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ASYMMETRIC INVARIANT SETS FOR COMPLETELY POSITIVE MAPS ON C*-ALGEBRAS

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Let A be a noncommutative C^*-algebra other than $M_2(\mathbb{C})$. We show that there exists a completely positive map ϕ of norm one on A and an element $a \in A$ such that $\phi(a) = a$, $\phi(a^*a) = a^*a$, but $\phi(aa^*) \neq aa^*$.

A linear map ϕ from a C^*-algebra A into itself is called a Schwarz map if, for all $a \in A$,

$$\phi(a)^*\phi(a) \leq \phi(a^*a) .$$

In [2] the invariant set $D_\phi = \{a \in A : \phi(a) = a , \phi(a^*a) = a^*a\}$ is studied. Limaye and Namboodiri prove that if $A = M_2(\mathbb{C})$ then D_ϕ is *-closed for any Schwarz map ϕ but that if $A \neq M_2(\mathbb{C})$ is a noncommutative C^*-algebra of compact operators or a type I factor then D_ϕ is not *-closed for some Schwarz map ϕ on A. The purpose of this note is to extend that result to arbitrary C^*-algebras, thereby answering a question posed in [2]. The map ϕ that we construct is even a completely positive contraction. It follows easily from Stinespring's theorem [3] that such a map is a Schwarz map.

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THEOREM. *Let* A *be a noncommutative* C^**-algebra other than* $M_2(\mathbb{C})$. *Then there exists a completely positive map* $\phi : A \longrightarrow A$ *such that* $\|\phi\| = 1$ *and* D_ϕ *is not* $*$*-closed.*

Proof. Since A is noncommutative, there exists an element $x \in A$ of norm one satisfying $x^2 = 0$ [1, 2.12.2]. We have $\{0,1\} \subseteq Sp(x^*x) = Sp(xx^*) \subseteq [0,1]$. There are two cases to consider.

Case 1. Suppose that $Sp(x^*x)$ contains at least three points. Choose $s \in Sp(x^*x)$ with $0 < s < 1$. Define continuous functions f , g , h on $[0,1]$ which vanish at 0 and satisfy $0 \le f \le g \le 1$, $0 \le h \le 1$, $fg = f$, $hg = 0$ and $f(s) = g(s) = h(1) = 1$. Thus $g = 1$ on the support of f and $g = 0$ on the support of h .

Note that $(x^*x)(xx^*) = 0$, so since f and g are uniform limits of polynomials on $[0,1]$ without constant terms we have that $f(x^*x)$, $g(x^*x)$ and $h(x^*x)$ are each orthogonal to all the elements $f(xx^*)$, $g(xx^*)$ and $h(xx^*)$. Let $y = f(xx^*)x\, f(x^*x)$. Then $y^2 = 0$. Also $y \neq 0$. For, considering polynomials approximating f , we see that $y = x\, f(x^*x)^2$ and, by definition of f , $x^*x\, f(x^*x)^2 \neq 0$.

Let $p = g(x^*x) + g(xx^*)$ and $q = h(x^*x)$. Then $pq = 0$. Now $py = y = yp$. Since $(y^*y)(yy^*) = 0$, there is a state σ of A such that $\sigma(y^*y) = 0$ but $\sigma(yy^*) > 0$. In addition, by the Cauchy-Schwarz inequality, we have $\sigma(y) = 0$. We define the completely positive map ϕ by

$$\phi(a) = pap + \sigma(a)q \ . \tag{1}$$

Then $\phi(y) = y$, $\phi(y^*y) = y^*y$, but $\phi(yy^*) \neq yy^*$. Therefore D_ϕ is not $*$-closed.

Case 2. $Sp(x^*x) = \{0,1\}$.

In this case x^*x and xx^* are orthogonal projections. If $x^*x + xx^* \neq 1$ or if A is not unital we can define ϕ as in (1) with $p = x^*x + xx^*$, q a positive element of norm one orthogonal to p , and σ a state of A satisfying $\sigma(x^*x) = 0$, $\sigma(xx^*) > 0$. Then $x \in D\phi$ but $x^* \notin D_\phi$. We can therefore suppose that $x^*x + xx^* = 1$, x^*x and xx^* being orthogonal equivalent projections in A . A can then be expressed

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as a matrix algebra $M_2(B)$, where the C^*-algebra B is *-isomorphic to the relative commutant of $\{x,x^*\}$ in A . Since $A \neq M_2(\mathbb{C})$, we can find an element $b \in B_+$ of norm one which contains at least two nonzero points in its spectrum. Then $a = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ satisfies $\|a\| = 1$, $a^2 = 0$ and the spectrum of a^*a strictly contains $\{0,1\}$. This returns us to the situation considered in Case 1, and completes the proof.

REMARK. It is clear that when A is a unital C^*-algebra the map ϕ can also be modified so as to be unital.

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