# Note on a Function of two Integral Arguments.

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1. In the part just issued of the *American Journal of Mathematics* (vii, pp. 3, 4) Professor Cayley has occasion to deal with the function

$$\left\{i, j\right\} \equiv \left[i\right]^j + \frac{j}{1}\left[i\right]^{j-1} + \frac{j(j-1)}{1.2}\left[i\right]^{j-2} + \ldots + 1$$

where $\left[i\right]^j$ stands for $i\,(i-1)\ldots(i-j+1)$. He points out that

$$\left\{i, j\right\} = \left\{j, i\right\},$$

expresses $\{i, 1\}$, $\{i, 2\}$, ... $\{i, 5\}$ each in descending powers of $i$, and tabulates the function from $\{1, 1\}$ to $\{5, 5\}$.

2. The series which defines the function may be looked on as a sum of terms, each of which is a product of a number of combinations ($C_{m,r}$) and a number of permutations ($P_{n,m-r}$): that is to say, we may write the definition as follows—

$$\left\{m, n\right\} \equiv P_{m,n} + C_{n,1}P_{m,n-1} + C_{n,2}P_{m,n-2} + \ldots\ldots + 1.$$

Now, multiplying both sides by $m+1$, and bearing in mind that of course

$$(m+1)P_{m,n-r} = P_{m+1,n-r+1},$$

we have

$$(m+1)\left\{m, n\right\} = P_{m+1,n+1} + C_{n,1}P_{m+1,n} + C_{n,2}P_{m+1,n-1} + \ldots\ldots + (m+1).$$

But by definition

$$\left\{m+1, n+1\right\} = P_{m+1,n+1} + C_{n+1,1}P_{m+1,n} + C_{n+1,2}P_{m+1,n-1} + \ldots\ldots + 1,$$

and therefore by subtraction and using the fact that

$$C_{n+1,r} - C_{n,r} = C_{n,r-1}$$

we have

$$\left\{m+1, n+1\right\} - (m+1)\left\{m, n\right\} = P_{m+1,n} + C_{n,1}P_{m+1,n-1} + \ldots + 1$$
$$= \left\{m+1, n\right\}:$$

and hence, as our law of recurrence,

$$\left\{m+1, n+1\right\} = (m+1)\left\{m, n\right\} + \left\{m+1, n\right\}. \qquad \text{(I.)}$$

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3. Now, since $\left\{m,\ 1\right\} = m+1$ the first line of our table is

$$2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots;$$

then from (I) $2{\cdot}2+3$, $3{\cdot}3+4$, $4{\cdot}4+5$, $5{\cdot}5+6$, ... give us the items of the second line, viz.

$$7,\ 13,\ 21,\ 31,\ 43,\ \ldots;$$

similarly $3{\cdot}7+13$, $4{\cdot}13+21$, $5{\cdot}21+31$, ... give us the items of the third line, viz.

$$34,\ 73,\ 136,\ 229,\ \ldots;$$

and so on; the table thus being

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | | 7 | 13 | 21 | 31 | 43 |
| 3 | | | 34 | 73 | 136 | 229 |
| 4 | | | | 209 | 501 | 1045 |
| 5 | | | | | 1546 | 4051 |
| 6 | | | | | | 13327 |

4. Further, the following properties of the function may be noted; several of them, especially the last three, may be used as a check upon the preceding process of calculation.

Since $\left\{m,\ n\right\} = m\left\{m-1,\ n-1\right\} + \left\{m, n-1\right\}$

and $\left\{n,\ m\right\} = n\left\{n-1,\ m-1\right\} + \left\{n, m-1\right\}$,

therefore by subtraction and division

$$\frac{\left\{n, m-1\right\} - \left\{m, n-1\right\}}{m-n} = \left\{m-1,\ n-1\right\}. \qquad \text{(II.)}$$

Similarly we have

$$\frac{m\left\{n,\ m-1\right\} - n\left\{m,\ n-1\right\}}{m-n} = \left\{m,\ n\right\}. \qquad \text{(III.)}$$

And consequently from these two

$$m\left\{m-1,\ n\right\} + \left\{m+1,\ n\right\} = n\left\{n-1,\ m\right\} + \left\{n+1, m\right\}; \text{(IV.)}$$

that is to say, $m\left\{m-1, n\right\} + \left\{m+1, n\right\}$ is symmetric with respect to $m$ and $n$, a fact which may also be seen from noting that either side of (IV) is equal to

$$\left\{m, n+1\right\} - \left\{m, n\right\} + \left\{m+1, n\right\}.$$

Again, since from (I) we have

$$\left\{n+2, n\right\} = \left\{n+2, n+1\right\} - (n+2)\left\{n+1, n\right\}$$

$$\left\{n+1, n+2\right\} = (n+1)\left\{n, n+1\right\} + \left\{n+1, n+1\right\}$$

and $$\left\{n+1, n+1\right\} = (n+1)\left\{n, n\right\} + \left\{n+1, n\right\},$$

therefore by addition

$$\left\{n+2, n\right\} = (n+1)\left\{n, n\right\} : \qquad \text{(V.)}$$

that is to say, *the numbers in the third diagonal of the above table are known multiples of the corresponding numbers in the first diagonal.*

Again from (II) by writing $2n$ for $m$, and $n+1$ for $n$, we have

$$\left\{n+1, 2n-1\right\} - \left\{2n, n\right\} = (n-1)\left\{2n-1, n\right\}$$

and from (I)

$$\left\{n, 2n\right\} = n\left\{n-1, 2n-1\right\} + \left\{n, 2n-1\right\}$$

therefore by addition

$$\left\{2n-1, n+1\right\} = n\left[\left\{2n-1, n\right\} + \left\{2n-1, n-1\right\}\right] \text{(VI.)}$$

that is to say, *in every odd column of the table there is one item which is a known multiple of the sum of the preceding pair of items.*

Also, *the first item of the said pair is the arithmetic mean of the item immediately to the right and the item immediately above the latter.* (VII.)

This follows from (III) by putting $m = 2n$ and dividing by $n$.

5. Lastly, it may be remarked that the function $\{m, n\}$ viewed as one whose arguments proceed not by finite differences but by differentials is a case of Gauss' function $F(\alpha, \beta, \gamma, x)$, viz., that case where $\alpha = -m$, $\beta = -n$, $\gamma = x = \infty$.

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