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A FINITE INDEX PROPERTY OF CERTAIN SOLVABLE GROUPS

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ABSTRACT. A group G is said to have the FINITE INDEX property (G is an FI-group) if, whenever $H \leq G$, $x^p \in H$ for some x in G and $p > 0$, then $|\langle H, x\rangle : H|$ is finite. Following a brief discussion of some locally nilpotent groups with this property, it is shown that torsion-free solvable groups of finite rank which have the isolator property are FI-groups. It is deduced from this that a finitely generated torsion-free solvable group has an FI-subgroup of finite index if and only if it has finite rank.

1. **Introduction.** Let us say that a group G has the FINITE INDEX property—or G is an FI-group—if, given any subgroup H and any element x in G such that $x^p \in H$ for some positive integer p, then H has finite index in $\langle H, x\rangle$. It is easily seen that an FI-group G has the isolator property; that is, the isolator of any subgroup H in G is itself a subgroup. The isolator of a subgroup H in G is, by definition, the set $\{x \in G; x^n \in H \text{ for some } n > 0\}$ and denoted by $\sqrt{H}$. However the converse to this is not true, since there are nilpotent groups without the finite index property: for instance, the wreath product G of an infinite elementary abelian p-group H by a cyclic group $\langle x\rangle$ of order p, a prime.

An alternate characterisation of FI-groups is given thus; if H, K are subgroups of G and A, B are subgroups of finite index in H, K respectively, then $|\langle H, K\rangle : \langle A, B\rangle|$ is finite. It is known (see e.g. [7], Lemma 3) that nilpotent groups of finite rank have this property. In fact, for groups of finite rank this property holds for any pair of subgroups H and K which are subnormal in their join ([7], Theorem 1). Using this one easily proves:

THEOREM 1. *The following are* FI-*groups.*

(i) *Baer groups with finite abelian section rank.*

(ii) *Locally nilpotent groups of finite abelian section rank whose torsion subgroups are reduced* (*i.e. have no quasicyclic subgroups*)

Note that the wreath product of a quasicyclic p-group by a cyclic group of order p is locally nilpotent of finite rank, but is not an FI-group. Thus

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"reduced" is necessary in part (ii) of the above theorem. Our main result here is

THEOREM 2. *Let G be a torsion-free solvable group of finite rank. Then G is an* FI-*group if it has the isolator property.*

Since FI-groups have the isolator property, the following result is an immediate consequence of Theorem C of [1].

THEOREM 3. *If a finitely generated torsion-free solvable group is an* FI-*group, then it has finite rank.*

Theorem 2 and Theorem B of [3] give:

THEOREM 4. *Let G be a torsion-free solvable group of finite rank. Then G has an* FI-*subgroup of finite index in G.*

Let π be any finite non-empty set of primes. Call G a *finite π-index* group if for all $H \leq G$ and $x \in G$ such that $x^n \in H$ for some π-number n, the index $|\langle H, x\rangle : H|$ is a finite π-number. In view of Theorem A of [4], a finitely generated solvable finite π-index group is finite-by-nilpotent with the finite subgroup being the direct product of its Hall π-subgroup and π'-subgroup. The converse is also true viz. a finitely generated finite-by-nilpotent group, with the finite subgroup being the direct product of a π-subgroup and π'-subgroup, is a finite π-index group.

Most of the standard notation used comes from [5]. We have already defined the term "isolator property" at the beginning of this section. A subgroup H of G is called isolated if $\sqrt{H} = H$. If H, K are subgroups of G then we write $H \sim K$ to mean $\sqrt{H} = \sqrt{K}$.

2. **Proofs.** Theorem 1 is easy to establish and we only give a sketch of the proof. We require two preliminary results, (the first of which is well known) before proceeding to the proof of Theorem 2. Theorems 3 and 4 require no formal proofs.

Sketch proof of Theorem 1. Suppose G belongs to one of the given classes. We may assume that $G = \langle H, x\rangle$ and $x^p \in H$.

If π denotes the set of primes which do not divide p, then H is π-isolated in G. Then factoring by the π-component of G, we obtain in either case a nilpotent group of finite rank (see [5, Chapter 6]) and hence an FI-group.

LEMMA 1. *Let A be a torsion-free abelian group of finite rank and θ a* 1–1 *endomorphism of A. then $A/\theta(A)$ is finite.*

This result is due to Fuchs. A proof appears in [6], 15.2.3.

LEMMA 2. *Let N be a torsion-free nilpotent group of finite rank and $\tau \in$* Aut(N), *the group of automorphisms of N. If $\langle N, \tau\rangle = G$ is torsion-free and has*

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the isolator property, $H \leq N$ *such that* H *is* $\langle \tau^p \rangle$ *invariant for some* $p > 0$, $H \sim N$ *and* $N = \langle H^{\langle \tau \rangle} \rangle$, *then* $|N : H| < \infty$.

Proof. Use induction on the Hirsch length $h(N)$ of N. There is nothing to prove if $h(N) = 0$. Let $h(N) = h$ and assume the result has been established in the case $h(N) < h$. Let M be a minimal τ-invariant isolated normal subgroup of N (such a subgroup exists since N satisfies the minimal condition on isolated subgroups). Clearly $M \leq Z(N)$ the centre of N. Let $K = H \cap M$, and let $M \otimes Q$, $K \otimes Q$ denote the tensor products over Z. Then the action of τ on $M \otimes Q = K \otimes Q$ is that of multiplication by an algebraic number, say t (see, for example, Lemma 3.1, page 546 of [2]). Since $\langle N, \tau \rangle$ satisfies the isolator property, M is also a minimal $\langle \tau^p \rangle$ invariant isolated subgroup of N. Thus the irreducible polynomials f, g for t and t^p have the same degree over Q and there exist integers $k_i \geq 1$, $i = 1, 2, \ldots, p-1$ such that $k_i t^i \in Z(t^p, t^{-p})$. If $k = \mathrm{lcm}(k_i; i = 1, 2, \ldots, p-1)$, then $kZ(t, t^{-1}) \subseteq Z(t^p, t^{-p})$. Hence for every $a \in K^{\langle \tau \rangle}$, $a^k \in K$. In particular, $K^{\langle \tau \rangle}/K$ has bounded exponent and hence $|K^{\langle \tau \rangle} : K|$ is finite.

Induction hypothesis enables us to assume that $|N : MH|$ is finite; otherwise replace N by N/M, H by MH/M and get a contradiction. Thus there is a finite set X such that $N = M\langle H, X \rangle$. Let $J = \langle H, X \rangle$. Since J is nilpotent of finite rank, $H \sim J$ and X is finite, $|J : H| = n$, say, is finite. Now let $J_1 = \langle x \in N, x^n \in H \rangle$. Then $J_1 \geq J$ and J_1 is $\langle \tau^p \rangle$ invariant. Since J_1 is nilpotent of finite rank, $|J_1 : H|$ is finite. Replace H by J_1 if necessary and assume $N = MH = \langle H^{\langle \tau \rangle} \rangle$. Then $N' = H'$ and any normal subgroup of H is also normal in N. Also $N = MH^{\tau^i}$ for all i so that $I = \bigcap_{i=1}^{p} H^{\tau^i} \geq N'$, N/I is a periodic abelian group of finite rank and $I \lhd G$. For any positive integer e, the subgroup $P(e)/I$ of N/I generated by elements whose exponent divide e, is finite and there exists a positive integer $\lambda = \lambda(e)$ such that τ^λ centralizes $P(e)/I$. Thus $[P(e), \tau^\lambda] \leq I \leq H$. In particular, for any element $z \in M$ such that $z^e \in I$, $[z, \tau^\lambda] = z^{(\tau^\lambda - 1)} \in K$. But $z^{f(\tau)} = 1$ as shown earlier in the proof. If $(t^\lambda - 1, f(t)) = 1$, then there exist polynomials u, v such that $(t^\lambda - 1)u(t) + f(t)v(t) = 1$. Thus $z^{(\tau^\lambda - 1)u(\tau)} z^{f(\tau)v(\tau)} = z$ and it follows that $z^{(\tau^\lambda - 1)u(\tau)} = z$. But $z^{(\tau^\lambda - 1)} \in K$ and so $z \in K^{\langle \tau \rangle}$, which is of finite index over K as shown earlier. The other alternative is that $(t^\lambda - 1, f(t)) \neq 1$ then $f(t)$ divides $t^\lambda - 1$ so that $z^{\tau^\lambda - 1} = 1$ and hence $[z, \tau] = 1$ since G has the isolator property (see [1], Proof of Theorem C, Step 1). But $[z, \tau] = 1$ for some $z \neq 1$ in M implies $M \leq Z(G)$ by our choice of M. Thus the alternatives are (i) $M \leq Z(G)$ or (ii) for every $z \in M$, $z \in K^{\langle \tau \rangle}$ and hence $M \leq K^{\langle \tau \rangle}$ which leads to $|MH : H| < \infty$ so we may suppose that (i) holds.

Let $R < N$ be a maximal normal isolated subgroup of G, properly contained in N. Let $S = H \cap R$. Since $S \lhd H$ and $M \leq Z(G)$, $S \lhd MH$. Thus $S^\tau \lhd MH$. But $MH = N = MH^\tau$, hence $S^{\langle \tau \rangle} \lhd N$. Let $S_1 = S^{\langle \tau \rangle}$. By induction hypothesis, S is of finite index in S_1. So we may assume that $H \geq S_1$. For any $h \in H \backslash S_1$, $h^\tau \equiv z_1 h^{u(\tau^p)} \bmod S_1$ where u is a suitable polynomial over Q, and $z_1 \in M$. This is so because H/S_1 is torsion-free abelian and every non-trivial $\langle \tau^p \rangle$ invariant subgroup of H/S_1 has the same rank as that of H/S_1 by our choice of R. Thus if l is the lcm of the denominators of the coefficients of u, then $h^{l\tau} \equiv zh^{v(\tau^p)}$ mod S_1 where $z = z_1^l$, and v is a polynomial with integer coefficients. Let $\mu = v(1)$, the sum of the coefficients of v. Then mod $S_1, h^{l^2\tau^2} \equiv z^l(zh^{v(\tau^p)})^{v(\tau^p)} \equiv z^l z^\mu h^{(v(\tau^p))^2}$; and $h^{l^p\tau^p} \equiv z^{l^{p-1}} z^{l^{p-2}\mu} \cdots z^{\mu^{p-1}} h^{(v(\tau^p))^p}$. Thus $z^\lambda \in K$ where $\lambda = l^{p-1} + \mu l^{p-2} + \cdots + \mu^{p-1}$. It follows that $H^\tau H/H$ is of exponent λ and being of finite rank, it is finite. From this it is easily deduced that N/H has finite exponent and is therefore finite.

Proof of Theorem 2. Let G be a torsion-free solvable group of finite rank with the isolator property and suppose $G = \langle H, \tau \rangle$, where $\tau^p \in H$. So $H \sim G$. We must show that $|G:H|$ is finite.

Reduction Step 1. By induction, we may assume that any such group with Hirsch length less than that of G has the finite index property. In particular if $M \neq 1$ is an isolated normal subgroup of G of minimal Hirsch length, we have that HM/M is of finite index in G/M, hence

$$(1.1) \qquad |G:HM| \text{ is finite.}$$

Since the result is trivial if G is abelian, we may assume that $M \leq J = \sqrt{G'}$. Write $K = H \cap J$. Then J is nilpotent, $M \leq Z(J)$ and $MH \cap J = MK$ is of finite index in J by 1.1. Thus $|J:N_J(K)|$ is finite and hence for some integer λ (depending on the nilpotency class of J) $[J, K]^\lambda \leq K$. Since J has finite rank, it follows that $|K^J:K|$ is finite. Now K^J is normalized by H since $K^{JH} = K^{HJ} = K^J$. Thus τ^p normalizes K^J. Moreover $K^{J\langle\tau\rangle} \leq J$ and $J \sim K$. Hence, by Lemma 2, K^J is of finite index in $K^{J\langle\tau\rangle}$. Writing $L = K^{J\langle\tau\rangle}$, we have that $L \lhd G$ and $|L:K|$ is finite and therefore $|HL:H|$ is finite. Replacing H by HL if necessary, we may assume that

$$(1.2) \qquad H \cap \sqrt{G'} \lhd G$$

Reduction Step 2. Let $C = \text{core}_G(MH)$, the largest normal subgroup of G contained in MH. Then G/C is finite, $M \leq C = M(H \cap C)$ and $H \cap \sqrt{G'} \leq C$. Also $H/H \cap C$ is finite. Let $K = \langle C \cap H, \tau \rangle$. Then $|K:C \cap H|$ is infinite; for suppose this were not so, and let $L = \text{Core}_K(C \cap H)$. Then $H' \leq H \cap \sqrt{G'} \leq L \leq H$, and so $L \lhd H$ and hence $L \lhd \langle H, K \rangle = G$. Since $|K:C \cap H|$ is supposed finite, $|C \cap H:L|$ is also finite. Thus H/L is a finite subgroup of $G/L = \langle H, \tau \rangle / L$ which is therefore a finitely generated periodic solvable group and hence finite, a contradiction. We may therefore replace H by $C \cap H$ if necessary and assume

$$(2.1) \qquad H^G \leq HM$$

Final Step. For any $h \in H$, $[h, \tau] = zh_1$, for some $z \in M$ and $h_1 \in (H \cap \sqrt{G'})$ since $[h, \tau] \in \sqrt{G'} \cap MH = M(\sqrt{G'} \cap H)$. Moreover $[h, \tau^p] = [h, \tau][h, \tau]^\tau \cdots [h, \tau]^{\tau^{p-1}} \equiv zz^\tau \cdots z^{\tau^{p-1}} \text{mod}(H \cap \sqrt{G'})$. But $\tau^p \in H$, so $[h, \tau^p] \in$

$H' \leq H \cap \sqrt{G'} \lhd G$ by (1.2). Thus $zz^{\tau} \cdots z^{\tau^{p-1}} \in H \cap \sqrt{G'} \cap M = H \cap M$. Since $H \cap \sqrt{G'}$
also in $H \cap M$. Now define E_1 to be the set $M \cap HH^{\tau} = \{M \cap H[h, \tau]; h \in H\}$. We have just seen that for any $z \in E_1$, $zz^{\tau} \cdots z^{\tau^{p-1}}$ and all its conjugates are in $H \cap M$. Hence for any $x \in E = \langle E_1^G \rangle$, $xx^{\tau} \cdots x^{\tau^{p-1}}$ is in $H \cap M$. Let θ be the map $x \to xx^{\tau} \cdots x^{\tau^{p-1}}$, $x \in E$. Then θ is an endomorphism of E and $\theta(E) \leq H \cap E$. Also $\theta(x) = 1$ implies $x^{\tau^p} = x$ and since G satisfies the isolator property, it is an R-group (see [1], Proof of Theorem C, Step 1) so that $x^{\tau} = x$ and $\theta(x) = x^p = 1$. But as G is torsion-free, $x = 1$. Thus θ is a 1–1 and since $E \cap M$ is abelian of finite rank, $E/\theta(E)$ is finite by Lemma 1.

Now $HH^{\tau} \subseteq \langle H^{\langle \tau \rangle} \rangle \leq HM$ (by 2.1). Hence $HH^{\tau} = HH^{\tau} \cap HM = H(HM \cap H^{\tau}) \subseteq HE$. Likewise, $HH^{\tau}H^{\tau^2} = H(HH^{\tau})^{\tau} \subseteq HH^{\tau}E \subseteq HEE = HE$. And inductively, $HH^{\tau} \cdots H^{\tau^{p-1}} \subseteq HE$. Thus $\langle HH^{\tau} \cdots H^{\tau^{p-1}} \rangle / H \leq HE/H \cong E/H \cap E$, which is finite since it is a quotient of $E/\theta(E)$. Hence H^G/H is finite and hence $|G:H|$ is finite.

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