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THE THICKNESS OF SCHUBERT CELLS AS INCIDENCE STRUCTURES

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Abstract

This paper explores the possible use of Schubert cells and Schubert varieties in finite geometry, particularly in regard to the question of whether these objects might be a source of understanding of ovoids or provide new examples. The main result provides a characterization of those Schubert cells for finite Chevalley groups which have the first property (*thinness*) of ovoids. More importantly, perhaps this short paper can help to bridge the modern language barrier between finite geometry and representation theory. For this purpose, this paper includes very brief surveys of the powerful lattice theory point of view from finite geometry and the powerful method of indexing points of flag varieties by Chevalley generators from representation theory.

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1. Introduction

This paper is the result of an effort to create 'interdisciplinary' communication and collaboration between the finite geometry and representation theory communities in Australia. The idea was that Chevalley groups could be a bridge between the two languages and the problems of interest to the two communities. Among others, the books of Taylor [Tay92] and Buekenhout and Cohen [BC13] are already existing, useful and important contributions to this dialogue. Although we have not used the language of buildings in this paper, the inspiring oeuvre of Tits [Tits74, Tits13a, Tits13b] is the pinnacle of the powerful connections between these different points of view. See, for example, [PR08] for a brief survey of how these points of view combine to give insight into the relationship between walks in buildings and representations of complex algebraic groups and groups over local fields.

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We chose to use the finite geometry question of finding ovoids as a framework for our investigation. The goal was to shape the language of algebraic groups and Chevalley groups to provide tools for studying the question. The work of Tits [Tits61] and Steinberg [St16, Example (c) before Theorem 34] on the Suzuki–Tits ovoid indicated that this was a fruitful research direction.

To describe further the results and methodology of this paper, let us review the definitions of ovoids (in finite geometry) and Schubert cells (in representation theory).

Ovoids. Let V be a vector space and let $\mathcal{P}(V)$ be the lattice of subspaces of V with inclusion $\subseteq$ as the partial order. A *point* is a one-dimensional subspace of V, a *line* is a two-dimensional subspace and a *hyperplane* is a codimension-one subspace of V. Let $\mathcal{O}$ be a set of points in $\mathcal{P}(V)$. A *tangent line to* $\mathcal{O}$ is a line in $\mathcal{P}(V)$ that contains exactly one point of $\mathcal{O}$. Then [Tits62, Section 1] defines an *ovoid of* $\mathcal{P}(V)$ as a set $\mathcal{O}$ of points of $\mathcal{P}(V)$ such that the following conditions hold.

(O1) If ℓ is a line in $\mathcal{P}(V)$, then ℓ contains 0, 1 or 2 points of $\mathcal{O}$ (thinness).
(O2) If $p \in \mathcal{O}$, then the union of the tangent lines to $\mathcal{O}$ through p is a hyperplane (maximality).

'Thinness' and 'maximality' characterize the definitions of ovoids, ovals and hyperovals lying inside projective spaces, projective planes, polar spaces and generalized quadrangles that can be found in the finite geometry literature (see, for example, [Br00, Section 1] and [BW11, Sections 2.1, 4.2 and 4.4]).

Schubert cells. Let $G(\mathbb{F})$ be a Chevalley group over $\mathbb{F}$ and let B be a Borel subgroup. The quotient $G(\mathbb{F})/B$ is the (generalized) flag variety. If $G(\mathbb{F}) = \mathsf{GL}_n(\mathbb{F})$, then $G(\mathbb{F})/B$ is the set of maximal chains $0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V$ in $\mathcal{P}(V)$, where V is an $\mathbb{F}$-vector space of dimension n. The flag varieties are studied with the use of the Bruhat decomposition,

$$G(\mathbb{F}) = \bigsqcup_{w \in W} BwB,$$

and the *Schubert cells* are

$$X_w = BwB,$$

viewed as subsets of the set of cosets $G(\mathbb{F})/B$. In the case of $\mathsf{GL}_n(\mathbb{F})/B$ the X_w are collections of maximal chains in $\mathcal{P}(V)$ and thus, when $\mathbb{F} = \mathbb{F}_q$ is a finite field, the X_w are natural objects in finite geometry. From the point of view of representation theory, the closures of the Schubert cells are the Schubert varieties of the projective variety $G(\overline{\mathbb{F}})/B$, and this makes them tools in the framework of geometric representation theory.

In pursuit of the question of what causes the 'thinness' that distinguishes ovoids we prove the following result (Theorem 1.1), which is a computation of the 'thickness' of the incidence structures that come from Schubert cells.

THEOREM 1.1 (Main theorem). *Let $G(\mathbb{F})$ be a Chevalley group with Weyl group W. Let P_i and P_j be standard maximal parabolic subgroups of $G(\mathbb{F})$ and let $w \in W$. Let $(X_w)_{ij}$*

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be the incidence structure associated to the Schubert cell X_w and let gP_j be a line in $(X_w)_{ij}$. Then the number of points in $(X_w)_{ij}$ incident to gP_j is

$$q^{\ell(z)},$$

where $w = uzv$ with $u \in W^j$, $zv \in W_j$, $z \in (W_j)^{i,j}$ and $v \in W_{i,j}$.

The objects in Theorem 1.1 will be defined in forthcoming sections, and, in particular, the incidence structure $(X_w)_{ij}$ will be introduced in Section 4. (It suffices to say here that its 'points' are certain left cosets gP_i, its 'lines' are certain left cosets of hP_j, and a point and line are incident if the ratio of their canonical coset representatives lies in the Borel subgroup of $G(\mathbb{F})$.) As an application of this theorem we determine the Schubert cell incidence structures coming from finite Chevalley groups which have the thinness property; see Corollary 4.4.

In this paper we first review the background finite geometry of incidence structures and projective geometries and the notation and framework for working with Chevalley groups and generalized flag varieties (Sections 2 and 3). In Section 4, we define an incidence structure for each Schubert cell and pair of maximal parabolic subgroups of the Chevalley group. This provides a way of analyzing the Schubert cell from the viewpoint of finite projective geometry. The main theorem (Theorem 1.1) is a consequence of Proposition 4.2.

2. Lattices and incidence structures

In this section we review the equivalence between subspace lattices of a vector space, projective lattices and projective incidence structures. An inspiring modern textbook is [Shu11]. A classic reference to lattice theory is [Birk48]. The definition of a modular lattice is given in [Birk48, Ch. V, Section 1]. The equivalence between projective incidence structures, complemented modular lattices and the subspace lattice of a vector space over a division ring, which is stated as Theorem 2.1 below, is proved (even in the infinite-dimensional case) in [Birk48, Ch. VIII, Theorem 15]. A classic reference to finite geometries is [Dem68], and the definition of an incidence structure is given in [Dem68, Section 1.1]. The definition of a projective incidence structure (often called a projective geometry) is found in [Birk48, Ch. VIII, Section 3], [Cam00, Section 3.3] and [Tay92, page 16].

2.1. The subspace lattice $\mathcal{P}(V)$ of a vector space V. Let $\mathbb{F}$ be a field or division ring and let V be a finite-dimensional vector space over $\mathbb{F}$. The *subspace lattice* $\mathcal{P}(V)$ of V is the set of subspaces of V with partial order given by subspace inclusion. More generally, one could consider a ring R, a (left) R-module M and the lattice of (left) R-submodules of M. At this level of generality, the situation is substantially more involved and complicated than that of a subspace lattice of a vector space (see [Vel95]). In the finite geometry literature, a (Desarguesian) *projective space* is $\mathsf{PG}(n, q) = \mathcal{P}(\mathbb{F}_q^{n+1})$, where $\mathbb{F}_q$ is the finite field with q elements. In the algebraic

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geometry literature (see [Har77, page 8]), *projective space* is the quotient

$$\mathbb{P}^n = \frac{\mathbb{F}^{n+1} - \{(0,\dots,0)\}}{\langle (a_0,\dots,a_n) = (ca_0,\dots,ca_n) \mid c \in \mathbb{F}^\times \rangle}.$$

These terminologies are conflicting and should, therefore, be used with care in the context of this paper.

2.2. Lattices. A *lattice* is a partially ordered set $\mathcal{P}$ that is closed under the operations of meet and join defined by $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$, for all $x, y \in cP$. A *modular lattice* is a lattice $\mathcal{L}$ such that, for all $x, y, z \in \mathcal{L}$ such that $x \leqslant z$, we have

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

Let $\mathcal{L}$ be a finite lattice with a unique minimal element 0 and a unique maximal element 1.

- An *atom* is $a \in \mathcal{L}$ such that there does *not* exist $a' \in \mathcal{L}$ with $0 < a' < a$.
- An *atomic lattice* is a lattice $\mathcal{L}$ such that every element is a join of atoms.
- A *maximal chain* is a maximal-length sequence $0 < a_1 < a_2 < \cdots < a_\ell < 1$ in $\mathcal{L}$.
- A lattice $\mathcal{L}$ is *ranked* if all maximal chains in $\mathcal{L}$ have the same length.

Let $\mathcal{L}$ be a ranked lattice and let $a \in \mathcal{L}$. The *rank* of a, written $\mathrm{rank}(a)$, is the integer i for which there exists a maximal chain

$$0 < a_1 < a_2 < \cdots < a_\ell < 1$$

with $a_i = a$. A *projective lattice* is an atomic ranked modular lattice such that, for all $x, y \in \mathcal{L}$, we have the *Grassmann identity*:

$$\mathrm{rank}(x \vee y) + \mathrm{rank}(x \wedge y) = \mathrm{rank}(x) + \mathrm{rank}(y).$$

Two lattices $\mathcal{L}$ and $\mathcal{L}'$ are *isomorphic* if there is an order-preserving bijection from $\mathcal{L}$ to $\mathcal{L}'$. The following theorem provides an equivalence between projective lattices and subspace lattices of a vector space over a division ring.

Theorem 2.1 (see [HP47, Ch. V and VI]).

(a) *Let V be a finite-dimensional vector space over a division ring. Then $\mathcal{P}(V)$ is a projective lattice.*

(b) *If $\mathcal{L}$ is a projective lattice, then there exist a division ring $\mathbb{F}$ and $n \in \mathbb{Z}_{>0}$ such that $\mathcal{L} \cong \mathcal{P}(\mathbb{F}^n)$.*

2.3. Incidence structures. An *incidence structure* is a triple (P, L, I) where P and L are sets and $I \subseteq P \times L$. Let $\mathsf{pr}_1 : P \times L \to P$ and $\mathsf{pr}_2 : P \times L \to L$ be the projections onto the first and second factors. We have the following interface between geometric language and its algebraic formalism.

- A point $p \in P$ is *contained in a line* $\ell \in L$ if $(p, \ell) \in I$.

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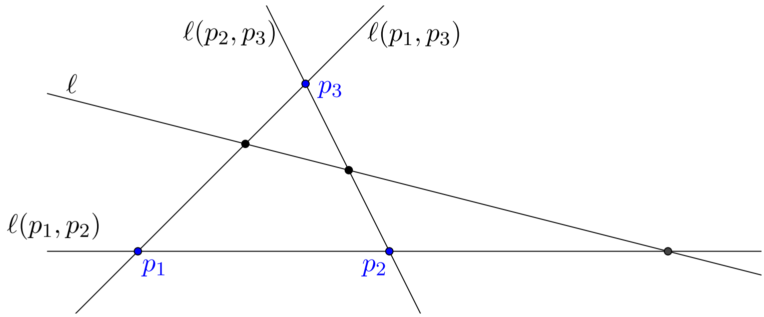
- A subset $S \subseteq P$ is *collinear* if there exists $\ell \in L$ such each element p of S is contained in ℓ.

Often it is convenient to identify $\ell \in L$ with the set of points $\mathrm{pr}_1(\mathrm{pr}_2^{-1}(\ell))$; the points *contained* in the line ℓ. A *projective incidence structure* is an incidence structure $I \subseteq P \times L$ such that the following statements hold.

(a) If $p_1, p_2 \in P$ and $p_1 \neq p_2$, then there exists a unique line $\ell(p_1, p_2) \in L$ containing p_1 and p_2 (any two points lie on a unique line).

(b) If $p_1, p_2, p_3 \in P$ are not collinear and ℓ is a line intersecting $\ell(p_1, p_3)$ and $\ell(p_2, p_3)$, then ℓ also intersects $\ell(p_1, p_2)$ (Veblen–Young axiom).



(c) Any line contains at least three points (thickness condition).

(d) There exist three noncollinear points in P (dimension $\geqslant 2$ condition).

(e) Any increasing sequence of subspaces has finite length (finite dimensionality condition).

Assume that $I \subseteq P \times L$ is an incidence structure such that any two points lie on a unique line. A *subspace* is a set $S \subseteq P$ such that S contains any line connecting two of its points, that is, if $p_1, p_2 \in S$, then $\mathrm{pr}_1(\mathrm{pr}_2^{-1}(\ell(p_1, p_2))) \subseteq S$. The *subspace lattice* $\mathcal{P}(I)$ of $I \subseteq P \times L$ is the set of subspaces $S \subseteq P$ partially ordered by inclusion.

The following 'Veblen–Young theorem' provides an equivalence between projective incidence structures and projective lattices.

Theorem 2.2 (see [HP47, Ch. V and VI]).

(a) *If $\mathcal{G}$ is a projective incidence structure, then $\mathcal{P}(\mathcal{G})$ is a projective lattice.*

(b) *Let $\mathcal{P}$ be a ranked lattice. Let*

$$\mathcal{P}_1 = \{p \in \mathcal{P} \mid \mathrm{rank}(p) = 1\},$$
$$\mathcal{P}_2 = \{\ell \in \mathcal{P} \mid \mathrm{rank}(\ell) = 2\},$$

and let I be the incidence relation inherited from $\mathcal{P}$; so $(p, \ell) \in \mathcal{P}_1 \times \mathcal{P}_2$ lies in I if and only if $p \leqslant \ell$ in $\mathcal{P}(\mathcal{G})$. If $\mathcal{P}$ is a projective lattice, then $(\mathcal{P}_1, \mathcal{P}_2, I)$ is a projective incidence structure.

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3. Flag varieties and Chevalley groups

In this section we review the formalism and establish our notation for working with (generalized) flag varieties. A classic reference to Chevalley groups and flag varieties is [St16]. Good supportive references are [Sesh14, Section 2.1] and [FH91, Section 23.3]. The first step in our review is to identify the flag variety as the set of maximal chains in the subspace lattice $\mathcal{P}(V)$.

3.1. Flag varieties and $\mathsf{GL}_n(\mathbb{F})$. Let $\mathbb{F}$ be a field (or division ring) and let V be a finite-dimensional $\mathbb{F}$-vector space. The *flag variety* $\mathcal{F}(V)$ is the set of maximal chains in $\mathcal{P}(V)$. By choosing a basis $\{e_1, \dots, e_n\}$ in V, the *standard flag*

$$F_0 = (0 \subseteq \mathrm{span}\{e_1\} \subseteq \mathrm{span}\{e_1, e_2\} \subseteq \cdots \subseteq \mathrm{span}\{e_1, \dots, e_n\} = V)$$

has stabilizer the *Borel subgroup* B consisting of all upper triangular matrices of $\mathsf{GL}_n(\mathbb{F})$. We then obtain a bijection, and an equivalence of group actions (of $\mathsf{GL}_n(\mathbb{F})$ on $\mathsf{GL}_n(\mathbb{F})/B$ and on $\mathcal{F}(V)$):

$$\begin{aligned} \mathsf{GL}_n(\mathbb{F})/B &\longrightarrow \mathcal{F}(V) \\ gB &\longmapsto gF_0. \end{aligned}$$

A *parabolic subgroup* of $\mathsf{GL}_n(\mathbb{F})$ is the stabilizer of a subspace $W \subseteq V$, and the *standard maximal parabolic subgroups* are

$$P_i = \mathrm{Stab}(\mathrm{span}\{e_1, e_2, \dots, e_i\}),$$

for $i \in \{1, 2, \dots, n\}$.

Let E_{ij} denote the $n \times n$ matrix with 1 in the (i, j)th position and 0 in all other positions. Let $\mathfrak{h}^* = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ be the free $\mathbb{Z}$-module with basis $\varepsilon_1, \dots, \varepsilon_n$ and let

$$R = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\}.$$

The group $\mathsf{GL}_n(\mathbb{F})$ is generated by the *elementary matrices*

$$x_{\varepsilon_i-\varepsilon_j}(c) = I + cE_{ij}, \quad s_{\varepsilon_i-\varepsilon_j} = I + E_{ij} + E_{ji} - E_{ii} - E_{jj}, \quad h_{\lambda^\vee}(d) = \mathrm{diag}(d^{\lambda_1}, \dots, d^{\lambda_n}),$$

for $\varepsilon_i - \varepsilon_j \in R$ and $c \in \mathbb{F}$, and for $\lambda^\vee = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ and $d \in \mathbb{F}^\times$. The *root subgroups* are

$$\mathcal{X}_{\varepsilon_i-\varepsilon_j} = \{x_{\varepsilon_i-\varepsilon_j}(c) \mid c \in \mathbb{F}\}$$

and the set of *positive roots* is

$$R^+ = \{\alpha \in R \mid \mathcal{X}_\alpha \subseteq B\}.$$

The *simple roots* $\alpha_1, \dots, \alpha_{n-1}$ are given by

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1},$$

and, setting $s_i = s_{\alpha_i}$, the *Weyl group* is

$$W = \langle s_1, \dots, s_n \mid s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\rangle$$

(which is the symmetric group S_n here). The *Bruhat decomposition* (see [St16, Example (a) after Theorem 4′], [FH91, Theorem 23.59] or [Sesh14, Section 4.2.4]) is

$$\mathsf{GL}_n(\mathbb{F}) = \bigsqcup_{w \in W} BwB.$$

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3.2. Chevalley groups and generalized flag varieties $G(\mathbb{F})/B$. In the same way that $\mathsf{GL}_n(\mathbb{F})$ is generated by elementary matrices, a Chevalley group $G(\mathbb{F})$ is generated by Chevalley generators $x_\alpha(c)$, $h_{\lambda^\vee}(d)$, for $\alpha, \lambda \in R$, $c \in \mathbb{F}$, $d \in \mathbb{F}^\times$, which satisfy specified relations [St16, Relations (R), Ch. 3, page 23]. The set R of *roots* is a labelling set for the *root subgroups*

$$\mathcal{X}_\alpha = \{x_\alpha(c) \mid c \in \mathbb{F}\} \quad \text{for } \alpha \in R.$$

The set R is endowed with a chosen decomposition into positive and negative roots

$$R = R^+ \sqcup (-R^+) \quad \text{where } -R^+ = \{-\alpha \mid \alpha \in R^+\}.$$

Defining

$$U = \langle \mathcal{X}_\alpha \mid \alpha \in R^+ \rangle, \quad T = \langle h_{\lambda^\vee}(d) \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}},\ d \in \mathbb{F}^\times \}, \quad \text{and} \quad B = UT,$$

we call $G(\mathbb{F})/B$ the *generalized flag variety*. The *simple roots* $\alpha_1, \ldots, \alpha_n$ provide a minimal set of root subgroup generators for

$$U = \langle \mathcal{X}_{\alpha_1}, \ldots, \mathcal{X}_{\alpha_n} \rangle.$$

The *standard maximal parabolic subgroups* are

$$P_i = \langle \mathcal{X}_{-\alpha_1}, \ldots, \mathcal{X}_{-\alpha_{i-1}}, \mathcal{X}_{-\alpha_{i+1}}, \ldots, \mathcal{X}_{-\alpha_n}, B \rangle \quad \text{for } i \in \{1, \ldots, n\}.$$

3.2.1. Labelling the points of the flag variety. Letting $N = \langle n_\alpha \mid \alpha \in R \rangle$, the *Weyl group* is $W = N/T$. For $i \in \{1, \ldots, n\}$, define

$$x_i(c) = x_{\alpha_i}(c), \quad n_i = x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1) \quad \text{and} \quad s_i = n_i T.$$

The Weyl group W has a Coxeter presentation with generators $s_1, \ldots, s_n$ and relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$, where m_{ij} is the order of $s_i s_j$ in W. A *reduced decomposition* for an element $w \in W$ is an expression $w = s_{i_1} \cdots s_{i_\ell}$ with ℓ minimal. The following proposition provides an explicit indexing of the points of the flag variety.

Proposition 3.1 ([St16, Theorems 4′, 15 and Lemma 43(a)]; see also [PRS, (7.3)]). *For each $w \in W$, fix a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$. Then*

$$G(\mathbb{F})/B = \bigsqcup_{w \in W} BwB \quad \text{with } BwB = \{x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B \mid c_1, \ldots, c_\ell \in \mathbb{F}\},$$

and $\{x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1} \mid c_1, \ldots, c_\ell \in \mathbb{F}\}$ is a complete set of representatives of the cosets of B in BwB.

4. Thickness in Schubert cells

Keeping the notation of Section 3.2, let $G(\mathbb{F})$ be a Chevalley group and let P_i and P_j be standard maximal parabolic subgroups of $G(\mathbb{F})$. Let $w \in W$. Define maps p_i^w and p_j^w as follows:

$$p_i^w\colon \begin{array}{ccc} BwB & \to & G/P_i \\ gB & \mapsto & gP_i \end{array} \quad \text{and} \quad p_j^w\colon \begin{array}{ccc} BwB & \to & G/P_j \\ gB & \mapsto & gP_j \end{array}.$$

Let $(X_w)_{ij}$ be the following incidence structure:

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(a) a *point* in $(X_w)_{ij}$ is an element gP_i of the image of p_i^w;
(b) a *line* in $(X_w)_{ij}$ is an element hP_j of the image of p_j^w; and
(c) a point gP_i is *incident* to a line hP_j if there exists $kB \in BwB$ such that $p_i^w(kB) = gP_i$ and $p_j^w(kB) = hP_j$.

Alternatively, it is not difficult to see that the incidence relation above can be simplified by stipulating that $gh^{-1} \in B$ instead.

Let

$$R_i^+ = \{\alpha \in R^+ \mid \mathcal{X}_{-\alpha} \in P_i\}, \quad R_j^+ = \{\alpha \in R^+ \mid \mathcal{X}_{-\alpha} \in P_j\}, \quad R_{\{i,j\}}^+ = R_i^+ \cap R_j^+,$$

and let

$$W_i = \langle s_\alpha \mid \alpha \in R_i^+\rangle, \quad W_j = \langle s_\alpha \mid \alpha \in R_j^+\rangle, \quad W_{\{i,j\}} = W_i \cap W_j.$$

For $z \in W$ the *inversion set of z* is

$$R(z) := \{\alpha \in R^+ \mid \mathcal{X}_{z\alpha} \notin B\},$$

and $\ell(z) := \mathrm{Card}(R(z))$ is the length of a reduced decomposition of z (in this definition $\mathcal{X}_{z\alpha} = z\mathcal{X}_\alpha z^{-1}$). Let W^j be the set of minimal-length coset representatives of W_j in W, and let $W_j^{\{i,j\}}$ be the set of minimal-length coset representatives of $W_i \cap W_j$ in W_j. So

$$W^j = \{z \in W \mid R(z) \cap R_j^+ = \varnothing\},$$
$$(W_j)^{\{i,j\}} = \{z \in W \mid R(z) \subseteq R_j^+ \text{ and } R(z) \cap R_{\{i,j\}} = \varnothing\}.$$

The following proposition is a slight generalization of Proposition 3.1.

Proposition 4.1. *For each $u \in W^j$, fix a reduced decomposition $u = s_{i_1} \cdots s_{i_k}$. Then*

$$G/P_j = \bigsqcup_{u \in W^j} BuP_j \quad \text{with } BuP_j = \{x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_k}(c_k)n_{i_k}^{-1}P_j \mid c_1, \ldots, c_k \in \mathbb{F}\},$$

and $\{x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_k}(c_k)n_{i_k}^{-1} \mid c_1, \ldots, c_k \in \mathbb{F}\}$ is a set of representatives of the cosets of P_j in BuP_j.

Proof. If $w \in W$, then there are unique $u \in W^j$ and $y \in W_j$ such that $w = uy$ (see [Bou02, Ch. 4, Section 1, Exercise 3]). If $u = s_{i_1} \cdots s_{i_k}$ and $y = s_{i_{k+1}} \cdots s_{i_\ell}$ are reduced decompositions, then $w = s_{i_1} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_\ell}$ is reduced. If $gB = x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B \in BwB$, then $gP_i = x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_k}(c_k)n_{i_k}^{-1}P_j$ since every factor of the product $x_{i_{k+1}}(c_{k+1})n_{i_{k+1}}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}$ is an element of P_j. □

Let $p_i\colon G/B \to G/P_i$ and $p_j\colon G/B \to G/P_j$ be the natural projection maps (e.g., $p_i(gB) = gP_i$ for all $g \in G$). Each $y \in W_j$ has a unique expression $y = zv$ with $z \in (W_j)^{\{i,j\}}$ and $v \in W_{\{i,j\}}$. For each $y \in W_j$, fix a reduced decomposition

$$y = s_{k_1} \cdots s_{k_r} s_{\ell_1} \cdots s_{\ell_t}, \quad \text{with } z = s_{k_1} \cdots s_{k_r} \in (W_j)^{\{i,j\}} \text{ and } v = s_{\ell_1} \cdots s_{\ell_t} (W_j)_{\{i,j\}}.$$

With

$$U_y = \{x_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1} x_{\ell_1}(e_1)n_{\ell_1}^{-1} \cdots x_{\ell_t}(e_t)n_{\ell_t}^{-1} \mid d_1, \ldots, d_r, e_1, \ldots, e_t \in \mathbb{F}\}, \tag{4.1}$$

we have

$$P_j = \bigsqcup_{y \in W_j} U_y B \quad \text{and} \quad p_j^{-1}(gP_j) = \bigsqcup_{y \in W_j} gU_y B. \tag{4.2}$$

With this notation in hand, we can now state the following proposition that determines the structure of each $p_i(p_j^{-1}(gP_j))$.

Proposition 4.2. *Let $gP_j \in G/P_j$. With notation as above, the map Φ from $p_i(p_j^{-1}(gP_j))$ to $\bigsqcup_{y \in W_j} \mathbb{F}^{\ell(z)}$ defined by*

$$\Phi(gx_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}P_i) := (d_1, \ldots, d_r)$$

is a bijection.

Proof. By (4.2), the set $p_j^{-1}(gP_j)$ is a disjoint union of the sets gU_y for $y \in W^j$. By (4.1), an element of gU_yB is of the form $gx_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}x_{\ell_1}(e_1)n_{\ell_1}^{-1} \cdots x_{\ell_t}(e_t)n_{\ell_t}^{-1}B$ and then

$$\begin{aligned} &p_i(gx_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}x_{\ell_1}(e_1)n_{\ell_1}^{-1} \cdots x_{\ell_t}(e_t)n_{\ell_t}^{-1}B) \\ &\quad = gx_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}x_{\ell_1}(e_1)n_{\ell_1}^{-1} \cdots x_{\ell_t}(e_t)n_{\ell_t}^{-1}P_i \\ &\quad = gx_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}P_i. \end{aligned}$$

Thus each element of $p_i(p_j^{-1}(gP_j))$ can be written in the form

$$gx_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}P_i.$$

Now let $z_1, z_2 \in (W_j)^{\{i,j\}}$ with chosen reduced decompositions

$$z_1 = s_{k_1} \cdots s_{k_r} \quad \text{and} \quad z_2 = s_{k'_1} \cdots s_{k'_m}.$$

Assume

$$gx_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}P_i = gx_{k'_1}(d'_1)n_{k'_1}^{-1} \cdots x_{k'_m}(d'_m)n_{k'_m}^{-1}P_i.$$

Then

$$x_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}P_i = x_{k'_1}(d'_1)n_{k'_1}^{-1} \cdots x_{k'_m}(d'_m)n_{k'_m}^{-1}P_i.$$

Since $z_1 \in (W_j)^{\{i,j\}}$ and $R^+_{\{i,j\}} \subseteq R^+_i$, we have $R(z) \cap R^+_i \subseteq R(z) \cap R^+_{\{i,j\}} = \varnothing$, yielding $z_1 \in W^i$. Similarly, $z_2 \in W^i$. Since $x_{k_1}(d_1)n_{k_1}^{-1} \cdots x_{k_r}(d_r)n_{k_r}^{-1}P_i = x_{k'_1}(d'_1)n_{k'_1}^{-1} \cdots x_{k'_m}(d'_m)$ $n_{k'_m}^{-1}P_i$, we have $z_1W_i = z_2W_i$. Since z_1 and z_2 are minimal-length coset representatives of the same coset in W/W_i, and since such coset representatives are unique (see [Bou02, Ch. 4, Section 1, Exercise 3]), we find that $z_1 = z_2$.

Since the reduced decompositions of elements of $(W_j)^{\{i,j\}}$ were fixed,

$$(k_1, \ldots, k_r) = (k'_1, \ldots, k'_m).$$

By Proposition 4.1, since $x_{k_1}(d_1)n_{k_1}^{-1}\cdots x_{k_r}(d_r)n_{k_r}^{-1}P_i = x_{k_1}(d_1')n_{k_1}^{-1}\cdots x_{k_r}(d_r')n_{k_r}^{-1}P_i$, we have

$$(d_1,\ldots,d_r) = (d_1',\ldots,d_r').$$

Thus each element of $p_i(p_j^{-1}(gP_j))$ has a *unique* expression of the form $gx_{k_1}(d_1)n_{k_1}^{-1}\cdots x_{k_r}(d_r)n_{j_k}^{-1}P_i$. □

Proof of Theorem 1.1. Let $w \in W$ and let gP_j be in the image of $p_j^w : BwB \to G/P_j$. The decomposition $w = uy = uzv$ is unique (see [Bou02, Ch. 4, Section 1, Exercise 3]). Thus z is determined. Hence by Proposition 4.2, the set

$$p_i^w(p_j^w)^{-1}(gP_j) = p_i^w(X_w) \cap p_i p_j^{-1}(gP_j)$$

has $q^{\ell(z)}$ elements. □

Example 4.3. Take $G = G(\mathbb{F}) = \mathsf{GL}_4(\mathbb{F})$ and the notation given in Section 3.1. Let $i = 1$ and $j = 2$. Then

$$W = S_4,\quad W_1 = S_1 \times S_3,\quad W_2 = S_2 \times S_2,\quad W_{1,2} = S_1 \times S_1 \times S_2$$

and

$$W^1 = \{1, s_1, s_2s_1, s_3s_2s_1\},\quad W^2 = \{1, s_2, s_1s_2, s_3s_2, s_1s_3s_2, s_2s_1s_3s_2\}$$

and

$$(W_2)^{1,2} = \{1, s_1\}.$$

Let $w = uzy = (s_1s_3s_2)(s_1)(s_3)$. Consider the incidence structure $(X_w)_{12}$ and

$$g = x_1(c_1)n_1^{-1}x_3(c_2)n_3^{-1}x_2(c_3)n_2^{-1}.$$

Then

$$\begin{aligned} p_1(p_2^{-1}(gP_2)) &= p_1(p_2^{-1}(x_1(c_1)n_1^{-1}x_3(c_2)n_3^{-1}x_2(c_3)n_2^{-1}P_2)) \\ &= \{x_1(c_1)n_1^{-1}x_3(c_2)n_3^{-1}x_2(c_3)n_2^{-1}x_1(d_1)n_1^{-1}P_1 \mid d_1 \in \mathbb{F}\}. \end{aligned}$$

This illustrates that $p_1(p_2^{-1}(gP_2)) \cong \mathbb{F}$ even though the elements of $p_1(p_2^{-1}(gP_2))$ as displayed are not the 'favourite' coset representatives of the cosets in G/P_1 given by Proposition 4.1. This provides a conceptual explanation of why Proposition 4.2 (and Theorem 1.1) are nontrivial. One needs to find the right coordinatization to succeed in displaying $p_1(p_2^{-1}(gP_2))$ naturally as an affine space.

Recall from the introduction that the first of the defining conditions for an ovoid $\mathcal{O}$ in $\mathcal{P}(V)$ is 'thinness' (O1): any ℓ of $\mathcal{P}(V)$ contains at most two points of $\mathcal{O}$. Using Theorem 1.1 to determine the Schubert incidence structures that are 'thin' produces the following result.

Corollary 4.4. *Let $G(\mathbb{F}_q)$ be a Chevalley group over a finite field $\mathbb{F}_q$. Then the Schubert incidence structures $(X_w)_{ij}$ such that there are at most two points incident with each line correspond to triples (w, i, j) such that*

$$\begin{cases} w \in W^jW_{i,j} & \text{if } q > 2, \\ w \in W^jW_{i,j} \cup W^js_iW_{i,j} & \text{if } q = 2. \end{cases}$$

Proof. Assume $w = uzy$ with $u \in W^j$, $z \in (W_j)^{i,j}$, $y \in W_{i,j}$. Then $\ell(z) = 0$ only when $z = 1$ and $\ell(z) = 1$, and this occurs only when $z = s_i$. □

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