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INFLUENCE OF STRONGLY CLOSED 2-SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

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Abstract. Let $H \leq K$ be subgroups of a group G. We say that H is strongly closed in K with respect to G if whenever $a^g \in K$, where $a \in H, g \in G$, then $a^g \in H$. In this paper, we investigate the structure of a group G under the assumption that every subgroup of order 2^m (and 4 if $m = 1$) of a 2-Sylow subgroup S of G is strongly closed in S with respect to G. Some results related to 2-nilpotence and supersolvability of a group G are obtained. This is a complement to Guo and Wei (*J. Group Theory* **13**(2) (2010), 267–276).

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1. Introduction. All groups are finite. Let $H \leq K$ be subgroups of a group G. We say that H is *strongly closed* in K with respect to G if whenever $a \in H, a^g \in K$, where $g \in G$ then $a^g \in H$. We also say that H is strongly closed in G if H is strongly closed in $N_G(H)$ with respect to G. The structure of groups which possess a strongly closed p-subgroup has been extensively studied. One of the most interesting results is due to Goldschmidt [**4**] that classified groups with an abelian strongly closed 2-subgroup. This result is a generalization of the celebrated Glauberman Z^*-theorem. These results play an important role in the proof of the classification of the finite simple groups. Recently, Bianchi et al. in [**3**], called a subgroup H, an $\mathcal{H}$-*subgroup* of G if $H^g \cap N_G(H) \leq H$ for all $g \in G$. It is easy to see that these two definitions coincide. With this concept, they gave a new characterization of supersolvable groups in which normality is a transitive relation which are called supersolvable $\mathcal{T}$-groups. In more detail, it is shown that every subgroup of G is strongly closed in G if and only if G is a supersolvable $\mathcal{T}$-group (see [**3**, Theorem 10]). Some local versions of this result have been studied in [**1**] and [**7**]. For example, Asaad ([**1**, Theorem 1.1]) proved that G is p-nilpotent if and only if every maximal subgroup of a p-Sylow subgroup P of G is strongly closed in G, and $N_G(P)$ is p-nilpotent. Guo and Wei ([**7**, Theorem 3.1]) showed that whenever p is odd and P is a p-Sylow subgroup of G, G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and either P is cyclic or every non-trivial proper subgroup of a given order of P is strongly closed in G. Also these results still hold without the p-nilpotence assumption on $N_G(P)$ if p is the smallest prime divisor of the order of G. The purpose of this paper is to prove the following theorem, which is a complement to [**7**, Theorem 3.1].

THEOREM 1.1. *Let $P \in Syl_2(G)$ and $D \leq P$ with $1 < |D| < |P|$. If P is either cyclic or every subgroup of P of order $|D|$ (and 4 if $|D| = 2$) is strongly closed in G, then G is 2-nilpotent.*

The following example shows that the additional assumption when $|D| = 2$ in Theorem 1.1 is necessary.

EXAMPLE. Let $G = SL_2(17)$. If $P \in Syl_2(G)$ then $P \cong Q_{32}$, a quaternion group of order 32. Moreover, P is maximal in G and hence $N_G(P) = P$ is 2-nilpotent in G. Clearly, the centre of G is a unique subgroup of order 2 and so it is strongly closed in G. However, G is not 2-nilpotent.

Theorem 1.1 above and [**7**, Theorem 3.4] now yield:

THEOREM 1.2. *Let p be the smallest prime divisor of $|G|$ and $P \in Syl_p(G)$. If P is cyclic or P has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ (and 4 if $|D| = 2$) is strongly closed in G, then G is p-nilpotent.*

We can now drop the odd order assumption on Theorems 3.5 and 3.6 in [**7**].

THEOREM 1.3. *If every non-cyclic Sylow subgroup P of G has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ (and 4 if $|D| = 2$) is strongly closed in G, then G is supersolvable.*

THEOREM 1.4. *Let E be a normal subgroup of G such that G/E is supersolvable. If every non-cyclic Sylow subgroup P of E has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ (and 4 if $|D| = 2$) is strongly closed in G, then G is supersolvable.*

2. Preliminaries. In this section, we collect some results needed in the proofs of the main theorems.

LEMMA 2.1. (Schur–Zassenhauss [**6**, Theorem 6.2.1]). *If P is a normal 2-Sylow subgroup of G then G possesses a complement Hall-$2'$-subgroup.*

LEMMA 2.2. ([**6**, Theorem 7.6.1]). *If a 2-Sylow subgroup of G is cyclic then G is 2-nilpotent.*

LEMMA 2.3. ([**1**, Corollary 1.2]). *Let P be a 2-Sylow subgroup of G. Then G is 2-nilpotent if and only if every maximal subgroup of P is strongly closed in G.*

LEMMA 2.4. *Suppose that H is a strongly closed p-subgroup of G.*
(a) If $H \leq L \leq G$ then H is strongly closed in L;
(b) If $\bar{G}$ is a homomorphic image of G, then $\bar{H}$ is strongly closed in $\bar{G}$ and $N_{\bar{G}}(\bar{H}) = \overline{N_G(H)}$;
(c) If H is subnormal in G then $H \trianglelefteq G$.

Proof. (a) is [**3**, Lemma 7(2)] and (c) is [**3**, Theorem 6(2)]. Finally, (b) is [**5**, (2.2)(a)]. □

LEMMA 2.5. ([**5**, Corollary B3]). *Suppose that H is a strongly closed 2-subgroup of G and $N_G(H)/C_G(H)$ is a 2-group. Then $H \in Syl_2(\langle H^G\rangle)$.*

LEMMA 2.6. ([**8**, Satz 4.5.5]). *If every element of order 2 and 4 of G are central then G is 2-nilpotent.*

LEMMA 2.7. ([**2**, Baumann]). *If G is a non-abelian simple group in which a 2-Sylow subgroup of G is maximal, then G is isomorphic to $L_2(q)$, where q is a prime number of the form $2^m \pm 1 \geq 17$.*

A *component* of G is a subnormal quasi-simple subgroup of G. Denote by $E(G)$, the subgroup of G generated by all components of G. Then the *generalised Fitting subgroup* $F^*(G)$ of G is a central product of $E(G)$ and the Fitting subgroup $F(G)$ of G.

LEMMA 2.8. ([**9**, Theorem 9.8]). $C_G(F^*(G)) \leq F^*(G)$.

LEMMA 2.9. ([**9**, Problem 4D.4, p. 146]). *Let A act via automorphisms on a 2-group P, where $|A|$ is odd. If A centralises every element of order 2 and 4 in P, then A acts trivially on P.*

The following result is a special case of [**7**, Lemma 2.10].

LEMMA 2.10. *Let P be an elementary abelian 2-subgroup of G and D a subgroup of P with $1 < |D| < |P|$. If every subgroup of P of order $|D|$ is normal in G, then every minimal subgroup of P is central in G.*

Proof. It follows from [**7**, Lemma 2.10] that every minimal subgroup of P is normal in G. As minimal subgroups of P are cyclic of order 2, they are all central. □

LEMMA 2.11. *Let A be an odd order group acting on a 2-group P. Let $D \leq P$ with $1 < |D| < |P|$. If every subgroup of P of order $|D|$ (and 4 if $|D| = 2$) is A-invariant, then A acts trivially on P.*

Proof. We can assume that $|D| \geq 4$. Let $\mathcal{D} = \{E \leq P \,:\, |E| = |D|\}$. Suppose that $\langle \mathcal{D} \rangle < P$. If $|\langle \mathcal{D} \rangle| > |D|$, then by inductive hypothesis, A centralises $\langle \mathcal{D} \rangle$, so that it centralises every subgroup of P of order 2 and 4, hence the result follows from Lemma 2.9. If $|\langle \mathcal{D} \rangle| = |D|$, then P has a unique subgroup of order $|D|$. As $2 < |D| < |P|$, P must be cyclic and thus A centralises P by applying Lemma 2.2 to the semi-direct product $A \ltimes P$. Therefore, we can assume that $\langle \mathcal{D} \rangle = P$. Next, if A centralises every element of $\mathcal{D}$, then as $|D| \geq 4$, A centralises every element of order 2 and 4, and we are done by using Lemma 2.9. Hence there exists $E \in \mathcal{D}$ such that $[E, A] \neq 1$. It follows that $\Phi(P) \leq E$, otherwise, $E < E\Phi(P) < P$, and by applying the inductive hypothesis for $E\Phi(P)$, A would centralise E, which contradicts the choice of E, thus prove the claim. If $\Phi(P)$ is trivial, then P is elementary abelian, and hence the result follows from Lemma 2.10. Thus $\Phi(P) > 1$. Assume that $|E/\Phi(P)| \geq 2$. By Lemma 2.10 again, A centralises $P/\Phi(P)$, and then $[P, A] \leq \Phi(P)$. By Coprime Action Theorem, A acts trivially on P and we are done. Thus we assume that $E = \Phi(P)$. For any $F \in \mathcal{D} - \{\Phi(P)\}$, we have $|F| = |\Phi(P)|$ and $\Phi(P) \neq F$, it follows that $F < F\Phi(P) < P$ and $F\Phi(P)$ is A-invariant. By inductive hypothesis, A centralises F, and hence P, as P is generated by $\mathcal{D} - \{\Phi(P)\}$. The proof is now complete. □

3. Proofs of the main results.

PROPOSITION 3.1. *Let $P \in Syl_2(G)$ and $D \leq P$ with $2 < |D| < |P|$. Assume that either P is cyclic or every subgroup of P of order $|D|$ is strongly closed in G, then G is 2-nilpotent.*

Proof. Suppose that the proposition is false. Let G be a minimal counter example. By Lemma 2.2, we can assume that P is non-cyclic.

Claim 1. $O_{2'}(G) = 1$. Assume that $O_{2'}(G) \neq 1$. Passing to $\bar{G} = G/O_{2'}(G)$, we see that $\bar{G}$ satisfies the hypothesis of the proposition by Lemma 2.4(*b*), so that by inductive hypothesis, $\bar{G}$ is 2-nilpotent and hence G is 2-nilpotent.

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Claim 2. If $L \trianglelefteq G$ and $L \neq G$, then $L \leq O_2(G)$. Assume that L is a proper normal subgroup of G which is not a 2-group. As $L \trianglelefteq G$, PL is a subgroup of G. Assume that $PL \neq G$. By Lemma 2.4(a) and the inductive hypothesis, PL is 2-nilpotent. Let $Q = O_{2'}(PL)$. Then $1 \neq Q \leq L \trianglelefteq G$ and since Q is characteristic in L, we have $Q \trianglelefteq G$ and hence $Q \leq O_{2'}(G) = 1$ by Claim 2, which is a contradiction. Thus $G = PL$. Let $U = P \cap L$. Then $U \in Syl_2(L)$. Suppose that U is not maximal in P. Let P_1 be a maximal subgroup of P that contains U. By comparing the order, we see that P_1L is a proper subgroup of $PL = G$. Then by Lemma 2.3, $2 < |D| < |P_1|$ and so P_1L is 2-nilpotent by induction. Arguing as above, we obtain $1 \neq O_{2'}(P_1L) \leq L \trianglelefteq G$ and hence $O_{2'}(P_1L) \leq O_{2'}(G) = 1$. This contradiction shows that U is maximal in P. Now by Lemma 2.3 again, $2 < |D| < |U|$. By induction again, L is 2-nilpotent which leads to a contradiction as above. This proves our claim.

Claim 3. $N_G(P)$ is 2-nilpotent. If $N_G(P) < G$, then it is 2-nilpotent by induction and we are done. Thus assume that $N_G(P) = G$. Then $P \trianglelefteq G$ and hence every subgroup of P of order $|D|$ is both subnormal and strongly closed in G so that they are normal in G by Lemma 2.4(c). By Schur–Zassenhaus Theorem, there exists a subgroup A of odd order such that $G = PA$. Since every subgroup of P of order $|D|$ with $2 < |D| < |P|$ is A-invariant, by Lemma 2.11, A centralises P and hence G is 2-nilpotent, which contradicts our assumption.

Claim 4. $F^*(G) = O_2(G)$. As $O_{2'}(G) = 1$, we have $F^*(G) = O_2(G)E(G)$. Assume that $E(G) \neq 1$. By Claim 2, we have $E(G) = G$ and then by applying that claim again, we see that G must be a quasi-simple group. Let $H \leq P$ be any subgroup of order $|D|$. Assume first that $H \nleq Z(G)$. Then H is not normal in G so that $\langle H^G \rangle = G$ and $P \leq N_G(H) < G$. By induction, $N_G(H)$ is 2-nilpotent so that $N_G(H)/C_G(H)$ is a 2-group. By Lemma 2.5, $H \in Syl_2(G)$, which is a contradiction as $|H| < |P|$. Thus $H \leq Z(G)$ and since $|D| > 2$, every subgroup of order 2 or 4 is central in G, whence the result follows from Lemma 2.6.

The final contradiction. We first show that P is maximal in G. Let L be any maximal subgroup of G that contains P. By induction, $L = PO_{2'}(L)$. Since $O_2(G) \trianglelefteq L$, we obtain $[O_2(G), O_{2'}(L)] \leq O_2(G) \cap O_{2'}(L) = 1$, hence $O_{2'}(L) \leq C_G(O_2(G)) \leq O_2(G)$ by Lemma 2.8. Thus $O_{2'}(L) = 1$, which implies that P is maximal in G. Moreover by Claim 2, $O_2(G)$ is a maximal normal subgroup of G, and then $\bar{G} = G/O_2(G)$ is a simple group with a nilpotent maximal subgroup $P/O_2(G)$. Assume that $\bar{G}$ is non-solvable. Then by Lemma 2.7, $\bar{G} \cong L_2(q)$, where q is a prime of the form $2^m \pm 1 \geq 17$. Let $\bar{M}$ be the maximal subgroup of $L_2(q)$ which is isomorphic to the dihedral group D_{2s}, where $s > 1$ is odd. Let M, K and A be the full inverse images of $\bar{M}$, the 2-Sylow subgroup and the cyclic subgroup of order s of $\bar{M}$ in G. By Schur–Zassenhauss Theorem, $A = O_2(G)T$, where $|T| = s$. Also $O_2(G) \leq K \in Syl_2(M)$ and $M = KT$, where $A \trianglelefteq M$. We next show that $|D| \leq |O_2(G)|$. Assume false. Then $|O_2(G)| < |D|$. Now if $|O_2(G)| < |D|/2$ then $\bar{G}$ satisfies the hypothesis of Proposition 3.1 with $|\bar{D}| = |D|/|O_2(G)|$, and hence $\bar{G}$ is 2-nilpotent, contradicts the simplicity of G. Thus we can assume that $|O_2(G)| = |D|/2$. Let $H \leq P$ be such that $O_2(G) \leq H$ and $|\bar{H}| = |H/O_2(G)| = 2$. In this case, $P \leq N_G(H) < G$ and so $N_G(H) = P$ as P is maximal in G. By Lemma 2.4(b), we have $N_{\bar{G}}(\bar{H}) = \bar{P}$. Thus $1 \neq \bar{H}$ is strongly closed in $\bar{G}$ and $N_{\bar{G}}(\bar{H})$ is a 2-group. By Lemma 2.5, $\bar{H} = \bar{P} \in Syl_2(G)$ and so by Lemma 2.2, $\bar{G}$ is 2-nilpotent. This contradiction shows that $|D| \leq |O_2(G)|$. Therefore, $2 < |D| \leq |O_2(G)| < |K|$, where $K \in Syl_2(M)$. By induction again, $M = KT$ is 2-nilpotent and thus $O_{2'}(M) = T \trianglelefteq M$. Hence $T \leq C_G(O_2(G)) \leq O_2(G)$ and then $T = 1$, which contradicts the fact that $|T| = s > 1$. We conclude that

$\bar{G}$ is solvable. Thus $\bar{G}$ must be a cyclic subgroup of prime order. Clearly $|\bar{G}| > 2$, otherwise G is a 2-group. Let $r = |\bar{G}|$ and $R \in Syl_r(G)$. Then $G = O_2(G)R$ and $r > 2$, which implies that $P = O_2(G) \trianglelefteq G$, and hence $G = N_G(P)$ is 2-nilpotent by Claim 3. The proof is now complete. □

Proof of Theorem 1.1. If P is cyclic or $|D| > 2$ or $|D| = 2$ but $|P| > 2|D| = 4$ then the theorem follows from Proposition 3.1. Thus we can assume that P is non-cyclic, $|D| = 2$ and $|P| = 4$. It follows that every maximal subgroup of P is strongly closed in G, hence G is 2-nilpotent by Lemma 2.3. The proof is now complete.

Proof of Theorem 1.3. By Theorem 1.2, G possesses a Sylow tower of supersolvable type. Let p be the largest prime divisor of $|G|$. If $p = 2$, then G must be a 2-group and hence it is supersolvable. Assume that $p > 2$. The proof now proceeds as in that of Theorem 3.5 in **[7]**.

Proof of Theorem 1.4. By Lemma 2.4 and Theorem 1.3, E is supersolvable. Let p be the largest prime divisor of $|E|$. If $p > 2$, then the result follows as in Theorem 3.6 in **[7]**. Hence we can assume that $p = 2$ and so E is a 2-group. As G is supersolvable whenever G is a 2-group, we also assume that G is not a 2-group. Since G/E is supersolvable, it has a Sylow tower of supersolvable type and so G/E is 2-nilpotent. Let K/E be the normal $2'$-complement of G/E. By Schur–Zassenhauss Theorem, $K = EA$, where A is of odd order. Let $E \leq P \in Syl_2(G)$. Then $G = AP$, where $AE \trianglelefteq G$. As $|A|$ is odd, $E \in Syl_2(AE)$ and AE satisfies the hypothesis of Theorem 1.1 so that AE is 2-nilpotent. Hence $A = O_{2'}(AE) \trianglelefteq AE \trianglelefteq G$, and so $A \trianglelefteq G$. We have $G/A \cong P$ is supersolvable and by hypothesis, G/E is also supersolvable. Since the class of supersolvable groups is a saturated formation, we have $G/(A \cap E) \cong G$ is supersolvable. This completes the proof.

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