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A COUNTEREXAMPLE TO A CLASSIFICATION THEOREM OF LINEARLY STABLE POLYTOPES

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1. Introduction. We give an example of a centrally symmetric 5-polytope which is linearly stable though its vertices do not form a subset of the vertices of a 5-cube. This example contradicts the "only if" part of the classification theorem on linearly stable polytopes stated by P. McMullen [**2**]. Moreover the example gives a 5-polytope, the vertices of which form a subset of a 5-cube while its dual does not possess the same property.

I wish to thank Professor M. Perles for directing me on my Masters' thesis where this example arose.

2. Notation and lemmas. We shall follow the notation and definitions of Grünbaum [**1**] and McMullen [**2**]. We shall write c.s. for centrally symmetric. A c.s. polytope P is called *linearly stable* if every c.s. polytope which is combinatorially equivalent to P is linearly equivalent to P. The regular d-cube will be denoted by C^d and is the d-polytope with the 2^d vertices of the form $(\xi_1, \ldots, \xi_d)$ where $\xi_i = \pm 1$. Any d-polytope linearly equivalent to C^d will be called a d-cube. For any c.s. d-polytope P we shall denote its dual by P^*. The set of vertices of a polytope P will be denoted by vert P and the convex hull of a set A will be denoted by conv A.

The following lemmas are proved by McMullen [**2**]:

(1) LEMMA. *If P is a linearly stable d-polytope then its dual P^* is linearly stable.*

(2) LEMMA. *Let P be a* c.s. *d-polytope. Then there is a d-cube C such that* vert $P \subset$ vert C *if and only if among the facets of P there are d linearly independent facets each of which contains half the vertices of P.*

(3) LEMMA. *If P is a* c.s. *d-polytope such that* vert $P \subset$ vert C *for some d-cube C, then P is linearly stable.*

(Lemma (3) is the result of the "if" part of the main theorem in [**2**]).

As a consequence of (2) and basic properties of P^* we have:

(4) LEMMA. *Let P be a* c.s. *d-polytope. Then there is a d-cube C such that* vert $P \subset$ vert C *if and only if among the vertices of P^* there are d linearly independent vertices, each of which is contained in half the facets of P^*.*

McMullen [**2**] incorrectly states as part of his main theorem that if P is a linearly stable c.s. d-polytope, then vert $P \subset$ vert C for some d-cube C.

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3. Example. Let P be the c.s. 5-polytope with vertices $\pm\nu_i$, $1 \leqq i \leqq 7$ where

$$\nu_1 = (1, 1, 1, -1, 1) \qquad \nu_2 = (-1, 1, -1, -1, 1)$$
$$\nu_3 = (-1, -1, 1, -1, 1) \qquad \nu_4 = (-1, -1, -1, -1, 1)$$
$$\nu_5 = (1, -1, -1, -1, 1) \qquad \nu_6 = (-1, 1, -1, 1, 1)$$
$$\nu_7 = (1, -1, 1, 1, 1)$$

Then vert $P \subset$ vert C^5 and thus by (3), P is linearly stable.

It is easily verified that $F_1 = \text{conv}\ \{\nu_1, \nu_2, \nu_3, -\nu_5, \nu_6\}$ and $F_2 = \text{conv}\ \{-\nu_1, \nu_3, \nu_4, \nu_6, \nu_7\}$ are facets of P (The corresponding facet hyperplanes are given by the equations $-X_1 + X_2 + X_3 + X_5 = 2$ and $-X_1 - X_2 + X_4 + X_5 = 2$), and that

$$\nu_4, \nu_7 \notin F_1 \cup -F_1 \text{ and } \nu_2, \nu_5 \notin F_2 \cup -F_2.$$

Thus the vertices $\pm\nu_2$, $\pm\nu_4$, $\pm\nu_5$, and $\pm\nu_7$ are not contained in half the facets of P, and so there are no 5 linearly independent vertices of P each of which is contained in half the facets of P. By (4), we deduce that there is no 5-cube C for which vert $P^* \subset$ vert C. Hence we see, using (1), that P^* is linearly stable but its vertices do not form a subset of the vertices of any 5-cube.

References

1. B. Grünbaum, *Convex polytopes* (Wiley, New York, 1967).

2. P. McMullen, *Linearly stable polytopes*, Can. J. Math., *21* (1969), 1427–1431.

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