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RELATIVIZED WEAK MIXING OF UNCOUNTABLE ORDER

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Introduction. We show that if Y is a metric minimal flow and $\theta: Y \to Z$ in an open homomorphism that has a section (i.e., a RIM), and if $S(\theta) = R(\theta)$, then ${}^{\circ}Y^{\Omega}$ contains a dense set of transitive points, where Ω is the first uncountable ordinal

$$Y^{\Omega} = \prod\{Y: 1 \leqq \alpha < \Omega \text{ and } \alpha \text{ not a limit ordinal}\}, \text{ and}$$

$${}^{\circ}Y^{\Omega} = \{y \in Y^{\Omega}: \theta(y_\alpha) = \theta(y_\beta) \text{ for } 1 \leqq \alpha, \beta < \Omega \text{ and } \alpha, \beta \text{ not limit ordinals}\},$$

$S(\theta)$ is the relativized equicontinuous structure relation, and

$$R(\theta) = \{(y_1, y_2) \in Y \times Y: \theta(y_1) = \theta(y_2)\}.$$

We use this to generalize a result of Glasner that a metric minimal flow whose enveloping semigroup contains finitely many minimal ideals is PI, [**5**].

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We use the methods developed in [**6**] to prove the above. We will use some definitions and notation from [**6**], [**3**], [**4**] and now introduce some further definitions and notations.

Preliminaries. Let (X, T) be a flow (transformation group) with compact Hausdorff phase space. We will write X for both the flow (X, T) and the phase space. If X is point-transitive, let X_m denote the set of transitive points in X; when X is metric, X_m is a dense G_δ set. $\phi: X \to Y$ will denote a homomorphism of X onto Y. For a homomorphism ϕ of X onto Y,

$$R_m(\phi) = \{(x, x') \in X_m \times X_m: \phi(x) = \phi(x')\},$$
$$Q_m(\phi) = \{(x, x') \in R_m(\phi): \text{there exist nets } t_n \text{ in } T \text{ and } (x_n, x_n') \text{ in } R_m(\phi) \text{ such that } (x_n, x'_n) \to (x, x') \text{ and } (x_n, x_n')t_n \to (x_0, x_0')\} \text{ for any } x_0 \text{ in } x_m.$$

$S_m(\phi)$ is the smallest closed (in $R_m(\phi)$) invariant equivalence relation containing $Q_m(\phi)$. When X is minimal, the subscript $_m$ is omitted.

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We will denote the enveloping semigroup of X by $E(X)$ and the set of idempotents in a minimal right ideal I of $E(X)$ by $J(I)$.

Suppose Y is minimal, $\theta: Y \to Z$, and λ is any ordinal. Let

$$Y^\lambda = \prod\{Y: 1 \leqq \alpha \leqq \lambda \text{ and } \alpha \text{ is not a limit ordinal})\}$$

and let

$$^\circ Y^\lambda = \{y \in Y^\lambda: \theta(y_\alpha) = \theta(y_\beta) \text{ for } 1 \leqq \alpha, \beta < \lambda \text{ and } \alpha, \beta \text{ not limit ordinals}\}$$

where y_α is the α-coordinate. Define $\theta^\lambda: {}^\circ Y^\lambda \to Z$ by $\theta^\lambda(y) = \theta(y_1)$.

Let $M(X)$ be the set of Borel probability measures on X. For μ in $M(X)$ define μt by $\mu t(A) = \mu(At^{-1})$ for every measureable set A.

A *section* λ for $\theta: X \to Y$ is a homomorphism $\lambda: Y \to M(X)$ such that $\hat{\phi}(\lambda_y) = \delta_y$ where $\hat{\phi}(\lambda_y)(A) = \lambda_y\phi^{-1}(A)$ for every Borel subset A of Y and δ_y is the point mass at y. (See [**4**] or [**6**].)

A homomorphism ϕ form Y onto Z is *stongly proximal* if for every measure $\mu \in M(X)$ with $\hat{\phi}(\mu) = \delta_y$ for some y in Y, there exists a net t_n in T such that $\lim \mu t_i = \delta_x$ for some x in X.

We say that a minimal flow is *strictly* SPI if there is an ordinal λ and flows X_α, $\alpha \leqq \lambda$ such that

(i) X_0 is the trivial flow

(ii) for every $\alpha < \lambda$ there exist a homomorphism $\phi_\alpha: X_{\alpha+1} \to X_\alpha$ which is strongly proximal or almost periodic,

(iii) for a limit ordinal $\alpha \leqq \lambda$, $X_\alpha = \text{inv lim}\,\{X_\beta: \beta < \alpha\}$

(iv) $X_\lambda = X$.

A minimal flow is an SPI flow if there exist a strictly SPI flow X' and a proximal homomorphism $\phi: X' \to X$.

1. Proposition. *Let X and Y be minimal flows. If $\theta: X \to Y$ is proximal, then the set of minimal ideals in $E(X)$ and $E(Y)$ have the same cardinality.*

Proof. Let ϕ be the induced semigroup homomorphism from $E(X)$ onto $E(Y)$. We need to show that ϕ is one-to-one. Suppose $I_1 \neq I_2$ are minimal ideals in $E(X)$ with $I = \phi(I_1) = \phi(I_2)$. Let $u \in J_1$ and $u^* \in J_2$ such that $uu^* = u$ where J_1 and J_2 are the idempotents in I_1 and I_2 respectively. Then

$$\phi(u) = \phi(uu^*) = \phi(u)\phi(u^*) = \phi(u^*) \in J,$$

the set of idempotents in $I \subseteq E(Y)$. So for every x in X,

$$\theta(xu) = \theta(x)\theta(u) = \theta(x)\theta(u^*) = \theta(xu^*),$$

so xu and xu^* are proximal and therefore $xu = xu^*$ since $xuu^* = xu$. But $xu = xu^*$ implies $u = u^*$ and

$$I_1 = uE(X) = u^*E(X) = I_2,$$

a contradiction.

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2. PROPOSITION. *Let λ be an ordinal. Suppose $\theta: Y \to Z$ is not a proximal extension and $y \in {}^\circ Y^\lambda$ is a point with dense orbit in ${}^\circ Y^\lambda$. Then* (a) *$E(Y)$ has at least $2^{\lambda-1}$ minimal right ideals and* (b) *there exists a minimal right ideal I such that*

$$y_1 J(I) \supseteq \{y_\alpha: y_\alpha \text{ is the } \alpha\text{-coordinate of } y\}.$$

Proof. (a) Fix a minimal right ideal $I \subseteq E(Y)$. The set

$$B = \{y_\alpha: y_\alpha \text{ is the } \alpha\text{-coordinate of } y\}$$

is contained in $Y_0 = \theta^{-1}(\theta^\lambda(y)) \subseteq Y$. Since θ is not proximal there exist y', y^* in Y_0 such that $y' \neq y^*$ and $y'u = y'$, $y^*u = y^*$ for some u in $J(I)$. Let

$$F = \{\alpha: 2 \leqq \alpha \leqq \lambda, \alpha \text{ not a limit ordinal}\}$$

and let L be a non-empty proper subset of F. Since y is a transitive point, there is an element q in $E(Y)$ such that

$$y_\alpha q = y' \text{ if } \alpha \in L,$$
$$y_\alpha q = y^* \text{ if } \alpha \in F \backslash L, \text{ and}$$
$$y_1 q = y^*.$$

Let $I_L = qI$. We will now show that $L \to I_L$ is a one-to-one map and thus (a) follows. Indeed if $I_{L_1} = I_{L_2}$ and $f \in L_2 \backslash L_1$ (say) and q_1, q_2 are the associated q's, then $q_1 I = q_2 I$; so $q_2 u = q_2 p$ for some p in I and therefore

$$y^* = y^*u = y_1 q_2 u = y_1 q_1 p = y^* p \text{ and}$$
$$y' = y'u = y_f q_2 u = y_f q_1 p = y^* p;$$

a contradiction.

(b) Let $p \in E(Y)$ such that $y_\alpha p = y_1$ for all α. Let u be an idempotent of some minimal right ideal in $E(Y)$ for which $y_1 u = y_1$. Then $puE(Y) = I$ is a minimal right ideal and $y_\alpha pu = y_1 u = y_1$ for all α. So for every q in I, $\{y_\alpha q\}$ is a singleton. Also for each y_α there is a v_α in $J(I)$ such that $y_\alpha v_\alpha = y_\alpha$. Therefore $y_\alpha = y_1 v_\alpha$ and $\{y_\alpha\} \subseteq y_1 J(I)$.

Remark. The assumption that y is a transitive point is stronger than we need.

3. LEMMA. *Suppose X and N are point-transitive, metric flows and Z is a common factor. Suppose $\phi: X \to Z$ is an open homomorphism, $\Psi: N \to Z$ has a section μ, and z_0 is an element of Z for which the support of μ_{z_0} equals $N_0 = \Psi^{-1}(z_0)$. Suppose*

$$S_m(\phi) \cap X_0 \times X_0 = R_m(\phi) \cap X_0 \times X_0 \text{ where } X_0 = \phi^{-1}(z_0).$$

Suppose $X_0 \cap X_m$ is dense in X_0 and $N_0 \cap N_m$ is dense in N_0. Then for

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each $x_0 \in X_0 \cap X_m$, the set

$$D(x_0) = \{n \in N_0 \colon (x_0, n) \text{ has dense orbit in } Z \circ {}^z N\}$$

is a dense G_δ subset of N_0.

Proof. (This proof is similar to that of Lemma 1.10 of [**6**].) Fix $x_0 \in X_0 \cap X_m$. Note $R_m(\theta)(x_0) = X_0 \cap X_m$, so $S_m(\phi)(x_0) = X_0 \cap X_m$. Let $\{U_i\}$, $\{V_i\}$ be countable families of open sets in X, N respectively such that the set of $U_i \circ^z V_i$ is a countable base of non-empty sets for the topology on $X \circ^z N$. Fix i. Let W be any non-empty, relative open subset of N_0 and

$$N^* = \text{cls }((\{x_0\} \times W)T).$$

Then $\{x'\} \times W \subseteq N^*$ for all $x' \in S_m(\phi)(x_0)$ by Corollary 1.4 of [**6**]. So $(X_0 \cap X_m) \times W \subseteq N^*$. So $X_0 \times W \subseteq N^*$. Now there exists w in W with dense orbit, so for some t in T and x' in X_0, $(x', w)t \in U_i \times V_i$ since ϕ is open. So $N^* \cap (U_i \times V_i) \neq \emptyset$ and there exists s in T and w' in W such that $(x_0, w')s \in U_i \times V_i$. Now since W was an arbitrary open set in N_0, the set

$$A_i = \{a \in N_0 \colon (x_0, a)t \in U_i \times V_i \text{ for some } t \text{ in } T\}$$

is dense in N_0, clearly it is open in N_0. Then $D(x_0) = \bigcap_1^\infty A_i$ is a dense G_δ subset of N_0.

4. THEOREM. *Suppose Y is a metric minimal flow, and suppose $\theta \colon Y \to Z$ is open, has a section, and $S(\theta) = R(\theta)$. Then ${}^\circ Y^\Omega$ contains a dense set D of transitive points, where Ω is the first uncountable ordinal. In addition for each y in Y there exists (y_α) in D with $y_1 = y$.*

Proof. Let μ be a section for θ. Note that by 3.3 of [**3**] there exists a residual set of points z_0 in Z such that $Y_0 = \theta^{-1}(z_0)$ equals the support of μ_{z_0}. Fix one such z_0. Let

$$H\lambda = (\theta^\lambda)^{-1}(z_0) \text{ and } K\lambda \subseteq H\lambda$$

be the set of points in $H\lambda$ with dense orbit in ${}^\circ Y^\lambda$. (Note: $H_1 = Y_0$, $H\lambda = Y_0{}^\lambda$).

We are going to wish to apply Lemma 3 with X and N replaced by ${}^\circ Y^\lambda$ and Y respectively for every $\lambda < \Omega$ (note that for $\lambda < \Omega$, ${}^\circ Y^\lambda$ is metric, also ${}^\circ Y^\lambda \to Z$ has a section). To do this we will need to establish that $K\lambda$ is dense in $H\lambda$ and that

$$S_m(\theta^\lambda) \cap H\lambda \times H\lambda = R_m(\theta^\lambda) \cap H\lambda \times H\lambda.$$

It is easy to see that $K\lambda$ is dense in $H\lambda$ for finite λ by applying Lemma 3 with X and N replaced by Y and ${}^\circ Y^\lambda$ respectively (in reverse of that above).

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Now to show $S_m(\theta^\lambda) \cap H\lambda \times H\lambda = R_m(\theta^\lambda) \cap H\lambda \times H\lambda$ for λ finite, note

$$R(\theta^\lambda) \cap H\lambda \times H\lambda = H2\lambda \text{ and}$$
$$P_m(\theta^\lambda) \cap H\lambda \times H\lambda \supseteq K2\lambda$$

and so is dense in $R(\theta^\lambda) \cap H\lambda \times H\lambda$ and thus in $R_m(\theta^\lambda) \cap H\lambda \times H\lambda$. So clearly

$$S_m(\theta^\lambda) \cap H\lambda \times H\lambda = R_m(\theta^\lambda) \cap H\lambda \times H\lambda.$$

Now for λ countably infinite we note that once it is shown for the first countably infinite ordinal, ω, that $K\omega$ is dense in $H\omega$, it follows that $K\lambda$ is dense in $H\lambda$ since $H\lambda$ is simply a reordering of components of $H\omega$. Also then it follows that

$$S_m(\theta^\lambda) \cap H\lambda \times H\lambda = R_m(\theta^\lambda) \cap H\lambda \times H\lambda$$

as above.

Now, to show that $K\omega$ is dense in $H\omega$, let A be any relative open set in $H\omega$, then $A \supseteq \prod_{\alpha<\omega} A_\alpha$ where all but finitely many of the A_α's equal Y_0. Let $\lambda_0 < \omega$ be larger than the last α with $A_\alpha \neq Y_0$. Then let

$$y^{\lambda_0} \in K\lambda_0 \cap \prod\{A_\alpha : \alpha \leqq \lambda_0\}.$$

Now apply (3) to $X = {}^\circ Y^{\lambda_0}$, $N = Y$ and extend y^{λ_0} to a transitive point y^{λ_0+1} in $H(\lambda_0 + 1)$ with $y_\alpha{}^{\lambda_0+1} = y_\alpha{}^{\lambda_0}$ for $\alpha \leqq y_0$. Continue by extending y^{λ_0+1} to a y^{λ_0+2}, etc. Define y^ω by $y_\lambda{}^\omega = y_\lambda{}^\lambda$ for $\omega > \lambda \geqq \lambda_0$ and $y_\lambda{}^\omega = y_\lambda{}^{\lambda_0}$ for $\lambda < \lambda_0$. Then $y^\omega \in A$ and is a transitive point. So $K\omega$ is dense in $H\omega$.

The proof that K^Ω is dense in H^Ω is similar.

(Note the proof for ω could be simplified by using the notions of topological transitivity and the fact that ${}^\circ Y_0{}^\omega$ is metric; the above approach is used since it clearly generalizes to the non-metric case of Ω.)

In addition, we see that for each $y \in Y_0$, one can construct a transitive point whose first coordinate is y. Note Y_0 is a fiber that equals the support of the measure of that fiber. To show this is true for all y in Y, we will provide the first two steps of an induction from which it will be clear how one proceeds.

Fix y_0 in Y, let $z_0 = \theta(y_0)$, $Y_0 = \theta^{-1}(z_0)$, and let μ be a section for $\theta: Y \rightarrow Z$ and B_z be the support of μ_z. Let W be any relative open subset of B_{z_0} and consider

$$N^* = \text{cls}((\{y_0\} \times W)T).$$

Then $\{y\} \times W \subseteq N^*$ for y in Y_0 by Corollary 1.4 of [**6**] since $S(\theta) = R(\theta)$. Continuing as in the proof of Lemma 3 we see that there exists a dense G_δ set of points y in B_{z_0} for which (y_0, y) is a transitive point in ${}^\circ Y^2$.

Now consider a transitive point (y_0, y_0') in ${}^\circ Y^2$ and let W be a relative open subset of B_{z_0}. Consider

$$N^* = \text{cls}((\{(y_0, y_0')\} \times W)T).$$

Then $\{(y, y')\} \times W \subseteq N^*$ for every transitive point (y, y') in $Y_0 \times Y_0$ by Corollary 1.4 of [**6**] since (y, y', y_0, y_0') is in the orbit closure of a transitive point in ${}^\circ Y^4$ (the existence of which we showed in Theorem 4), and thus in $S_m(\theta)$. So

$$(Y_0 \times B_{z_0}) \times W \subseteq N^*.$$

In particular,

$$(B_{z_0} \times B_{z_0}) \times \{W\} \subseteq N^* \text{ for } w \text{ in } W.$$

Let $Y^* \in Y$, $z^* = \theta(y^*)$, then we see that Proposition 2.2 of [**6**] implies

$$B_{z^*} \times B_{z^*} \times \{y^*\} \subseteq N^*.$$

Now by choosing z^* such that $B_{z^*} = \theta^{-1}(z^*)$ and using the openness of θ we see that $(Y \circ Y) \circ Y \subseteq N^*$. Then by proceeding as in the proof of Lemma 3 we see that there exists a dense G_δ subset of points y in B_z such that (y_0, y_0', y) is a transitive point in ${}^\circ Y^3$. Continuing in this manner we can extend y_0 to a transitive point in ${}^\circ Y^\Omega$.

5. Lemma. *Let X, Y, Z be minimal sets. Suppose $\phi: X \to Z$ has a section μ and θ: $Y \to Z$ is strongly proximal. Let W be a minimal subset of $X \circ {}^Z Y$ and π_1, π_2 be the projections of W onto X, Y respectively. Then π_2 has a section λ_y such that $\lambda_y = \mu_{\theta(y)} \times \delta_y$ on W.*

Proof. Fix any z_0 in Z. Let ν be a Borel probability measure on W such that $\hat{\pi}_1(\nu) = \mu_{z_0}$ (at least one exists). Now $\hat{\pi}_2(\nu)$ is a measure on Y with $\hat{\theta}(\hat{\pi}_2(\nu)) = \delta_{z_0}$, so for some y in Y (and thus every y in Y) there is a net t_n in T such that

$$\lim \hat{\pi}_2(\nu)t_n = \delta_y.$$

We may assume $\lambda = \lim \nu t_n$ exists. Clearly

$$\theta(y) = \lim z_0 t_n.$$

Also

$$\lim \hat{\pi}_1(\nu)t_n = \lim \mu_{z_0} t_n = \lim \mu_{z_0 t_n} = \mu_{\theta(y)}.$$

If $y \in B$,

$$\begin{aligned}\lambda((A \times B) \cap W) &= \lambda([(A \times Y).\cap W] \cap [(X \times B) \cap W]) \\ &= \lambda[A \times Y) \cap W] + \lambda[(X \times B) \cap W] - \lambda([(A \times Y) \cap W] \\ &\quad \cup [(X \times B) \cap W]) = \mu_{\theta(y)}(A) \text{ since}\end{aligned}$$

$$\lambda[(X \times B) \cap W] = \delta_y(B) = 1 \text{ and } y[(A \times Y) \cap W] = \mu_{\theta(y)}(A).$$

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If $y \notin B$, $\lambda((A \times B) \cap W) \leqq \lambda((X \times B) \cap W) = \delta_y(B) = 0$; so

$$\lambda((A \times B) \cap W) = \mu_{\theta(y)}(A) \cdot \delta_y(B).$$

So we see $\lambda = \mu_{\theta(y)} \times \delta_y$ restricted to W. Note that therefore

$$\text{supp } \mu_{\theta(Y)} \times \{y\} \subseteq W$$

since $\lambda(W) = 1$.

SPI STRUCTURE THEOREM. *For any minimal flow X there exist minimal flows Y and Z and homomorphisms $\theta: Y \to Z$, $\phi: Y \to X$ such that Z is strictly* SPI, *ϕ is strongly proximal, θ is open and has a section, and $S(\theta) = R(\theta)$. If X is metric, then there exist Y and Z that are metric.*

Proof. The theorem without the statement that θ is open follows easily from 4.1 of [**4**]. That one could require θ to be open was noted in [**2**] and follows easily from Lemma 5 above and from Theorem 3.1 of [**7**] in the metric case; in the nonmetric case see [**1**].

6. THEOREM. *If in the* SPI *structure theorem with X metric, θ is not proximal, then there exists at least 2^Ω minimal right ideals in the enveloping semigroup of X.*

Proof. This follows easily from the Theorem 4, Proposition 2 and Proposition 1.

7. COROLLARY. *Let X be a metric minimal flow. If $E(X)$ has less than 2^Ω minimal ideals, then X is* PI.

8. THEOREM. *If X is a metric minimal flow that is not an* SPI *flow, then for every x in X and every minimal right ideal in $E(X)$, the set $xJ(I)$ is uncountable.*

Proof. Let Y, Z, θ, and ϕ be as in the SPI structure theorem. Start with any y in Y and any minimal right ideal I and take a ν_1 in $J = J(I)$ for which $y\nu_1 = y$. By Lemma 1.8 of [**6**], there is a point y_2 such that (y, y_2) is a transitive point in $Y \circ {}^Z Y$. Let $\nu_2 \in J$ with $y_2\nu_2 = y_2$. Then by taking intersections, there is a point y_3 such that (y, y_3) and $(y\nu_2, y_3)$ are transitive points in $Y \circ {}^Z Y$. Continuing we see there is an uncountable set $\{y_\alpha: \alpha < \Omega\}$ such that $(y\nu_\alpha, y_\beta)$ in a transitive point in $Y \circ {}^Z Y$, where $\nu_\alpha \in J$ with $y_\alpha\nu_\alpha = y_\alpha$. Now we wish to show that the $\phi(y\nu_\alpha)$'s are all distinct. Suppose not, suppose $\phi(y\nu_\alpha) = \phi(y\nu_\beta)$ for some $\alpha < \beta$. Since $(y\nu_\alpha, y_\beta)$ is a transitive point there is a p in $E(Y)$ for which $y\nu_\alpha p = y_\beta p$ and so

$$\phi(y_\beta p) = \phi(y\nu_\alpha p) = \phi(y\nu_\beta p).$$

But $y_\beta \nu = y_\beta$, so we must have that $y\nu_\beta = y_\beta$, in which case we have a

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transitive point (yv_α, y_β) in $Y \circ {}^zY$ with $\phi(yv_\alpha) = \phi(y_\beta)$. So $R(\theta) \subseteq R(\phi)$ and so X is a factor of Z; that is X is an SPI flow, a contradiction.

9. COROLLARY. *If X is a metric minimal flow and if $xJ(I)$ is countable for some x in X and minimal right ideal I in $E(X)$, then X is an* SPI *flow.*

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