2$, ϕ_D will stand for $\phi \circ b$, where b is an affine mapping which takes D onto Q. If D is a rectangle with dimensions $a_1 \times a_2$, then a dual rectangle of D is any rectangle with the same axis directions and with dimensions $a_1^{-1} \times a_2^{-1}$.

Lemma 2.1 *Suppose that μ is a non-negative Borel measure on $\mathbb{R}^2$ satisfying (1.1). Suppose D is a rectangle with dimensions $R_2 \times R_1$, where $R_2 \gtrsim R_1$, and let D_{dual} be the dual of D centered at the origin. Then, if $\widetilde{\mu}(E) = \mu(-E)$,*

$$(\widetilde{\mu} * |\widehat{\phi_D}|)(y) \lesssim R_2^{2-\alpha}, \ y \in \mathbb{R}^2, \tag{2.8}$$

and if $K \gtrsim 1, \ y_0 \in \mathbb{R}^2$, then

$$\int_{K \cdot D_{\mathrm{dual}}} (\widetilde{\mu} * |\widehat{\phi_D}|)(y_0 + y)\, dy \lesssim K^{\alpha} R_2^{1-\alpha} R_1^{-1}. \tag{2.9}$$

Now if $I \in \mathcal{I}$ and $\operatorname{supp} f_I \subset D$ as above, the identity $\widehat{f_I} = \widehat{f_I} * \widehat{\phi_D}$ implies that

$$|\widehat{f_I}| \leq (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2} \|\widehat{\phi_D}\|_1^{1/2} \lesssim (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2},$$

and so

$$\begin{aligned} \int |\widehat{f_I}(y)|^2 \, d\mu(y) &\lesssim \int (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)(y)\, d\mu(y) \\ &= \int |\widehat{f_I}(y)|^2 (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\, dy \lesssim \|f_I\|_2^2 \, R^{1-\alpha/2+2\delta} m^{2-\alpha}, \end{aligned} \tag{2.10}$$

where the last inequality follows from (2.8) and the fact that, since $2^{-n} \approx R^{-1/2}$, D has dimensions $(R^{1/2+\delta} m) \times R^{\delta}$. Thus the estimate

$$\sum_{I \in \mathcal{I}} \int |\widehat{f_I}(y)|^2 \, d\mu(y) \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha} \sum_{I \in \mathcal{I}} \|f_I\|_2^2 \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha} \tag{2.11}$$

follows from $\|f\|_2 = 1$ and the finite overlap of the intervals $I \in \mathcal{I}$ (which implies finite overlap for the supports of the $f_I, I \in \mathcal{I}$).

To bound the principal term of the right-hand side of (2.7), fix n with $4 \leq 2^n \leq R^{1/2}$ and a pair I, J of dyadic intervals with $|I| = |J| = 2^{-n}$ and $I \sim J$. Since $I \sim J$, the support of $f_I * f_J$ is contained in a rectangle D with dimensions

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$(R^{1+\delta}2^{-n}m) \times (R^{1+\delta}2^{-2n})$. For later reference, let v be a unit vector in the direction of the longer side of D. As in (2.10),

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\,d\mu(y) \lesssim \int (|\widehat{f_I}\,\widehat{f_J}| * |\widehat{\phi_D}|)(y)\,d\mu(y) = \int |\widehat{f_I}(y)\widehat{f_J}(y)|\,(\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\,dy. \tag{2.12}$$

Now tile $\mathbb{R}^2$ with rectangles P having exact dimensions $C \times (C2^{-n}m^{-1})$ for some large $C > 0$ to be chosen later and having shorter axis in the direction of v. Let ψ be a fixed nonnegative Schwartz function satisfying $1 \le \psi(y) \le 2$ if $y \in Q$, $\widehat{\psi}(x) = 0$ if $x \notin Q$, and

$$\psi \le C_M \sum_{j=1}^{\infty} 2^{-Mj}\chi_{2^jQ}. \tag{2.13}$$

Since $\sum_P \psi_P^3 \approx 1$, it follows from (2.12) that if $f_{I,P}$ is defined by $\widehat{f_{I,P}} = \psi_P \cdot \widehat{f_I}$, then

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\,d\mu(y) \lesssim \sum_P \Big(\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2\,dy\Big)^{1/2}\Big(\int |(\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)|^2\,dy\Big)^{1/2}. \tag{2.14}$$

To estimate the first integral in this sum, we begin by noting that the support of $f_{I,P}$ is contained in $\mathrm{supp}(f_I) + P_{\mathrm{dual}}$, where P_{dual} is a rectangle dual to P and centered at the origin. Let $\widetilde{I}$ be the interval with the same center as I but lengthened by $2^{-n}/10$ and let $\widetilde{J}$ be defined similarly. Since $I \sim J$, it follows that $\mathrm{dist}(\widetilde{I}, \widetilde{J}) \ge 2^{-n}/2$. Now the support of f_I is contained in $\Gamma_{R,I} + B(0, R^{\delta})$ and P_{dual} has dimensions $(m2^nC^{-1}) \times C^{-1}$ with the longer direction at an angle $\lesssim 2^{-n}/m$ to any of the tangents to the curve $(t, \phi(t))$ for $t \in \widetilde{I}$ (or $t \in \widetilde{J}$). Recalling that $2^n \lesssim R^{1/2}$, one can check that if C is large enough, $\mathrm{supp}(f_{I,P}) \subset \Gamma_{R,\widetilde{I}} + B(0, CR^{\delta})$ and, similarly, $\mathrm{supp}(f_{J,P}) \subset \Gamma_{R,\widetilde{J}} + B(0, CR^{\delta})$. The following lemma will be proved at the end of this section.

Lemma 2.2 *Suppose ϕ satisfies the estimates $0 < \phi' \le m_1$ and $\phi'' \ge m_2$ with $m_1 \ge 1$ and*

$$m_1, m_2 \approx m. \tag{2.15}$$

Suppose that the closed intervals $\tilde{I}, \tilde{J} \subset [1, 2]$ satisfy $\mathrm{dist}\,(\tilde{I}, \tilde{J}) \ge c\,2^{-n}$. *Then for $\delta > 0$ and $x \in \mathbb{R}^2$, there is the following estimate for the two-dimensional Lebesgue measure of the intersection of translates of tubular neighborhoods of $\Gamma_{R,\tilde{I}}$ and $\Gamma_{R,\tilde{J}}$:*

$$|x + \Gamma_{R,\tilde{I}} + B(0, CR^{\delta}) \cap \Gamma_{R,\tilde{J}} + B(0, CR^{\delta})\,| \lesssim R^{2\delta}2^n m. \tag{2.16}$$

The implicit constant in (2.16) depends only on the implicit constants in (2.15) and the positive constants c and C.

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It follows from Lemma 2.2 that for $x \in \mathbb{R}^2$ we have

$$|x + \operatorname{supp}(f_{I,P}) \cap \operatorname{supp}(f_{J,P})| \lesssim R^{2\delta} 2^n m. \tag{2.17}$$

Now

$$\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2\, dy = \int |\widetilde{f_{I,P}} * f_{J,P}(x)|^2\, dx$$

and

$$\begin{aligned}|\widetilde{f_{I,P}} * f_{J,P}(x)| &\le \int |f_{I,P}(w-x)\, f_{J,P}(w)|\, dw \\ &\le |x + \operatorname{supp}(f_{I,P}) \cap \operatorname{supp}(f_{J,P})|^{1/2} \left(|\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x)\right)^{1/2}.\end{aligned}$$

Thus, by (2.17),

$$\begin{aligned}\Big(\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2\, dy\Big)^{1/2} &\lesssim R^{\delta} 2^{n/2} m^{1/2} \Big(\int |\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x)\, dx\Big)^{1/2} \\ &= R^{\delta} 2^{n/2} m^{1/2} \|f_{I,P}\|_2 \|f_{J,P}\|_2.\end{aligned} \tag{2.18}$$

To estimate the second integral in the sum (2.14), we use (2.13) to observe that

$$\psi_P \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j P}.$$

Thus

$$\int (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\, dy \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \int_{2^j P} (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\, dy.$$

Noting that $2^j P \subset y_P + K D_{\mathrm{dual}}$ for some $K \lesssim R^{1+\delta} 2^{-2n+j}$ and some $y_P \in \mathbb{R}^2$, we apply (2.9) to obtain

$$\begin{aligned}\int (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\, dy &\lesssim \sum_{j=1}^{\infty} 2^{-Mj} (R^{1+\delta} 2^{-2n+j})^{\alpha} (R^{1+\delta} 2^{-n} m)^{1-\alpha} (R^{1+\delta} 2^{-2n})^{-1} \\ &\lesssim 2^{-n(\alpha-1)} m^{1-\alpha}.\end{aligned}$$

Since $(\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \lesssim (R^{1+\delta} 2^{-n} m)^{2-\alpha}$ by (2.8) and since $\psi_P(y) \lesssim 1$, it follows that

$$\Big(\int \big((\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\big)^2\, dy\Big)^{1/2} \lesssim R^{1-\alpha/2+\delta(1-\alpha/2)} 2^{-n/2} m^{3/2-\alpha}. \tag{2.19}$$

Now (2.18) and (2.19) imply by (2.14) that

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\, d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \Big(\sum_P \|f_{I,P}\|_2^2\Big)^{1/2} \Big(\sum_P \|f_{J,P}\|_2^2\Big)^{1/2}.$$

Since

$$\sum_P \|\widehat{f_{I,P}}\|_2^2 = \int |\widehat{f}_I(y)|^2 \sum_P |\psi_P(y)|^2\, dy,$$

it follows from $\sum_P \psi_P^2 \lesssim 1$ that

$$\int |\widehat{f}_I(y)\widehat{f}_J(y)|\, d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \|f_I\|_2 \|f_J\|_2.$$

Thus

$$\begin{aligned}(2.20)\qquad \sum_{\substack{|I|=|J|=2^{-n}\\ I\sim J}} \int |\widehat{f}_I(y)\widehat{f}_J(y)|\, d\mu(y) &\lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \sum_{\substack{|I|=|J|=2^{-n}\\ I\sim J}} \|f_I\|_2 \|f_J\|_2 \\ &\lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \|f\|_2^2.\end{aligned}$$

Now (2.4) follows from (2.7), (2.11), (2.20), and the fact that the first sum in (2.7) has $\lesssim \log R$ terms.

Proof of Lemma 2.2 Fix $t \in \tilde{I},\ s \in \tilde{J}$ such that

$$(2.21)\qquad x + R\big(t, \phi(t)\big) + \overline{B(0, CR^{\delta})} \cap R\big(s, \phi(s)\big) + \overline{B(0, CR^{\delta})} \neq \varnothing$$

and such that t is minimal subject to (2.21). Without loss of generality, assume that $t < s$. Suppose that v and w satisfy

$$(2.22)\quad x + R(t+w, \phi(t+w)) + \overline{B(0, CR^{\delta})} \cap R(s+v, \phi(s+v)) + \overline{B(0, CR^{\delta})} \neq \varnothing.$$

We will begin by observing that

$$(2.23)\qquad w \le \frac{8C2^n R^{\delta-1} m_1}{c\, m_2}.$$

From (2.21) and (2.22) it follows that

$$(2.24)\qquad |w-v|,\ \Big|\big(\phi(s+v)-\phi(s)\big) - \big(\phi(t+w)-\phi(t)\big)\Big| \le 4CR^{\delta-1}.$$

Now

$$(2.25)\ \big(\phi(s+v)-\phi(s)\big) - \big(\phi(t+w)-\phi(t)\big) = \int_t^{t+w} \big(\phi'(u+s-t)-\phi'(u)\big)\, du + e,$$

where the error term e satisfies $|e| \le 4CR^{\delta-1} m_1$ because of the first inequality in (2.24) and the bound on ϕ'. Since $s - t \ge c2^{-n}$, the lower bound on ϕ'' shows that

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the integral in (2.25) exceeds $wc2^{-n}m_2$. Thus if $wc2^{-n}\, m_2 > 8CR^{\delta-1}\, m_1$ (that is, if (2.23) fails) then, since $m_1 \geq 1$, (2.25) exceeds $4CR^{\delta-1}$, contradicting (2.24).

To see (2.16), define $\tilde{t}$ by

$$\tilde{t} = t + \frac{8C2^n R^{\delta-1} m_1}{c\, m_2}$$

and note that by (2.23) the intersection in (2.16) is contained in a translate of

$$\{R(u, \phi(u)) : t \leq u \leq \tilde{t}\} + B(0, CR^{\delta}) \doteq \Gamma + B(0, CR^{\delta}).$$

Using $\phi' \lesssim m$, the length of the curve Γ is $\lesssim 2^n R^{\delta} m$. Thus Γ is contained in $\lesssim 2^n m$ balls of radius R^{δ}. This implies (2.16). ■

3 Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2 First suppose $1 < \alpha < 2$. Choose $\epsilon > 0$ such that $\gamma + 2\epsilon < \alpha(p - 1/2) - p$. Then apply Theorem 1.1 with $\phi(t) = R^{p-1}t^p$ and $m = R^{p-1}$ to conclude that

$$\int_1^2 \left| \widehat{\mu}\big(Rt, (Rt)^p\big) \right|^2 dt \lesssim R^{-\alpha/2+\epsilon} R^{(p-1)(1-\alpha)}$$

and so

$$\int_R^{2R} |\widehat{\mu}(t, t^p)|^2\, t^{\gamma}\, dt \lesssim R^{-\epsilon}.$$

Now (1.6) follows by taking $R = 2^n$.

To deal with the remaining cases we note that if $d\nu$ is dt on the curve $(t, R^{p-1}t^p)$, $1 \leq t \leq 2$, then there is the estimate $|\widehat{\nu}(\xi)| \lesssim |\xi|^{-1/2}$. It follows from Theorem 1 in [1] that

$$\int_1^2 \left| \widehat{\mu}\big(Rt, (Rt)^p\big) \right|^2 dt \lesssim R^{-\min(\alpha, 1/2)}.$$

This implies the conclusions of Theorem 1.2 in cases (ii) and (iii), exactly as in the preceding paragraph. ■

Proof of Theorem 1.3 We begin by observing that if the conclusion (1.6) of Theorem 1.2 holds for $\alpha \in (0, 2)$ with C depending only on the size of the support of the nonnegative measure μ and the implied constant in (1.1), then the same conclusion holds (with C replaced by $16C$) for complex measures whose total variation measure $|\mu|$ satisfies (1.1).

We consider first the case $\alpha \in (1, 2)$. Suppose R is large and positive. An argument like the one in the paragraph following (2.7) shows that the set

$$\{(t, t^p) : R \leq t \leq R + \sqrt{R}\}$$

is contained in a rectangle D with (approximate) dimensions $1 \times R^{p-1/2}$. Let ν be a unit vector in the direction of the long axis of D and c_D be the center of D. Also, denote the dual of D centered at the origin by D_{dual}. Note that D_{dual} is a rectangle

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with dimensions $1 \times R^{1/2-p}$ with short axis in the direction v. Fix a function $\psi \in C_c^\infty$ supported in D_{dual} such that $\widehat{\psi} \gtrsim R^{(p-1/2)(1-\alpha)}$ on D and $\|\psi\|_\infty \lesssim R^{(p-1/2)(2-\alpha)}$. Let $T \approx R^{(p-1/2)(\alpha-1)}$ be a natural number and define μ by

$$\mu(y) \doteq e^{2\pi i y \cdot c_D} \sum_{k=1}^{T} \psi(y - kT^{-1}v). \tag{3.1}$$

It is easy to check that $|\mu|$ satisfies (1.1) independently of R. Also note that

$$|\widehat{\mu}(x)| \gtrsim R^{(p-1/2)(1-\alpha)} \chi_D(x) \Big| \sum_{k=1}^{T} e^{-2\pi i \frac{k}{T} v \cdot (x - c_D)} \Big|.$$

Now if $jT \le v \cdot (x - c_D) \le jT + 1/4$ for any integer j, then we have

$$\Big| \sum_{k=1}^{T} e^{-2\pi i \frac{k}{T} v \cdot (x - c_D)} \Big| \gtrsim T.$$

Therefore there are $N \approx R^{p-1/2}/T \approx R^{(p-1/2)(2-\alpha)}$ subrectangles $P_1, \dots, P_N$ of D with dimensions $1 \times 1/4$ whose centers are in an arithmetic progression with distance T between the adjacent points such that

$$|\widehat{\mu}(x)| \gtrsim R^{(p-1/2)(1-\alpha)} T \sum_{k=1}^{N} \chi_{P_k}(x) \approx \sum_{k=1}^{N} \chi_{P_k}(x).$$

Using this, we obtain

$$\int_R^{R+\sqrt{R}} |\widehat{\mu}(t, t^p)|^2 t^\gamma \, dt \gtrsim R^\gamma \int_R^{R+\sqrt{R}} \sum_{k=1}^{N} \chi_{P_k}(t, t^p) \, dt$$
$$\gtrsim R^\gamma \frac{N}{R^{p-1}} \approx R^{\gamma - \alpha p + \alpha/2 + p}.$$

This implies that $\gamma \le \alpha p - \alpha/2 - p$, and so gives conclusion (i) of Theorem 1.3.

Conclusion (ii) of Theorem 1.3 also follows from the examples just constructed: since the support of μ above is contained in a ball of radius ≈ 1, if $|\mu|$ satisfies (1.1) for some $\alpha > 1$, then the same is certainly true for all $\alpha \in (0, 1]$. Taking $\alpha = 1 + \delta$ for arbitrary $\delta > 0$ gives $\gamma \le -1/2$.

To conclude, suppose $\alpha \in (0, 1/2)$ and $R > 0$ is large. Let D be a rectangle with dimensions $R \times R^p$ that contains $\{(t, t^p) : R \le t \le 2R\}$, and let v, C_D, and D_{dual} be as above. Note that now D_{dual} is a rectangle with dimensions $R^{-1} \times R^{-p}$ with short axis in the direction v. Fix a function $\psi \in C_c^\infty$ supported in D_{dual} and satisfying $\widehat{\psi} \gtrsim R^{-\alpha}$ on D and $\|\psi\|_\infty \lesssim R^{p+1-\alpha}$. Fix a natural number T with $T \approx R^\alpha$ and again define μ by (3.1). As before, $|\mu|$ satisfies (1.1) independently of R and there are

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$N \approx R^p/T \approx R^{p-\alpha}$ disjoint subrectangles $P_1, \dots, P_N$ of D of dimensions $1 \times 1/4$ such that

$$|\widehat{\mu}(x)| \gtrsim R^{-\alpha} T \sum_{k=1}^{N} \chi_{P_k}(x) \approx \sum_{k=1}^{N} \chi_{P_k}(x).$$

As above, that leads to

$$\int_R^{2R} |\widehat{\mu}(t, t^p)|^2 \, t^\gamma \, dt \gtrsim R^\gamma \int_R^{2R} \sum_{k=1}^{N} \chi_{P_k}(t, t^p) \, dt$$
$$\gtrsim R^\gamma \frac{N}{R^{p-1}} \approx R^{\gamma+p-\alpha-(p-1)}.$$

This gives conclusion (iii) of Theorem 1.3. ∎

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