ON POLYNOMIAL EXPANSIONS OF ANALYTIC FUNCTIONS [1]

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(Received 5 October 1967)

1. Introduction

A set of polynomials $p_0(z), p_1(z), \cdots$ is said to form a basic set if every polynomial can be expressed in one and only one way as a finite linear combination of them.

Given any family F of polynomials we shall let $U(n)$ denote the number of polynomials in F of degree less than n. It is clear that any linearly independent set of polynomials satisfying the condition $U(n) = n$ is a basic set. Such a basic set is called a simple set.

Suppose that $\{p_i(z)\}$ $i = 0, 1, 2, \cdots$ is a simple set of polynomials. We may write

$$p_i(z) = \sum_{j=0}^{i} p_{ij} z^j, \text{ where } p_{ii} = 1 \qquad (i = 0, 1, 2, \cdots) \tag{1}$$

$$z^i = \sum_{j=0}^{i} \pi_{ij} p_j(z) \qquad (i = 0, 1, 2, \cdots), \tag{2}$$

where $\pi_{ii} = 1$.

Let us define the operator Π_i as

$$\sum_{k=i}^{\infty} \frac{\pi_{ki}}{k!} D^k,$$

where D denotes the differential operator.

The purpose of this paper is to generalize the following result ([1], Theorem 2).

THEOREM 1. *Let*

$$p_i(z) = \sum_{i=0}^{i} p_{ij} z^j, \textit{ where } p_{ii} = 1 \qquad (i = 0, 1, 2, \cdots)$$

be a simple set of polynomials whose coefficients satisfy the inequality

[1] The author is indebted to Professor E. G. Straus, who suggested some of the ideas in this paper.

$$|p_{ij}| \leqq M$$

and let $f(z)$ *be analytic in* $|z| < R$, *where* $R > 1+M$. *Then the basic series*

$$\sum_{i=0}^{\infty} p_i(z)(\Pi_i f)(0)$$

converges absolutely to $f(z)$ *in* $|z| < R$, *where* $(\Pi_i f)(0)$ *is defined as* $[\Pi_i f(z)]_{z=0}$.

More specifically, let $\{p_i^k(z)\}$ $(k = 0, 1, 2, \cdots, n;\ i = 0, 1, 2, \cdots)$ be a finite family of simple sets of polynomials such that $p_i^0(z) = z^i$ and for $j \leqq 1$ define p_{ij}^k by

$$p_i^{k+1} = \sum_{j=0}^{i} p_{ij}^k p_j^k(z) \tag{3}$$

$(k = 0, 1, 2, \cdots, n-1;\ i = 0, 1, 2, \cdots)$ where $p_{ii}^k = 1$. For $j > i$ define $p_{ij}^k = 0$.

Let π_{ij}^n be defined by

$$z^i = \sum_{j=0}^{i} \pi_{ij}^n p_j^n(z) \qquad (i = 0, 1, 2, \cdots),$$

so that $\pi_{ii}^n = 1$. Furthermore, let Π_i^n denote the operator

$$\sum_{k=i}^{\infty} \frac{\pi_{ki}^n}{k!} D^k.$$

We shall show that if $|p_{ij}^k| \leqq M$ and if $f(z)$ is analytic in $|z| < R$ where $R > 1+M$, then the basic series

$$\sum_{i=0}^{\infty} p_i^n(z)(\Pi_i^n f)(0)$$

converges absolutely to $f(z)$ in $|z| < R$.

We shall show further that the boundedness condition of theorem 1 is not a necessary condition and that for certain simple sets of polynomials the uniform boundedness of the zeros of the polynomials is a necessary and sufficient condition for the theorem to hold.

Finally, we remark that for a suitably restricted class of entire functions Whittaker [1 p. 11] needs no condition on the p_{ij} to assure that a basic series converges to $f(z)$. We are, however, throughout concerned with the convergence of a basic series to $f(z)$ for arbitrary f, in which case it is necessary to restrict the polynomials, though not necessarily as severaly as in theorem 1.

2. An extension of theorem 1

With p_{ij} and π_{ij} defined for $j \leqq i$ by (1) and (2), and $p_{ij} = \pi_{ij} = 0$ for $j > i$, Whittaker [1, pp. 6, 15] shows that $(\pi_{ij})(p_{ij}) = I$ (the unit matrix) and that if also $|p_{ij}| \leqq M$ (a constant) then

$$|\pi_{ij}| < (1+M)^{i-j} \qquad (0 \leqq j \leqq i,\ i = 0, 1, 2, \cdots). \tag{4}$$

Lemma 1 below is a generalization of (4) and will be used together with Lemma 2 to prove Theorem 2. Before proceeding, however, we would like to make some comments about the notation $(p_{ij})^{-1}$ to be used in the sequel. Indeed (π_{ij}) is the *unique* left inverse of (p_{ij}) *among row-finite matrices* — conceivably (p_{ij}) could have some other (non-row-finite) left inverse. But, a lower triangular matrix A with non-zero diagonal elements has a *unique right inverse* A^{-1} (which is also lower triangular), and A^{-1} is also a left inverse (and the only row-finite left inverse) — consult, for example, Cooke [3, p. 22].

Lemma 1. *Let* $T_0 = I$ *and* $T_m = P_{m-1} \cdots P_1 P_0$ $(m = 1, 2, \cdots, n)$, *where* $P_k = (p_{ij}^k)$ *is defined by* (3) *and satisfies, for some constant* M,

$$|p_{ij}^k| \leqq M \qquad (i, j = 0, 1, 2, \cdots;\ k = 0, 1, \cdots, n-1). \tag{5}$$

Then $T_m^{-1} = (\pi_{ij}^m)$ *has the property*

$$|\pi_{ij}^m| < (i-j+1)^{m-1}(1+M)^{i-j} \qquad (0 \leqq j \leqq i,\ i = 0, 1, 2, \cdots).$$

Proof. The result is trivial for $m = 0$, and reduces to (4) for $m = 1$. Suppose the inequality holds for some $m < n$.

Now $T_{m+1} = P_m T_m$, so that $T_{m+1}^{-1} = T_m^{-1} P_m^{-1}$; using the inductive hypothesis on T_m^{-1}, and (4) on P_m^{-1}, we then obtain

$$\begin{aligned} |\pi_{ij}^{m+1}| &< \sum_{k=j}^{i} (i-k+1)^{m-1}(1+M)^{i-k} \cdot (1+M)^{k-j} \\ &\leqq (i-j+1) \cdot (i-j+1)^{m-1}(1+M)^{i-j} \end{aligned}$$

and the lemma follows.

Lemma 2. *If* (5) *holds and* $R > 1+M$ *then*

$$M_i^k(R) \leqq (i+1)^k R^i \qquad (k = 0, 1, \cdots, n;\ i = 0, 1, 2, \cdots),$$

where $M_i^k(R) = \max_{|z|=R} |p_i^k(z)|$.

Proof. Since $p_i^0(z) = z^i$ we have $M_i^0(R) = R^i$, so that the result holds for $k = 0$. Suppose it holds for some $k < n$. Then, by (3) and (5),

https://doi.org/10.1017/S1446788700007576 Published online by Cambridge University Press

$$M_i^{k+1}(R) \leqq M \sum_{j=0}^{i-1} M_j^k(R) + M_i^k(R)$$
$$\leqq M \sum_{j=0}^{i-1} (j+1)^k R^j + (i+1)^k R^i$$
$$\leqq (i+1)^k M \sum_{j=0}^{i-1} R^j + (i+1)^k R^i$$
$$\leqq (i+1)^k i R^i + (i+1)^k R^i = (i+1)^{k+1} R^i$$

and the lemma follows.

Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$ be analytic in the region $|z| < R$ with $R > M+1$. We have

$$z^i = \sum_{j=0}^{i} \pi_{ij}^n p_j^n(z),\ \pi_{ii}^n = 1.$$

Let

$$E(z) = \sum_{j=0}^{\infty} p_j^n(z) \sum_{k=j}^{\infty} a_k \pi_{kj}^n = \sum_{j=0}^{\infty} p_j^n(z) (\Pi_j^n f)(0).$$

We can now prove

Theorem 2. *$E(z)$ converges absolutely to $f(z)$ in $|z| < R$.*

Proof. If the order of summation is reversed in the double series defining $E(z)$, we obtain $f(z)$. Consequently the theorem will be proved if we can show that, for $|z| < R$ (and $R > 1+M$), the series

$$S \equiv \sum_{i=0}^{\infty} |a_i| \sum_{j=0}^{i} |\pi_{ij}^n| |p_j^n(z)|$$

converges. First choose R_0 such that $M+1 < R_0 < R$; then, if $|z| \leqq R_0$, $|p_j^n(z)|$ is majorized by $M_j^n(R_0)$, and using Lemmas 1 and 2 we obtain

$$S \leqq \sum_{i=0}^{\infty} |a_i| \sum_{j=0}^{i} (i-j+1)^{n-1}(1+M)^{i-j} \cdot (j+1)^n R_0^j$$
$$\leqq \sum_{i=0}^{\infty} |a_i| (i+1)^{2n} R_0^i.$$

The last series converges, since if we choose R_1 in $R_0 < R_1 < R$, we can make $(i+1)^{2n} R_0^i < R_1^i$ for all sufficiently large i; and this proves the theorem.

We now show that the condition of theorem 1 that $|p_{ij}| < M$ is not a necessary condition. Though the following lemma is not really essential to prove this fact, nevertheless it is of independent interest and is worth mentioning.

Lemma 3. *Given a sequence of polynomials $z^n - c_1 z^{n-1} - c_2 z^{n-2} \cdots - c_n$*

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$(n = 0, 1, 2, \cdots)$, *where the coefficients are uniformly bounded, the zeros of these polynomials must be uniformly bounded.*

PROOF. If the coefficients are uniformly bounded by M but the conclusion is false, then for some n, there is a zero z with $|z| > M+1$. But then

$$|z|^n \leqq M(|z|^{n-1} + \cdots + 1) = M(|z|^n - 1)/(|z|-1) < |z|^n - 1$$

and this contradiction establishes the lemma.

LEMMA 4. *There exist simple sets of polynomials $\{p_n(z)\}$ such that their zeros are not uniformly bounded and yet every analytic function f is representable in terms of these polynomials in its region of analyticity.*

PROOF. Let $p_{2n}(z) = z^{2n}$ and $p_{2n+1}(z) = z^{2n+1} - (2n+1)z^{2n}$. Clearly $2n+1$ is a zero of $p_{2n+1}(z)$, so that the zeros are unbounded. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k z^k = \sum_{n=0}^{\infty} (a_{2n} z^{2n} + a_{2n+1} z^{2n+1}) \\ &= \sum_{n=0}^{\infty} (a_{2n} + (2n+1) a_{2n+1}) z^{2n} + \sum_{n=0}^{\infty} a_{2n+1} (z^{2n+1} - (2n+1) z^{2n}). \end{aligned}$$

These last two series clearly converge for $|z| < R$ whenever $\sum a_k z^k$ does so, and the lemma follows.

Thus it follows that

THEOREM 3. *There exist simple sets of polynomials $\{p_n(z)\}$ such that their coefficients are not uniformly bounded and yet every analytic function f is representable in terms of these polynomials in its region of analyticity.*

Now let $\{z_n\}$ be a sequence of complex numbers such that the set consisting of its distinct elements has no limit point. We consider the simple set S of polynomials whose elements $p_n(z)$ are given by

$$\begin{aligned} p_0(z) &= 1 \\ p_1(z) &= (z - z_1) \\ &\vdots \\ p_n(z) &= p_{n-1}(z)(z - z_n). \end{aligned}$$

THEOREM 4. *Let S be as above and $f(z)$ be analytic for $|z| < R$. Then $f(z)$ can be expressed in $|z| < R$ as*

$$a_0 + a_1(z - z_1) + a_2(z - z_1)(z - z_2) + a_3(z - z_1)(z - z_2)(z - z_3) + \cdots \tag{6}$$

if and only if the z_i are bounded (i.e., $\{z_n\}$ as a set is finite).

PROOF. Sato [2] showed that for every bounded set of $\{z_n\}$ (even if they have a limit point) such a representation is possible. On the other hand assume that z_n is unbounded, then one can find an entire function f which vanishes at z_n with the appropriate multiplicity. Such an f cannot be represented by (6), since all a_i in the series would have to vanish.

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