Canad. Math. Bull. Vol. 19 (2), 1976

A CONVEXITY RESULT FOR WEAK DIFFERENTIAL INEQUALITIES

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Introduction. In this note we present a natural "weak" form of a certain convexity estimate for evolution inequalities as given in Agmon-Nirenberg's paper [1], p. 139 (see also A. Friedman [2], Theorem 4.2 and 4.3). Our proof will follow that given in [1] and [2] with the natural modifications due to the enlargement of the class of solutions which are taken into account.

1. Let us consider a Hilbert space H, and B; $\mathcal{D}(B)\subseteq H\to H$ be a self-adjoint—generally unbounded—operator in H with domain $\mathcal{D}(B)$.

A class of test-functions $K_B[a, b]$ associated to B and to a given interval $[a, b]$ is defined as follows:

A function $\varphi(t)$, $a\le t\le b\to H$ belongs to $K_B[a, b]$ if and only if it is: once continuously differentiable in H; has a compact support in the open interval (a, b); belongs to $\mathcal{D}(B)$ for any $t\in(a, b)$; $(B\varphi)(t)$ is H-continuous in $[a, b]$.

Now, if $u(t)$ is a function, $a\le t\le b\to H$ which belongs to $\mathcal{D}(B)$ for any $t\in[a, b]$, continuously differentiable in H with $(Bu)(t)-H$ continuous in $[a, b]$, then the function $f(t)=u'(t)-Bu(t)$ is also H-continuous.

If we assume that an inequality of the form

$$\|u'(t)-Bu(t)\|_H=\|f(t)\|_H\le\phi(t)\,\|u(t)\|_H,\ t\in[a, b] \tag{1.1}$$

is satisfied, where $\phi(t)$ is a given non-negative scalar function defined for $t\in[a, b]$ then we say that $u(t)$ is a "strong" solution of an abstract differential inequality or of an "evolution inequality".

Let us take now the equality $u'(t)-Bu(t)=f(t)$ and multiply scalarly with an arbitrary function $\varphi(t)\in K_B[a, b]$. We get then

$$\langle u'(t), \varphi(t)\rangle_H-\langle Bu(t), \varphi(t)\rangle_H=\langle f(t), \varphi(t)\rangle_H \tag{1.2}$$

or also

$$\frac{d}{dt}\langle u(t), \varphi(t)\rangle_H-\langle u(t), \varphi'(t)\rangle_H-\langle u(t), B\varphi(t)\rangle_H=\langle f(t), \varphi(t)\rangle_H,\ t\in[a, b]$$

If we integrate this last equality between a and b, we obtain, because $\varphi(t)$ is

(Received by the editors December 23, 1974.)

(1) This research is supported by a grant of the National Research Council, Canada.

null near a and b, the equality

$$
(1.3)\qquad -\int_a^b \langle u(t), \varphi'(t)\rangle_H\, dt = \int_a^b \langle u(t), B\varphi(t)\rangle_H\, dt + \int_a^b \langle f(t), \varphi(t)\rangle_H\, dt, \forall \varphi \in K_B[a, b]
$$

We see that this last expression can be written with a general H-continuous function $u(t)$ and this leads us to the following

DEFINITION. A H-continuous function $u(t)$ verifies a weak evolution inequality (1.1) if there exists a H-continuous function $f(t)$ defined on $[a, b]$, such that (1.3) holds for all test-functions and also that the estimate

$$
(1.4)\qquad \|f(t)\|_H \le \phi(t)\, \|u(t)\|_H,\ t \in [a, b]
$$

is satisfied, where $\phi(t)$ is an everywhere defined non-negative scalar function on $[a, b]$.

In the present paper we prove the following

THEOREM. *Let us assume that the H-continuous function $u(t)$ verifies the weak evolution inequality* (1.1) *with a function $\phi(t)$ which is integrable on $[a, b]$ and if $\int_a^b \phi(t)\, dt \le 1/2\sqrt{2}$ then the estimate*

$$
(1.5)\qquad \|u(t)\| \le 2\sqrt{2}\, \|u(a)\|^{(b-t)/(b-a)}\, \|u(b)\|^{(t-a)/(b-a)},\ a \le t \le b
$$

is also satisfied.

2. **Proof of the theorem** (I). To start the proof, which follows the main lines in [1], [2] with the appropriate modifications for the "weak" case, we let $\{E_\lambda\}_{-\infty}^{\infty}$ to be the spectral family of the self-adjoint operator B, so that $Bx = \int_{-\infty}^{\infty} \lambda\, dE_\lambda x, \forall x \in \mathcal{D}(B)$, in the well-known sense (see [3] for the spectral theorem).

Let then E be the projection operator defined by $Ex = \int_0^\infty dE_\lambda x,\ x \in H$, so that $E = I - E_0$. Define then two continuous H-valued functions $u_1(t)$, $u_2(t)$ through the relations $u_1(t) = (Eu)(t)$, $u_2(t) = (I-E)u(t) = E_0 u(t)$ (here I is the identity operator in H). In the same way, consider the H-continous functions:

$$
f_1(t) = (Ef)(t), \qquad f_2(t) = (I-E)f(t)
$$

where $f(t) = u'(t) - Bu(t)$ in the above defined weak sense (as in 1.3). It will follow that $u_1'(t) - Bu_1(t) = f_1(t)$ and $u_2'(t) - Bu_2(t) = f_2(t)$ in the same weak sense. More precisely, the following is true:

LEMMA 1. *The relations*

$$
(1.6)\qquad -\int_a^b \langle u_j(t), \varphi'(t)\rangle_H\, dt = \int_a^b \langle u_j(t), (B\varphi)(t)\rangle_H\, dt + \int_a^b \langle f_j(t), \varphi(t)\rangle_H\, dt
$$

are verified for $j = 1, 2$ and for every test-function $\varphi(t) \in K_B[a, b]$.

In order ot prove this Lemma it is obviously sufficient to consider just $j=1$ or $j=2$. If, say $j=1$, we have the following

$$\text{If} \quad \varphi \in K_B[a, b] \quad \text{then} \quad E\varphi \in K_B[a, b] \quad \text{too.}$$

In fact, the strong H-derivative $dE\varphi/dt$ exists and equals $E\, d\varphi/dt$, so it is also strongly continuous; also $E\varphi = \theta$ where $\varphi = \theta$, hence $E\varphi$ has compact support in (a, b); furthermore, the range of $E\varphi$ is in the domain of B when $t \in (a, b)$: in fact, it is known that $h \in H$ belongs to $\mathcal{D}(B)$ is and only if

$$\int_{-\infty}^{\infty} |\lambda|^2\, d\langle E_\lambda h, h\rangle = \int_{-\infty}^{\infty} |\lambda|^2\, d\, \|E_\lambda h\|^2 < \infty$$

Now, if $h \in \mathcal{D}(B)$ then $Eh \in \mathcal{D}(B)$ because

$$\int_{-\infty}^{\infty} |\lambda|^2\, d\, \|E_\lambda E h\|^2 = \int_0^{\infty} \lambda^2\, d\|(E_\lambda - E_0)h\|^2$$
$$= \int_0^{\infty} \lambda^2\, d\, \|E_\lambda h\|^2 < \infty$$

Hence, $(E\varphi)(t) \in \mathcal{D}(B)$ for any $t \in [a, b]$; we need also that $B(E\varphi)$ is H-continuous as is for $B\varphi$. But $BE\varphi = EB\varphi$ (as B commutes with any of E_λ). So, if $B\varphi$ is continuous, $BE\varphi$ is too.

At this stage we write

$$\begin{aligned}
\int_a^b \langle u_1(t), (B\varphi)(t)\rangle\, dt + \int_a^b \langle f_1(t), \varphi(t)\rangle\, dt
&= \int_a^b \langle Eu(t), B\varphi(t)\rangle\, dt + \int_a^b \langle Ef(t), \varphi(t)\rangle\, dt \\
&= \int_a^b \langle u(t), B(E\varphi)(t)\rangle\, dt + \int_a^b \langle f(t), (E\varphi)(t)\rangle\, dt \\
&= -\int_a^b \langle u(t), (E\varphi)'(t)\rangle\, dt = -\int_a^b \langle u(t), E\varphi'(t)\rangle\, dt \\
&= -\int_a^b \langle Eu(t), \varphi'(t)\rangle\, dt = -\int_a^b \langle u_1(t), \varphi'(t)\rangle\, dt
\end{aligned}$$

which gives Lemma for $j=1$.

3. **Proof of the Theorem** (II). Let us consider now a sequence of scalar-valued functions $\{\alpha_n(t)\}_{n=1}^{\infty}$ which are non-negative C^1-functions, vanishing for $|t| \geq 1/n$, with $\int_{-1/n}^{1/n} \alpha_n(\tau)\, d\tau = 1$ and then form the convolution

$$(u_1 * \alpha_n)(t) = \int_{|t-\tau| \leq 1/n} u_1(\tau)\alpha_n(t-\tau)\, d\tau$$

which is well-defined for $a + 1/n \leq t \leq b - 1/n$, and is continuously differentiable there. As proved in our paper [4], after use of (1.6) we find that $(u_1 * \alpha_n)(t) \in \mathcal{D}(B)$ for $t \in [a+1/n, b-1/n]$, and in the same interval it is

$$(u_1 * \alpha_n)'(t) = B(u_1 * \alpha_n)(t) + (f_1 * \alpha_n)(t)$$

where

$$(f_1 * \alpha_n)(t) = \int_{|t-\tau| \le 1/n} f_1(\tau)\alpha_n(t-\tau)\, d\tau$$

Now we see that

$$(u_1 * \alpha_n)(t) = \int_{|t-\tau| \le 1/n} (Eu)(\tau)\alpha_n(t-\tau)\, d\tau = E(u * \alpha_n)(t), \forall t \in \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

Hence, $(u_1 * \alpha_n)(t) \in E(H) \forall t \in [a+1/n, b-1/n]$, and then, remarking that $B \ge 0$ on $E(H)$, it is: $\langle B(u_1 * a_n)(t), (u_1 * a_n)(t)\rangle_H \ge 0 \forall t$ in this interval.

Now we see that, on $[a+1/n, b-1/n]$

$$\frac{d}{dt}\langle u_1 * \alpha_n, u_1 * \alpha_n\rangle = 2\,\mathrm{Re}\langle B(u_1 * \alpha_n), (u_1 * \alpha_n)\rangle$$
$$+ 2\,\mathrm{Re}\langle f_1 * \alpha_n, u_1 * \alpha_n\rangle \ge 2\,\mathrm{Re}\langle f_1 * \alpha_n, u_1 * \alpha_n\rangle$$

If we integrate between $t \in (a-1/n, b-1/n)$ and $b-1/n$, we get

$$(1.7)\quad \|(u_1 * \alpha_n)(b-1/n)\|^2 - \|(u_1 * \alpha_n)(t)\|^2 \ge 2\,\mathrm{Re} \int_t^{b-1/n} \langle f_1 * \alpha_n, u_1 * \alpha_n\rangle\, ds$$

$a + \frac{1}{n} < t < b - \frac{1}{n}$.

Now we can prove

LEMMA 2. *The estimate*

$$\|u_1(b)\|^2 - \|u_1(t)\|^2 \ge 2\,\mathrm{Re} \int_t^b \langle f_1(\tau), u_1(\tau)\rangle\, d\tau$$

is valid, $\forall t \in (a, b)$.

First we prove that $\lim_{n\to\infty}(u_1 * \alpha_n)(b-1/n) = u_1(b)$. In fact

$$(u_1 * \alpha_n)\left(b - \frac{1}{n}\right) = \int_{b-2/n}^{b} u_1(\tau)\alpha_n\left(b - \frac{1}{n} - \tau\right) d\tau,$$

and

$$u_1(b) = \int_{b-2/n}^{b} u_1(b)\alpha_n\left(b - \frac{1}{n} - \tau\right) d\tau$$

because

$$\int_{|\tau|<1/n} \alpha_n(\tau)\, d\tau = 1$$

https://doi.org/10.4153/CMB-1976-037-4 Published online by Cambridge University Press

Then

$$\|(u_1*\alpha_n)\left(b-\frac{1}{n}\right)-u_1(b)\|\le\int_{b-2/n}^{b}\|u_1(\tau)-u_1(b)\|\,\alpha_n\left(b-\frac{1}{n}-\tau\right)d\tau$$

$$\le\sup_{b-2/n\le\tau\le b}\|u_1(\tau)-u_1(b)\|\int_{b-2/n}^{b}\alpha_n\left(b-\frac{1}{n}-\tau\right)d\tau$$

$$=\sup_{b-2/n\le\tau\le b}\|u_1(\tau)-u_1(b)\|,\ \forall n=1,2,\ldots$$

and this $\to 0$ as $n\to\infty$ by continuity of $u_1(\tau)$ for $\tau=b$.

Hence, we have also:

$$\lim_{n\to\infty}\left\|(u_1*\alpha_n)\left(b-\frac{1}{n}\right)\right\|=\|u_1(b)\|.$$

But the estimate

$$\left\|(u_1*\alpha_n)\left(b-\frac{1}{n}\right)\right\|\le\sup_{[a,b]}\|u_1(\tau)\|$$

is also valid, hence we get too:

$$\lim_{n\to\infty}\left\|(u_1*\alpha_n)\left(b-\frac{1}{n}\right)\right\|^2=\|u_1(b)\|^2.$$

Furthermore:

$$\lim_{n\to\infty}\|(u_1*\alpha_n)(t)\|^2=\|u_1(t)\|^2$$

for $t\in(a,b)$ and

$$\lim_{n\to\infty}\int_t^{b-1/n}\langle f_1*\alpha_n, u_1*\alpha_n\rangle\,ds=\int_t^b\langle f_1(s), u_1(s)\rangle\,ds.$$

This last limit holds because of the following: consider the difference

$$\int_t^b\langle f_1(s), u_1(s)\rangle\,ds-\int_t^{b-1/n}\langle f_1*\alpha_n, u_1*\alpha_n\rangle\,ds$$

Now, denote

$$\langle f_1(s), u_1(s)\rangle=\phi_1(s), \langle (f_1*\alpha_n)(s), (u_1*\alpha_n)(s)\rangle=\phi_n(s)$$

we see that $\phi_1(s)$ is continuous on $t\le s\le b$, and $\phi_n(s)$ are continuous on $t\le s\le b-1/n$.

Then our expression equals

$$\int_t^b\phi_1(s)\,ds-\int_t^{b-1/n}\phi_n(s)\,ds.$$

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Let us extend $\phi_n(s)$ as:

$$\tilde{\phi}_n(s)=\begin{cases}\phi_n(s),\ t\le s<b-1/n\\ 0,\ b-1/n\le s\le b\end{cases}$$

It follows

$$\int_t^{b-1/n}\phi_n(s)\,ds=\int_t^b\tilde{\phi}_n(s)\,ds,$$

so that

$$\lim_{n\to\infty}\int_t^b[\phi_1(s)-\tilde{\phi}_n(s)]\,ds,$$

must be null.

We can apply here Lebesgue's theorem:

(i) $\tilde{\phi}_n(s)\to\phi_1(s)$ almost-everywhere on $[t,b]$.

In fact, for any $s\in[t,b)$, $\tilde{\phi}_n(s)=\phi_n(s)$ when n is big enough, such that $b-1/n>s$: furthermore $\phi_n(s)\to\phi_1(s)$ for any $a<s<b$ because $(f_1*\alpha_n)(s)\to f_1(s)$, $(u_1*\alpha_n)(s)\to u_1(s)$; hence, $\tilde{\phi}_n(s)\to\phi_1(s)$ for any $s>a$, with possible exception of $s=b$. ($\phi_1(b)$ need not be null, whereas $\tilde{\phi}_n(s)$ are all null for $s=b$.)

(ii) $\tilde{\phi}_n(s)$ are uniformly bounded on $[t,b]$. In fact

$$\sup_{t\le s\le b}|\tilde{\phi}_n(s)|\le\sup_{t\le s\le b-1/n}|\phi_n(s)|\le\sup_{t\le s\le b-1/n}\|(f_1*\alpha_n)(s)\|\|(u_1*\alpha_n)(s)\|$$

$$\le\sup_{a\le s\le b}\|f_1(s)\|\sup_{a\le s\le b}\|u_1(s)\|$$

REMARK. We can also avoid Lebesgue's theorem as follows: take an arbitrary $\delta>0$. Then

$$\begin{aligned}\int_t^b[\phi_1(s)-\tilde{\phi}_n(s)]\,ds&=\int_t^{b-\delta}[\phi_1(s)-\tilde{\phi}_n(s)]\,ds\\&\quad+\int_{b-\delta}^b[\phi_1(s)-\tilde{\phi}_n(s)]\,ds\\&=\int_t^{b-\delta}[\phi_1(s)-\phi_n(s)]\,ds+\int_{b-\delta}^b[\phi_1(s)-\tilde{\phi}_n(s)]\,ds,\end{aligned}$$

for $\delta>\frac{1}{n}$.

The second integral estimates by $C\cdot\delta$, $C=2\sup_{a\le s\le b}\|u_1(s)\|\|f_1(s)\|$.

Then given $\varepsilon>0$, take first $\delta(\varepsilon)$ such that $C\delta<\varepsilon/2$. Then, because $\phi_n(s)\to\phi_1(s)$ uniformly on $[t,b-\delta]$, there exist an integer $N(\varepsilon)$ such that $\delta>1/N$ and $n>N\Rightarrow$

$$\left\|\int_t^{b-\delta}[\phi_1(s)-\phi_n(s)]\,ds\right\|<\frac{\varepsilon}{2}$$

so that $n>N\Rightarrow$

$$\left\|\int_t^b [\phi_1(s)-\tilde{\phi}_n(s)]\,ds\right\|<\varepsilon$$

Hence, from (1.7), Lemma 2 follows for $a<t<b$. However, the Lemma is true also for $t=a$, or $t=b$, as follows by continuity.

In exactly same way we see that the following is true.

LEMMA 3. *The estimate*

$$\|u_2(t)\|^2-\|u_2(a)\|^2\le 2\,\mathrm{Re}\int_a^t \langle f_2(s), u_2(s)\rangle\,ds$$

holds, $\forall t\in[a, b]$.

4. **Proof of Theorem** (III).

We see firstly that

$$\left|2\,\mathrm{Re}\int_t^b \langle f_1(s), u_1(s)\rangle\,ds\right|\le 2\left|\int_t^b \langle f_1(s), u_1(s)\rangle\,ds\right|\le 2\int_t^b \|f_1(s)\|\|u_1(s)\|\,ds$$
$$\le 2\int_t^b \|f(s)\|\|u(s)\|\,ds.$$

It follows:

$$-2\int_t^b \|f(s)\|\|u(s)\|\,ds\le 2\,\mathrm{Re}\int_t^b \langle f_1(s), u_1(s)\rangle\,ds.$$

Hence, applying Lemma 2, we obtain

$$\|u_1(b)\|^2-\|u_1(t)\|^2\ge -2\int_t^b \|f(s)\|\|u(s)\|\,ds.$$

Then it is:

$$\|u_1(t)\|^2\le\|u_1(b)\|^2+2\int_t^b \|f(s)\|\,u(s)\|\,ds\le\|u_1(b)\|^2+2M\int_t^b \|f(s)\|\,ds,$$

where $M=\sup_{a\le s\le b}\|u(s)\|$.

Also, from Lemma 3, we get

$$\|u_2(t)\|^2\le\|u_2(a)\|^2+2M\int_a^t \|f(s)\|\,ds$$

and by addition

$$\|u_1(t)\|^2+\|u_2(t)\|^2\le\|u_1(b)\|^2+\|u_2(a)\|^2+2M\int_a^b \|f(s)\|\,ds$$

As $u_1(t)=Eu(t)$, $u_2(t)=(I-E)u(t)$, $\langle u_1(t), u_2(t)\rangle=\langle Eu(t),(I-E)u(t)\rangle=\langle u(t),(E-E)u(t)\rangle=0$; so $\|u_1(t)+u_2(t)\|^2=\langle u_1+u_2, u_1+u_2\rangle=\|u_1(t)\|^2+\|u_2(t)\|^2$. Hence

$$\|u(t)\|^2 \le \|u_1(b)\|^2+\|u_2(a)\|^2+2M\int_a^b \|f(s)\|\, ds$$

If we use inequality

$$2MN \le \left(\frac{M}{\sqrt{2}}\right)^2+(\sqrt{(2)}N)^2 \quad \text{where} \quad N=\int_a^b \|f(s)\|\, ds,$$

we have

$$\|u(t)\|^2 \le \|u_1(b)\|^2+\|u_2(a)\|^2+\frac{M^2}{2}+2\left(\int_a^b \|f(s)\|\, ds\right)^2, \forall t\in[a, b].$$

Hence

$$M^2 \le \|u_1(b)\|^2+\|u_2(a)\|^2+\frac{M^2}{2}+2\left(\int_a^b \|f(s)\|\, ds\right)^2$$

and finally we have the estimate

$$(*) \qquad M^2 \le 2(\|u(a)\|^2+\|u(b)\|^2)+4\left(\int_a^b \|f(s)\|\, ds\right)^2$$

Let us define now, for any real σ, the H-continuous function $w_\sigma(t)=e^{\sigma t}u(t)$, and let $B_\sigma=B+\sigma I$ which is again self-adjoint, with $\mathcal{D}(B_\sigma)=\mathcal{D}(B)\forall$ real σ. Then we have

LEMMA 4. *The relation $w'_\sigma(t)-B_\sigma w_\sigma(t)=e^{\sigma t}f(t)$ holds in the weak sense over (a, b).*

So, we must prove that, $\forall\varphi\in K_{B_\sigma}[a, b]=K_B[a, b]$ is

$$-\int_a^b \langle w_\sigma(s), \varphi'(s)\rangle\, ds=\int_a^b \langle w_\sigma(s), B_\sigma\varphi(s)\rangle\, ds+\int_a^b \langle e^{\sigma s}f(s), \varphi(s)\rangle\, ds$$

or

$$-\int_a^b \langle u(s), e^{\sigma s}\varphi'(s)\rangle\, ds=\int_a^b \langle u(s), e^{\sigma s}B_\sigma\varphi(s)\rangle\, ds+\int_a^b \langle f(s), e^{\sigma s}\varphi(s)> ds.$$

Now $e^{\sigma s}\varphi'(s)=(e^{\sigma s}\varphi(s))'-\sigma e^{\sigma s}\varphi(s)$. Also $\psi_\sigma(s)=e^{\sigma s}\varphi(s)\in K_B[a, b]$ as obviously. We have to prove

$$-\int_a^b \langle u(s), \psi'_\sigma(s)-\sigma\psi_\sigma(s)\rangle\, ds=\int_a^b \langle u(s), B\psi_\sigma(s)+\sigma\psi_\sigma(s)\rangle\, ds$$
$$+\int_a^b \langle f(s), \psi_\sigma(s)\rangle\, ds,$$

that is to prove

$$-\int_a^b \langle u(s), \psi'_\sigma(s)\rangle\, ds = \int_a^b \langle u(s), B\psi_\sigma(s)\rangle\, ds + \int_a^b \langle f(s), \psi_\sigma(s)\rangle\, ds$$

which is true because $u' - Bu = f$ in weak sense on (a, b).

Once this is established, we apply $(*)$ to this slightly changed situation and obtain the estimate

$$\left(\sup_{a\le t\le b} \|e^{\sigma t}u(t)\|\right)^2 \le 2(\|e^{\sigma a}u(a)\|^2 + \|e^{\sigma b}u(b)\|^2) + 4\left(\int_a^b \|e^{\sigma s}f(s)\|\, ds\right)^2$$

We use now the main assumption (1.4)

$$\|f(s)\| \le \phi(s)\,\|u(s)\|, \qquad s \in [a, b].$$

Then

$$\int_a^b \|e^{\sigma s}f(s)\|\, ds \le \int_a^b e^{\sigma s}\phi(s)\,\|u(s)\|\, ds \le \sup_{[a,b]} (e^{\sigma s}\,\|u(s)\|) \int_a^b \phi(s)\, ds$$

$$\le \frac{1}{2\sqrt{2}} \sup_{[a,b]} (e^{\sigma s}\,\|u(s)\|)$$

and squaring get

$$\left(\int_a^b \|e^{\sigma s}f(s)\|\, ds\right)^2 \le \tfrac{1}{8}\left(\sup_{[a,b]} (e^{\sigma s}\,\|u(s)\|)\right)^2$$

Hence

$$(\sup\|e^{\sigma s}u(s)\|)^2 \le 2(\|e^{\sigma a}u(a)\|^2 + \|e^{sb}u(b)\|^2) + \tfrac{1}{2}(\sup e^{\sigma s}\,\|u(s)\|)^2$$

and

$$\left(\sup_{[a,b]} \|e^{\sigma s}u(s)\|\right)^2 \le 4(\|e^{\sigma a}u(a)\|^2 + \|e^{\sigma b}u(b)\|^2)$$

Then $\forall t \in [a, b]$, $\|e^{\sigma t}u(t)\| \le \sup_{[a,b]} \|e^{\sigma s}u(s)\|$ and

$$|e^{\sigma t}u(t)\|^2 \le \left(\sup_{[a,b]} \|e^{\sigma s}u(s)\|\right)^2;$$

so

$$(**)\ \|e^{\sigma t}u(t)\|^2 \le 4(\|e^{\sigma a}u(a)\|^2 + \|e^{\sigma b}u(b)\|^2),\ \forall t \in [a, b]$$

We pass now to the final part of the proof for (1.5). First we consider the case when $u(a) = \theta$ or $u(b) = \theta$. If both are, from $(**) \Rightarrow \|u(t)\| = 0\ \forall t \in [a, b]$, so (1.5) holds. If say, $u(a) = \theta$, from $(**)$ we get

$$\|e^{\sigma t}u(t)\| \le 2\,\|e^{\sigma b}u(b)\|, \qquad \|u(t)\| \le 2e^{\sigma(b-1)}\,\|u(b)\|$$

As σ was chosen arbitrarily, we deduce, when $t<b$ and $\sigma\to-\infty$, that $\|u(t)\|=\theta$, $a\le t<b$ and hence $u(b)=\theta$ also and (1.5) holds. The non-trivial case is when both $u(a)$ and $u(b)$ are $\neq\theta$. We can choose σ so that $\|e^{\sigma a}u(a)\|=\|e^{\sigma b}u(b)\|$

$$e^{\sigma(b-a)}=\frac{\|u(a)\|}{\|u(b)\|},\qquad \sigma(b-a)=\log\frac{\|u(a)\|}{\|u(b)\|},\qquad \sigma=\log\left(\left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{1}{b-a}}\right).$$

In that case, $e^{\sigma t}=(\|u(a)\|/\|u(b)\|)^{t/(b-a)}$. Hence $(**)$ becomes

$$\left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{2t/(b-a)}\|u(t)\|^2\le 8\left(\|u(a)\|^2\left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{2a/(b-a)}\right)=8\,\frac{\|u(a)\|^{\frac{2b}{b-a}}}{\|u(b)\|^{\frac{2a}{b-a}}}$$

hence

$$\|u(t)\|\le 2\sqrt{2}\,\|u(a)\|^{b-t/b-a}\,\|u(b)\|^{t-a/b-a},\quad a\le t\le b$$

which proves our theorem.

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