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Level compatibility in Sharifi's conjecture

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Abstract. Romyar Sharifi has constructed a map ϖ_M from the first homology of the modular curve $X_1(M)$ to the K-group $K_2(\mathbf{Z}[\zeta_M + \zeta_M^{-1}, \frac{1}{M}]) \otimes \mathbf{Z}[1/2]$, where ζ_M is a primitive Mth root of unity. Sharifi conjectured that ϖ_M is annihilated by a certain Eisenstein ideal. Fukaya and Kato proved this conjecture after tensoring with $\mathbf{Z}_p$ for a prime $p \geq 5$ dividing M. More recently, Sharifi and Venkatesh proved the conjecture for Hecke operators away from M. In this note, we prove two main results. First, we give a relation between ϖ_M and $\varpi_{M'}$ when $M' \mid M$. Our method relies on the techniques developed by Sharifi and Venkatesh. We then use this result in combination with results of Fukaya and Kato in order to get the Eisenstein property of ϖ_M for Hecke operators of index dividing M.

1 Introduction and notation

Sharifi [8] has constructed a beautiful and explicit map (1.1) between modular symbols and a cyclotomic K-group. This map is conjecturally annihilated by a certain Eisenstein ideal. This conjecture, despite its apparent simplicity, turns out to be highly nontrivial and has led to much work in recent years, in particular by Fukaya and Kato [4] and more recently by Sharifi and Venkatesh [9].

This paper is devoted to the study of certain norm relations satisfied by Sharifi's map. This aspect has been studied before by Fukaya–Kato and Scott [11]. Their results are, however, quite restrictive (*cf.* Remark 1.5 for a detailed comparison between their results and ours). We use the techniques developed by Sharifi and Venkatesh to remove most of these restrictions.

Our main motivation is to apply the results of the present note to obtain results toward the Birch and Swinnerton–Dyer conjecture in the "Eisenstein" case [6]. We now set up some notation and describe our results in details.

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1.1 Homology of modular curves and Hecke operators

Let $M \geq 4$ be an integer. Let

$$\Gamma_1(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \text{ such that } a-1 \equiv c \equiv 0 \text{ (modulo } M) \right\},$$

and denote by $X_1(M)$ the compact modular curve (over $\mathbf{C}$) of level $\Gamma_1(M)$. Let

$$C_M = \Gamma_1(M) \backslash \mathbf{P}^1(\mathbf{Q})$$

be the set of cusps of $X_1(M)$, and let C_M^0 be those cusps in C_M of the form $\Gamma_1(M) \cdot \frac{a}{b}$ with $\gcd(a,b) = 1$ and $a \not\equiv 0$ (modulo M) (in the case $b=0$, we have the cusp $\Gamma_1(M) \cdot \infty$).

Let $H_1(X_1(M), C_M, \mathbf{Z})$ be the singular homology of $X_1(M)$ relative to C_M. If α and β are in $\mathbb{P}^1(\mathbf{Q})$, let

$$\{\alpha, \beta\} \in H_1(X_1(M), C_M, \mathbf{Z})$$

be the class of the hyperbolic geodesic from α to β in $X_1(M)$. The group $H_1(X_1(M), C_M, \mathbf{Z})$ carries an action of various Hecke operators, which we now recall.

If ℓ is a prime number, the ℓth Hecke operator T_ℓ is the double coset operator $\Gamma_1(M) \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \Gamma_1(M)$. As usual, we denote T_ℓ by U_ℓ is ℓ divides M. The Atkin–Lehner involution W_M is the involution of $H_1(X_1(M), C_M, \mathbf{Z})$ induced by the map $z \mapsto -\frac{1}{Mz}$ of the upper half-plane.

For any $x \in (\mathbf{Z}/M\mathbf{Z})^\times$, we denote by $\langle x \rangle$ the corresponding diamond operator, which is the automorphism of $X_1(M)$ induced by the action of any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$ such that

$$d \equiv x \text{ (modulo } M) .$$

This gives by functoriality an action of $(\mathbf{Z}/M\mathbf{Z})^\times / \pm 1$ on $H_1(X_1(M), C_M, \mathbf{Z})$. Note that diamond operators act on the set of cusps C_M and that this action preserves C_M^0. We say that two cusps c and c' are in the *same diamond orbit* if there exists $x \in (\mathbf{Z}/M\mathbf{Z})^\times$ such that $\langle x \rangle \cdot c = c'$.

There are also *dual Hecke operators*: if T is one of the operators defined above, we let $T^* = W_M^{-1} T W_M$. As is well known (*cf.* [1, Theorem 5.5.3]), we have $\langle x \rangle^* = \langle x \rangle^{-1}$ (for all $x \in (\mathbf{Z}/M\mathbf{Z})^\times$) and $T_\ell^* = \langle \ell \rangle^{-1} T_\ell$ (for all primes $\ell \nmid M$).

1.2 (Dual) Manin symbols

Let $\xi_M : \mathbf{Z}[\Gamma_1(M) \backslash \mathrm{SL}_2(\mathbf{Z})] \to H_1(X_1(M), C_M, \mathbf{Z})$ be the (modified) Manin map: it sends a coset $\Gamma_1(M) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\{-\frac{d}{Mb}, -\frac{c}{Ma}\}$ (it is the usual Manin map sending

$\Gamma_1(M) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\{\frac{b}{d}, \frac{a}{c}\}$ composed with the Atkin–Lehner involution W_M). Manin showed that $W_M \circ \xi_M$ is surjective, and therefore ξ_M is surjective (*cf.* [7, Section 1.6]).

Let $S_M^0 \subset \Gamma_1(M) \backslash \mathrm{SL}_2(\mathbf{Z})$ be the subset consisting of $\Gamma_1(M) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $M \nmid c$ and $M \nmid d$. The restriction

$$\xi_M^0 : \mathbf{Z}[S_M^0] \to H_1(X_1(M), C_M^0, \mathbf{Z})$$

is surjective (*cf.* [3, Section 2.1.3]).

1.3 Algebraic K-theory and motivic cohomology

Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. For any integer $M \geq 4$, choose a primitive Mth root of unity $\zeta_M \in \overline{\mathbf{Q}}$ such that for all $M' \mid M$, we have $\zeta_{M'} = \zeta_M^{M/M'}$.

We have a canonical group isomorphism

$$(\mathbf{Z}/M\mathbf{Z})^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbf{Q}(\zeta_M)/\mathbf{Q})$$

sending $a \in (\mathbf{Z}/M\mathbf{Z})^\times$ to the Galois automorphism characterized by $\zeta_M \mapsto \zeta_M^a$. The complex conjugation of $\mathrm{Gal}(\mathbf{Q}(\zeta_M)/\mathbf{Q})$ corresponds to the class of -1 in $(\mathbf{Z}/M\mathbf{Z})^\times$ under that isomorphism.

If A is a commutative ring, let $K_2(A)$ be the second K-group of A, as defined by Quillen. For any $x, y \in A^\times$, there is an element $\{x, y\}$ of $K_2(A)$, called the *Steinberg symbol* of x and y. It is bilinear in x and y and has the property that if $x + y = 1$, then $\{x, y\} = 1$.

There is an action of $\mathrm{Gal}(\mathbf{Q}(\zeta_M)/\mathbf{Q})$ (and in particular of the complex conjugation) on $K_2(\mathbf{Z}[\zeta_M, \frac{1}{M}])$. We denote by $\mathcal{K}_M$ the largest quotient of $K_2(\mathbf{Z}[\zeta_M, \frac{1}{M}]) \otimes \mathbf{Z}[\frac{1}{2}]$ on which the complex conjugation acts trivially. Note that $\mathcal{K}_M$ is a $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$-module.

1.4 Sharifi's ϖ_M map and summary of known results

The map $\mathbf{Z}[S_M^0] \to \mathcal{K}_M$ given by

$$\Gamma_1(M) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \{1 - \zeta_M^c, 1 - \zeta_M^d\}$$

factors through ξ_M^0 (*cf.* [3, Section 2.1.4]), and thus induces a map

$$\varpi_M : H_1(X_1(M), C_M^0, \mathbf{Z}) \to \mathcal{K}_M . \tag{1.1}$$

Let us note that our map ϖ_M is, in the notation of [9, Proposition 4.3.2], equal to $\Pi_M^\circ \circ W_M$. Sharifi made the following conjecture.

Conjecture 1.1 (Sharifi) *The restriction of ϖ_M to $H_1(X_0(M), \mathbf{Z})$ is annihilated by the Hecke operators $T_\ell - \ell\langle\ell\rangle - 1$ for primes ℓ not dividing M and by the Hecke operators $U_\ell - 1$ for primes $\ell \mid M$.*

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This is equivalent to [9, Conjecture 4.3.5(a)], where the authors use dual Hecke operators but use $\Pi_M^\circ = \varpi_M \circ W_M$ instead of ϖ_M. As Sharifi and Venkatesh mention right after [9, Theorem 4.3.6], it is expected that the conjecture holds without restricting ϖ_M to $H_1(X_0(M), \mathbf{Z})$.

This conjecture has a history of partial results: [4, 5] and most recently [9]. Let us recall the main results of Sharifi–Venkatesh and Fukaya–Kato on this conjecture.

Theorem 1.2 (Sharifi–Venkatesh) *The restriction of ϖ_M to $H_1(X_0(M), \mathbf{Z})$ is annihilated by the Hecke operators $T_\ell - \ell\langle\ell\rangle - 1$ for primes ℓ not dividing M.*

This follows from [9, Theorem 4.3.7]. Therefore, to prove Conjecture 1.1, it only remains to consider the Hecke operators $U_\ell - 1$ for primes $\ell \mid M$. Fukaya–Kato do get a result including $U_\ell - 1$, but they have to tensor with $\mathbf{Z}_p$ where $p \geq 5$ is a prime dividing M.

Theorem 1.3 (Fukaya–Kato) *Let $p \geq 5$ be a prime dividing M. The map*

$$\varpi_M \otimes 1 : H_1(X_1(M), C_M^0, \mathbf{Z}_p) \to \mathcal{K}_M \otimes \mathbf{Z}_p$$

is annihilated by the Hecke operators $T_\ell - \ell\langle\ell\rangle - 1$ for primes ℓ not dividing M and by the Hecke operators $U_\ell - 1$ for primes $\ell \mid M$.

We refer to [4, Theorem 5.2.3(1)] for this result. Let us note that Fukaya and Kato actually consider (the p-ordinary part of) $H^1(Y_1(M), \mathbf{Z}_p)$ instead of $H_1(X_1(M), C_M, \mathbf{Z}_p)$. These two groups are canonically isomorphic, but the isomorphism transfers *dual* Hecke operators (*i.e.*, T_ℓ^*, U_ℓ^* or $\langle x\rangle^{-1}$) to *usual* Hecke operators (*i.e.*, T_ℓ, U_ℓ or $\langle x\rangle$).

1.5 Our main results

Another important aspect of Sharifi's theory is the way in which the maps ϖ_M relate with each other when varying M. This has been studied under some assumptions in [4, 11]. If p is a prime, there are two degeneracy maps $\pi_1, \pi_2 : X_1(Mp) \to X_1(M)$ given on the upper half-plane by $\pi_1 : z \mapsto z$ and $\pi_2 : z \mapsto pz$. On the K-side, there is a norm map $\mathrm{Norm} : \mathcal{K}_{Mp} \to \mathcal{K}_M$. Our main result is the following.

Theorem 1.4 *Let $p \geq 2$ be a prime number, and let $M \geq 4$. Let $C \subset C_{Mp}^0$ be a subset of cusps which are all in the same orbit under the action of* $\mathrm{Ker}((\mathbf{Z}/Mp\mathbf{Z})^\times \to (\mathbf{Z}/M\mathbf{Z})^\times)$ *(the action being given by diamond operators as recalled above).*

(i) *Assume that p divides M. We have a commutative diagram*

$$\begin{array}{ccc} H_1(X_1(Mp), C, \mathbf{Z}) & \xrightarrow{\varpi_{Mp}} & \mathcal{K}_{Mp} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \mathrm{Norm}} \\ H_1(X_1(M), \mathbf{Z}) & \xrightarrow{\varpi_M} & \mathcal{K}_M \,. \end{array}$$

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(ii) *Assume that p does not divide M. We have a commutative diagram*

$$\begin{array}{ccc} H_1(X_1(Mp), C, \mathbf{Z}) & \xrightarrow{\varpi_{Mp}} & \mathcal{K}_{Mp} \\ \downarrow{\scriptstyle \pi_1 - \langle p \rangle \pi_2} & & \downarrow{\scriptstyle \text{Norm}} \\ H_1(X_1(M), \mathbf{Z}) & \xrightarrow{\varpi_M} & \mathcal{K}_M . \end{array}$$

Here, $\langle p \rangle$ is the pth diamond operator, induced by the action of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$ with $d \equiv p$ (modulo M) on $X_1(M)$.

Remark 1.5 (i) Theorem 1.4(i) has been proved by Fukaya and Kato in [4, Theorem 5.2.3(2)] after tensoring by $\mathbf{Z}_p$ for $p \geq 5$. They use $H^2(G_{\mathbf{Q}(\zeta_M)}, \mathbf{Z}_p(2))$ instead of $\mathcal{K}_M \otimes \mathbf{Z}_p$. The étale Chern class map (*cf.* [10]) provides an isomorphism

$$K_2\left(\mathbf{Z}\left[\zeta_M, \frac{1}{M}\right]\right) \otimes \mathbf{Z}_p \simeq H^2_{\text{ét}}\left(\mathbf{Z}\left[\zeta_M, \frac{1}{M}\right], \mathbf{Z}_p(2)\right) .$$

Since $H^2_{\text{ét}}(\mathbf{Z}[\zeta_M, \frac{1}{M}], \mathbf{Z}_p(2))$ is a subgroup $H^2(G_{\mathbf{Q}(\zeta_M)}, \mathbf{Z}_p(2))$ and $\mathcal{K}_M$ is identified with the fixed part by the complex conjugation in $K_2(\mathbf{Z}[\zeta_M, \frac{1}{M}]) \otimes \mathbf{Z}_p$, we get a canonical embedding

$$\mathcal{K}_M \otimes \mathbf{Z}_p \hookrightarrow H^2(G_{\mathbf{Q}(\zeta_M)}, \mathbf{Z}_p(2)) .$$

Fukaya and Kato's map actually takes values in $\mathcal{K}_M$ (by construction). They also do not need to restrict to the subset C of C^0_{Mp}. Their techniques rely on p-adic Hodge theory.

(ii) Similarly, Theorem 1.4(ii) has been proved (for the absolute homology) by Scott in [11, Theorem 7] after tensoring by $\mathbf{Z}_\ell$ for a prime $\ell \neq p$ dividing M (Scott's p is our ℓ and vice versa). Scott relies on the techniques of Fukaya and Kato.

(iii) Thus, the main novelty of our result is to work with $\mathbf{Z}$ coefficients. This is because we rely instead on the motivic techniques of Sharifi and Venkatesh.

(iv) It would be interesting to allow less restrictive conditions on C, and in particular replace $H_1(X_1(M), \mathbf{Z})$ in the bottom line of our diagrams by a relative homology group. We were actually able to improve slightly our result when C contains the cusp ∞ (*cf.* diagrams (5.7) and (5.11)). We were not able to go beyond these results because the techniques of Sharifi and Venkatesh essentially deal with the absolute homology of modular curves.

(v) The techniques of Sharifi and Venkatesh, combined with the result of Section 4 actually show that the restriction of ϖ_M to $H_1(X_1(M), C_\infty, \mathbf{Z})$ is annihilated by $T_\ell - \ell\langle \ell \rangle - 1$ for primes ℓ not dividing M, where C_∞ are the cusps of $X_1(M)$ in the same diamond orbit as ∞. This is a slight improvement on Theorem 1.2 (which holds for the restriction of ϖ_M to $H_1(X_1(M), \mathbf{Z})$).

By combining Theorem 1.4 and the results of Fukaya and Kato, one gets the following result.

Theorem 1.6 *Let $M \geq 4$. The map $H_1(X_1(M), \mathbf{Z}[\frac{1}{6}]) \to \mathcal{K}_M \otimes \mathbf{Z}[\frac{1}{6}]$ obtained by restricting ϖ_M to $H_1(X_1(M), \mathbf{Z})$ and inverting 6 is annihilated by the Hecke operator $U_\ell - 1$ for all primes ℓ dividing M. Here, U_ℓ is the classical Hecke operator of index ℓ, corresponding to the double coset of $\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}$.*

Remark 1.7 (i) As mentioned above, Theorem 1.6 goes beyond Theorem 1.2. Our result thus completes the proof of Conjecture 1.1 for the absolute homology, after inverting 6.

(ii) Fukaya and Kato proved Theorem 1.6 after tensoring with $\mathbf{Z}_p$ for a prime $p \geq 5$ dividing M (*cf.* Theorem 1.3). Our trick is to use Theorem 1.3 after adding p to the level, and then descend using Theorem 1.4(ii). The reason we have to invert 6 is that Fukaya and Kato assume that $p \nmid 6$ (note that 2 is inverted anyway in the definition of ϖ_M). It would be nice to be able to avoid inverting 3 in our result.

The plan of this paper is as follows: in Section 2, we recall some basic facts and notation about various kinds of homology and cohomology groups. In Section 3, we recall some constructions of Sharifi and Venkatesh. In Section 4, we explain how to use the cocycle of Sharifi and Venkatesh to produce a map on a certain relative homology group. Finally, in Section 5, we prove Theorems 1.4 and 1.6.

2 Background and notation regarding homology and cohomology

Let Γ be a torsion-free finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$ (*e.g.*, $\Gamma = \Gamma_1(M)$ for $M \geq 3$). Let $Y = \Gamma \backslash \mathfrak{h}$ be the open modular curve of level Γ, where $\mathfrak{h}$ is the upper half-plane. We denote by X the corresponding compactified modular curve: we have $X = Y \cup C$ where $C = \Gamma \backslash \mathbf{P}^1(\mathbf{Q})$ is the set of cusps of X.

We denote by $H_1(X, C, \mathbf{Z})$ the first singular homology group of X relative to C. We have the following exact sequence coming from the long exact sequence for the pair (X, C):

(2.1) $$0 \to H_1(X, \mathbf{Z}) \to H_1(X, C, \mathbf{Z}) \to \mathbf{Z}[C] \to \mathbf{Z} \to 0,$$

where the map $\mathbf{Z}[C] \to \mathbf{Z}$ is the degree map.

The Poincaré duality yields a perfect bilinear pairing $H_1(X, C, \mathbf{Z}) \times H_1(Y, \mathbf{Z}) \to \mathbf{Z}$ (also called the *intersection pairing*, due to its interpretation in terms of intersection number of cycles). Under this duality, (2.1) becomes

(2.2) $$0 \to \mathbf{Z} \to \mathbf{Z}[C] \to H_1(Y\, \mathbf{Z}) \to H_1(X, \mathbf{Z}) \to 0,$$

where the map $\mathbf{Z}[C] \to H_1(Y\, \mathbf{Z})$ sends $c \in C$ to the class of a little oriented circle around c.

If G is a group and T is a left G-module, we say that $c : G \to T$ is a 1-cocycle if for all $g, g' \in G$ we have $c(gg') = c(g) + g \cdot c(g')$. The first cohomology group $H^1(G, T)$ can be computed as the abelian group of 1-cocycles $c : G \to T$ modulo the cocycles of the form $c(g) = gx - x$ for some $x \in T$ (independent of g).

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Similarly, using the projective resolution of $\mathbf{Z}$ as a $\mathbf{Z}[G]$-module in terms of inhomogeneous chains, one can compute the first homology group $H_1(G,T)$ as

$$H_1(G,T) = Z/B, \tag{2.3}$$

where $Z \subset \mathbf{Z}[G] \otimes_{\mathbf{Z}} T$ is the kernel of the map sending $[g] \otimes x$ to $g^{-1}x - x$ (for $g \in G$ and $x \in T$) and B is generated by the elements of the form $[gg'] \otimes x - [g] \otimes x - [g'] \otimes g^{-1}x$ for $g, g' \in G$ and $x \in T$.

Finally, let us recall that since Γ is torsion-free, it is isomorphic to the fundamental group of Y, and therefore we have canonical group isomorphisms

$$H_1(\Gamma, \mathbf{Z}) \simeq \Gamma^{ab} \simeq H_1(Y, \mathbf{Z}) . \tag{2.4}$$

3 Reminders from the work of Sharifi and Venkatesh

Sharifi and Venkatesh constructed a 1-cocycle

$$\Theta : \mathrm{SL}_2(\mathbf{Z}) \to \mathbf{K}_2/\mathbf{Z}\cdot\{-z_1, -z_2\} .$$

Here,

$$\mathbf{K}_2 := K_2(\mathbf{Q}(\mathbf{G}_m^2)) = K_2(\mathbf{Q}(z_1, z_2))$$

carries a left action of $\mathrm{SL}_2(\mathbf{Z})$ induced by the natural right action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{G}_m^2$ given by

$$(z_1, z_2)\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(z_1^a z_2^c, z_1^b z_2^d\right) .$$

Furthermore, $\mathbf{Z}\cdot\{-z_1, -z_2\}$ is the subgroup of $\mathbf{K}_2$ generated by the Steinberg symbol of $-z_1$ and $-z_2$. The cocycle Θ actually takes values in $\mathbf{K}_2^{(0)}/\{-z_1, -z_2\}$, where $\mathbf{K}_2^{(0)}$ is the subgroup of $\mathbf{K}_2$ fixed by the pushforward $[m]_*$ of the multiplication by m map for all $m \in \mathbf{N}$ (*cf.* [9, Section 4.1.2]).

Let us recall a characterization of Θ. Let

$$\mathbf{K}_1 = \bigoplus_D K_1(\mathbf{Q}(D)) = \bigoplus_D \mathbf{Q}(D)^\times,$$

where D runs through all the irreducible divisors of $\mathbf{G}_m^2$. There is a divisor map $\partial : \mathbf{K}_2 \to \mathbf{K}_1$ sending a Steinberg symbol $\{f, g\}$ to the element of K_1 whose component in D is

$$(-1)^{\nu(f)\nu(g)} g^{\nu(f)} f^{-\nu(g)},$$

where ν is the valuation coming from D (*cf.* [9, equation (2.6)]). The map ∂ induces an embedding

$$\partial : \mathbf{K}_2^{(0)}/\mathbf{Z}\cdot\{-z_1, -z_2\} \hookrightarrow \mathbf{K}_1$$

(*cf.* [9, Section 3.2 and Lemma 4.1.2]). As in [9, Section 3.2], for any $a, c \in \mathbf{Z}$ with $\gcd(a, c) = 1$, there is a special element

$$\langle a, c\rangle \in \mathbf{K}_1, \tag{3.1}$$

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which is supported on the divisor $D : 1 - z_1^a z_2^c = 0$ and is given there by the function $1 - z_1^b z_2^d$ for any $b, d \in \mathbf{Z}$ such that $ad - bc = 1$ (this is independent of the choice of b and d).

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, $\Theta(\gamma)$ is characterized by the equality

$$\partial(\Theta(\gamma)) = \langle b, d\rangle - \langle 0, 1\rangle$$

in K_1 (*cf.* [9, Proposition 3.3.1]). As in the proof of [9, Proposition 3.3.4], one sees that $\Theta(\gamma) = 0$ if $\gamma(0) = 0$, *i.e.*, if $\gamma = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ for some $m \in \mathbf{Z}$ (this follows from the injectivity of $\partial : \mathbf{K}_2^{(0)}/\mathbf{Z}\cdot\{-z_1, -z_2\} \hookrightarrow \mathbf{K}_1$).

Finally, let us recall how Sharifi and Venkatesh (*cf.* [9, Section 4.2.1]) specialize Θ to a cocycle

$$\Theta_M : \Gamma_0(M) \to K_2\left(\mathbf{Z}\left[\zeta_M, \frac{1}{M}\right]\right)/\mathbf{Z}\cdot\{-1, -\zeta_M\}$$

(for every $M \geq 4$). Here, the action of $\Gamma_0(M)$ on $K_2(\mathbf{Z}[\zeta_M, \frac{1}{M}])$ is given as follows: we have a surjective group homomorphism

$$\Gamma_0(M) \to (\mathbf{Z}/M\mathbf{Z})^\times$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \text{ (modulo } M\text{)},$$

and $(\mathbf{Z}/M\mathbf{Z})^\times$ acts on $K_2(\mathbf{Z}[\zeta_M, \frac{1}{M}])$ via our identification $(\mathbf{Z}/M\mathbf{Z})^\times \simeq \mathrm{Gal}(\mathbf{Q}(\zeta_M)/\mathbf{Q})$.

The idea is to "evaluate" $\Theta(\gamma)$ at $(z_1, z_2) = (1, \zeta_M)$. To "evaluate" at $(z_1, z_2) = (1, \zeta_M)$, a naive idea would be to send a Steinberg symbol $\{f, g\} \in \mathbf{K}_2$ to

$$\{f(1, \zeta_M), g(1, \zeta_M)\} \in K_2(\mathbf{Q}(\zeta_M)) .$$

This does not make sense in general because $f(1, \zeta_M)$ or $g(1, \zeta_M)$ may not be well defined (f or g may have a pole or zero at $(1, \zeta_M)$).

The idea of Sharifi and Venkatesh is to prove that $\Theta(\gamma)$ is actually a combination of Steinberg symbols which can be evaluated at $(1, \zeta_M)$. They make this precise by using *motivic cohomology groups*. We refer to [9, Section 2.1] for the precise definition and results they are using regarding motivic cohomology groups. In particular, if $U \subset \mathbf{G}_m^2$ is an open subset, there is a motivic cohomology group $H^2(U, 2)$ (which is an abelian group). As explained in [9, Remark 2.2.3], the functorial map $H^2(U, 2) \to H^2(\mathbf{Q}(\mathbf{G}_m^2), 2)$ is injective and $H^2(\mathbf{Q}(\mathbf{G}_m^2), 2)$ is canonically identified with $\mathbf{K}_2$.

As noted in [9, Section 4.2.1], for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$, the element

$$\Theta(\gamma) \in \mathbf{K}_2/\mathbf{Z}\cdot\{-z_1, -z_2\}$$

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lies in the image of $H^2(U_\gamma, 2)/\{-z_1, -z_2\}$, where U_γ is the open subset of $\mathbf{G}_m^2$, which is the complement of $\{z_1^b z_2^d = 1\} \cup \{z_2 = 1\}$. Since $(1, \zeta_M) \in U_\gamma$, there is a functorial map

$$s_M^* : H^2(U_\gamma, 2) \to H^2(\mathbf{Q}(\zeta_M), 2) \simeq K_2(\mathbf{Q}(\zeta_M)) .$$

By [9, Corollary 4.2.5], $\Theta_M(\gamma) := s_M^*(\Theta(\gamma))$ actually belongs to the subgroup

$$K_2\left(\mathbf{Z}\left[\zeta_M, \frac{1}{M}\right]\right) / \mathbf{Z}\cdot\{-1, -\zeta_M\}$$

of $K_2(\mathbf{Q}(\zeta_M))/\mathbf{Z}\cdot\{-1, -\zeta_M\}$.

We therefore have a 1-cocycle

(3.2) $$\Theta_M : \Gamma_0(M) \to K_2(\mathbf{Q}(\zeta_M))/\mathbf{Z}\cdot\{-1, -\zeta_M\} .$$

By [9, Proposition 4.2.1], the cocycle Θ_M is parabolic. This means that if $c \in \mathbf{P}^1(\mathbf{Q})$ and $\Gamma_c \subset \Gamma_0(M)$ is the stabilizer of c, then the restriction of Θ_M to Γ_c is a coboundary, *i.e.*, of the form $\gamma \mapsto \gamma \cdot x - x$ for some $x \in K_2(\mathbf{Q}(\zeta_M))/\mathbf{Z}\cdot\{-1, -\zeta_M\}$ (depending on c a priori).

4 From cocycles to relative homology

In this section, we explain how the cocycle Θ_M defined in (3.2) gives rise to a group homomorphism

$$\tilde{\Theta}_M : H_1(X_1(M), C_0, \mathbf{Z}) \to \mathcal{K}_M,$$

where C_0 is the set of cusps of $X_1(M)$ which are in the same diamond orbit as the cusp $\Gamma_1(M) \cdot 0$.

Recall that we have denoted by $\mathcal{K}_M$ the largest quotient of $K_2(\mathbf{Z}[\zeta_M, \frac{1}{M}]) \otimes \mathbf{Z}[\frac{1}{2}]$ on which the complex conjugation acts trivially. Note that since the Steinberg symbol $\{-1, -\zeta_M\}$ has order dividing 2, its image in $\mathcal{K}_M$ is trivial. Furthermore, the action of $(\mathbf{Z}/M\mathbf{Z})^\times$ on $\mathcal{K}_M$ factors through $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$ (by definition).

Recall that we have a group homomorphism $\Gamma_0(M) \to (\mathbf{Z}/M\mathbf{Z})^\times$ given by $\gamma \mapsto \langle\gamma\rangle$, where if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we let $\langle\gamma\rangle = d$ (modulo M). Therefore, a (left) $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$-module can be considered naturally as a (left) $\Gamma_0(M)$-module. In particular, this is the case of $H_1(X_1(M), C_0, \mathbf{Z})$, on which $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$ acts via diamond operators.

There is a map

$$f : \Gamma_0(M) \to H_1(X_1(M), C_0, \mathbf{Z})$$

given by $\gamma \mapsto \{0, \gamma 0\}$. The map f is a 1-cocycle, since for all $\gamma, \gamma' \in \Gamma_0(M)$, we have

$$\begin{aligned} f(\gamma\gamma') &= \{0, \gamma\gamma' 0\} \\ &= \{0, \gamma 0\} + \{\gamma 0, \gamma\gamma' 0\} \\ &= f(\gamma) + \langle\gamma\rangle f(\gamma') . \end{aligned}$$

We shall need the following result, which allows us to transfer a 1-cocycle to a map on homology.

Proposition 4.1 *Let T be a (left) $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$-module (where $M > 3$). Let $u : \Gamma_0(M) \to T$ be a 1-cocycle satisfying $u(\gamma) = 0$ for any $\gamma \in \Gamma_1(M)$ such that there exists $c \in \mathbf{P}^1(\mathbf{Q})$ with $\gamma c = c$. Then u factors through the map $f : \Gamma_0(M) \to H_1(X_1(M), C_0, \mathbf{Z})$, thus inducing a morphism of left $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$-modules $\tilde{u} : H_1(X_1(M), C_0, \mathbf{Z}) \to T$.*

Proof For notational simplicity, let $G = (\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$, $\Gamma_0 = \Gamma_0(M)/\pm 1$, and $\Gamma_1 = \Gamma_1(M) \subset \Gamma_0$. Recall that if $\gamma \in \Gamma_0$, we let $\langle\gamma\rangle \in G$ be the class of the lower-right corner of γ. For any $c \in \mathbf{P}^1(\mathbf{Q})$, let $\gamma_c \in \Gamma_1$ be a generator of the stabilizer of c in Γ_1.

By assumption, we have $u(\gamma_c) = 0$ for all $c \in \mathbf{P}^1(\mathbf{Q})$. Thus, u induces a group homomorphism $u' : \mathbf{Z}[G \times \Gamma_0]/I \to T$ given by $u'(g, \gamma) = g \cdot u(\gamma)$, where I is the subgroup of $\mathbf{Z}[G \times \Gamma_0]$ generated by the elements $(g, \gamma\gamma') - (g, \gamma) - (g\langle\gamma\rangle, \gamma')$ and by the $(1, \gamma_c) - (1, 1)$ for all $g \in G$, $\gamma, \gamma' \in \Gamma_0(M)$ and $c \in \mathbf{P}^1(\mathbf{Q})$.

It suffices to prove that the map

$$\varphi : \mathbf{Z}[G \times \Gamma_0]/I \to H_1(X_1(M), C_0, \mathbf{Z})$$

sending (g, γ) to $g \cdot \{0, \gamma 0\}$ is an isomorphism. Note that φ is well defined since

$$\begin{aligned}\{0, \gamma_c 0\} &= \{0, c\} + \{c, \gamma_c c\} + \{\gamma_c c, \gamma_c 0\} \\ &= (1 - \langle\gamma_c\rangle) \cdot \{0, c\} \\ &= 0\end{aligned}$$

(we have used the fact that $\gamma_c \in \Gamma_1$, so the diamond operator $\langle\gamma_c\rangle$ is trivial).

Let us first prove that φ is surjective. Any element of $H_1(X_1(M), C_0, \mathbf{Z})$ is a combination of modular symbols of the form $\{\alpha, \beta\}$ where $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$ project onto C_0 in $X_1(M)$. This latter condition means that $\alpha = \gamma 0$ and $\beta = \gamma' 0$ for some $\gamma, \gamma' \in \Gamma_0(M)$. Note that

$$\begin{aligned}\{\alpha, \beta\} &= \{\alpha, 0\} + \{0, \beta\} \\ &= \{0, \gamma' 0\} - \{0, \gamma 0\},\end{aligned}$$

so $H_1(X_1(M), C_0, \mathbf{Z})$ is generated by the elements of the form $\{0, \gamma 0\}$ for $\gamma \in \Gamma_0(M)$. We have thus proved that φ is surjective. To prove that φ is injective, it is enough to show that $\mathbf{Z}[G \times \Gamma_0]/I$ is a free $\mathbf{Z}$-module of the same rank as $H_1(X_1(M), C_0, \mathbf{Z})$.

Since $M > 3$, the group Γ_1 is torsion-free and we have $H_1(\Gamma_1, \mathbf{Z}) \simeq H_1(Y_1(M), \mathbf{Z})$ (*cf.* (2.4)). By Shapiro's lemma for group homology, we have $H_1(\Gamma_1, \mathbf{Z}) \simeq H_1(\Gamma_0, \mathbf{Z}[G])$. Using the description of group homology in terms of inhomogeneous chains (*cf.* (2.3)), one gets a short exact sequence

$$0 \to H_1(\Gamma_0, \mathbf{Z}[G]) \to \mathbf{Z}[G \times \Gamma_0]/J \xrightarrow{\partial} \mathbf{Z}[G] \to \mathbf{Z} \to 0,$$

where $\partial(g, \gamma) = g \cdot \langle\gamma\rangle^{-1} - g$ and J is the subgroup of $\mathbf{Z}[G \times \Gamma_0]$ generated by

$$(g, \gamma\gamma') - (g, \gamma) - (g\langle\gamma\rangle^{-1}, \gamma')$$

for $g \in G$, $\gamma, \gamma' \in \Gamma_0(M)$. The last map $\mathbf{Z}[G] \to \mathbf{Z}$ is the augmentation (degree) map (note that J is indeed in the kernel of ∂).

As in (2.2), we have an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}[C_M] \to H_1(Y_1(M), \mathbf{Z}) \to H_1(X_1(M), \mathbf{Z}) \to 0,$$

where C_M is the set of cusps of $Y_1(M)$. Here, the map $\mathbf{Z}[C_M]$ sends a cusp c to the homology class of a small loop around c in $Y_1(M)$.

Under the isomorphism $H_1(Y_1(M), \mathbf{Z}) \simeq H_1(\Gamma_0, \mathbf{Z}[G])$ and the embedding $H_1(\Gamma_0, \mathbf{Z}[G]) \hookrightarrow \mathbf{Z}[G \times \Gamma_0]/J$ described above, the map $\mathbf{Z}[C_M] \to H_1(\Gamma_0, \mathbf{Z}[G])$ sends a cusp c to the class of $(1, \gamma_c) - (1, 1)$ in $\mathbf{Z}[G \times \Gamma_0]/J$.

Thus, we have an exact sequence

$$0 \to H_1(X_1(M), \mathbf{Z}) \to \mathbf{Z}[G \times \Gamma_0]/I' \xrightarrow{\partial} \mathbf{Z}[G] \to \mathbf{Z} \to 0,$$

where I' is the subgroup of $\mathbf{Z}[G \times \Gamma_0]$ generated by the elements

$$(g, \gamma\gamma') - (g, \gamma) - (g\langle\gamma\rangle^{-1}, \gamma')$$

and by the $(1, \gamma_c) - (1, 1)$ for all $g \in G$, $\gamma, \gamma' \in \Gamma_0(M)$ and $c \in \mathbf{P}^1(\mathbf{Q})$.

The involution $G \to G$ given by $g \mapsto g^{-1}$ induces an isomorphism $\mathbf{Z}[G \times \Gamma_0]/I' \xrightarrow{\sim} \mathbf{Z}[G \times \Gamma_0]/I$. This shows that $\mathbf{Z}[G \times \Gamma_0]/I$ is a free $\mathbf{Z}$-module of rank $\mathrm{rk}_{\mathbf{Z}} H_1(X_1(M), \mathbf{Z}) + \#G - 1$. We have $\#G = \#C_0$, and the exact sequence (*cf.* (2.1))

$$0 \to H_1(X_1(M), \mathbf{Z}) \to H_1(X_1(M), C_0, \mathbf{Z}) \to \mathbf{Z}[C_0] \to \mathbf{Z} \to 0$$

shows that $\mathrm{rk}_{\mathbf{Z}} \mathbf{Z}[G \times \Gamma_0]/I = \mathrm{rk}_{\mathbf{Z}} H_1(X_1(M), C_0, \mathbf{Z})$, as wanted. ∎

Let us apply Proposition 4.1 to $T = \mathcal{K}_M$ and $u : \Gamma_0(M) \to T$ induced by Θ_M. Since u is parabolic and $\Gamma_1(M)$ acts trivially on T, the condition that u vanishes on parabolic elements of $\Gamma_1(M)$ is satisfied. Therefore, we get a $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$-equivariant homomorphism

$$\tilde{\Theta}_M : H_1(X_1(M), C_0, \mathbf{Z}) \to \mathcal{K}_M\,.$$

5 Proofs of the theorems

We start with the following lemma (we thank Venkatesh for explaining this to us).

Lemma 5.1 *Let $M \geq 4$ and $p \geq 2$ be a prime. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Let $\phi_p : \Gamma_0(Mp) \to \Gamma_0(M)$ be the group homomorphism sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & pb \\ c/p & d \end{pmatrix}$. We have a commutative diagram*

$$\begin{array}{ccc} \Gamma_0(Mp) & \xrightarrow{\Theta} & \mathbf{K}_2^{(0)}/\mathbf{Z}\cdot\{-z_1, -z_2\} \\ \downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \alpha_*} \\ \Gamma_0(M) & \xrightarrow{\Theta} & \mathbf{K}_2^{(0)}/\mathbf{Z}\cdot\{-z_1, -z_2\}, \end{array}$$

where α_* is the trace map induced by α.

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Proof Since $\partial : \mathbf{K}_2^{(0)}/\mathbf{Z}\cdot\{-z_1, -z_2\} \hookrightarrow \mathbf{K}_1$ is injective, it suffices to prove that the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_0(Mp) & \xrightarrow{\partial\circ\Theta} & \mathbf{K}_1 \\ \downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \alpha_*} \\ \Gamma_0(M) & \xrightarrow{\partial\circ\Theta} & \mathbf{K}_1\,. \end{array}$$

In other words, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(Mp)$, then it suffices to check that

$$\alpha_*(\langle b, d\rangle - \langle 0, 1\rangle) = \langle pb, d\rangle - \langle 0, 1\rangle \tag{5.1}$$

(*cf.* (3.1) for the definition of the symbol $\langle b, d\rangle$).

As in [9, equation (3.2)], we have $\langle b, d\rangle = \gamma^*\langle 0, 1\rangle$ where $\gamma^* : \mathbf{K}_1 \to \mathbf{K}_1$ is the pullback induced by the right action of γ on $\mathbf{G}_m^2$. Note that we have $\gamma^* = (\gamma^{-1})_*$. Thus, we have

$$\begin{aligned} \alpha_*\langle b, d\rangle &= \alpha_*(\gamma^{-1})_*\langle 0, 1\rangle = (\gamma^{-1}\cdot\alpha)_*\langle 0, 1\rangle = (\alpha^{-1}\cdot\gamma^{-1}\cdot\alpha)_*\alpha_*\langle 0, 1\rangle \\ &= (\alpha^{-1}\cdot\gamma\cdot\alpha)^*\alpha_*\langle 0, 1\rangle\,. \end{aligned}$$

Let us prove that $\alpha_*\langle 0, 1\rangle = \langle 0, 1\rangle$. By definition, $\langle 0, 1\rangle$ is the function $1 - z_1^{-1}$ on the divisor $z_2 = 1$ of $\mathbf{G}_m^2$. Furthermore, right multiplication by α on $\mathbf{G}_m^2$ is given by $(z_1, z_2) \mapsto (z_1, z_2^p)$. Thus, the divisor $z_2 = 1$ is mapped to itself by right multiplication by α. Therefore (*cf.* [9, Remark 2.3.3]), $\alpha_*\langle 0, 1\rangle$ is the norm of $1 - z_1^{-1}$ via the identity map $K_1(\mathbf{Q}(z_1)) \to K_1(\mathbf{Q}(z_1))$, on the divisor $z_2 = 1$. This proves that $\alpha_*\langle 0, 1\rangle = \langle 0, 1\rangle$.

Since $\alpha^{-1}\cdot\gamma\cdot\alpha = \begin{pmatrix} a & pb \\ c/p & d \end{pmatrix}$, we get $\alpha_*\langle b, d\rangle = \langle pb, d\rangle$. This proves (5.1), and concludes the proof of Lemma 5.1. ∎

We are now ready to prove Theorem 1.4. Let

$$\gamma' \in \Gamma_0(Mp),$$

and let

$$\gamma = \phi_p(\gamma') \in \Gamma_0(M)$$

be as in Lemma 5.1. Let $f : \mathbf{G}_m^2 \to \mathbf{G}_m^2$ given by

$$f : (z_1, z_2) \mapsto (z_1, z_2^p)$$

(note that f is induced by the right action of the matrix α of Lemma 5.1).

Let $U = U_\gamma$ be as in Section 3 and

$$U' = f^{-1}(U) \cap U_{\gamma'}\,.$$

Both U and U' are open subschemes of $\mathbf{G}_m^2$, and we have $(1, \zeta_M) \in U$ and $(1, \zeta\cdot\zeta_{Mp}) \in U'$ for all pth root of unity ζ.

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Consider the following Cartesian diagram of schemes:

$$\begin{array}{ccc} X & \longrightarrow & U' \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathbf{Q}(\zeta_M)) & \xrightarrow{s_M} & U, \end{array} \tag{5.2}$$

where s_M is given by the closed point $(1, \zeta_M) \in U$, and X makes the diagram Cartesian by definition.

Lemma 5.2 *We have a natural isomorphism of schemes over* $\mathrm{Spec}(\mathbf{Q})$*:*

$$X \simeq \mathrm{Spec}(\mathbf{Q}(\zeta_M)) \times_{\mathrm{Spec}(\mathbf{Q}[T,T^{-1}])} \mathrm{Spec}(\mathbf{Q}[t, t^{-1}]),$$

where the maps $\mathrm{Spec}(\mathbf{Q}(\zeta_M)) \to \mathrm{Spec}(\mathbf{Q}[T, T^{-1}])$ *and* $\mathrm{Spec}(\mathbf{Q}[t, t^{-1}]) \to \mathrm{Spec}(\mathbf{Q}[T, T^{-1}])$ *are given by* $T \mapsto \zeta_M$ *and* $T \mapsto t^p$*, respectively.*

Under this isomorphism, the map $X \to \mathrm{Spec}(\mathbf{Q}(\zeta_M))$ *is the projection onto the first factor. The compositum map* $X \to U' \hookrightarrow \mathrm{Spec}(\mathbf{Q}[z_1, z_2, z_1^{-1}, z_2^{-1}])$ *is given by the compositum of the projection*

$$\mathrm{Spec}(\mathbf{Q}(\zeta_M)) \times_{\mathrm{Spec}(\mathbf{Q}[T,T^{-1}])} \mathrm{Spec}(\mathbf{Q}[t, t^{-1}]) \to \mathrm{Spec}(\mathbf{Q}[t, t^{-1}])$$

and of the map $\mathrm{Spec}(\mathbf{Q}[t, t^{-1}]) \to \mathrm{Spec}(\mathbf{Q}[z_1, z_2, z_1^{-1}, z_2^{-1}])$ *defined by* $z_1 \mapsto 1$ *and* $z_2 \mapsto t$*.*

Proof Let Y be such that the following diagram is Cartesian:

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{G}_m^2 \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathbf{Q}(\zeta_M)) & \xrightarrow{s_M} & \mathbf{G}_m^2\ . \end{array}$$

We claim that there is a natural isomorphism $Y \simeq X$. To prove that, it is enough to prove that we have a commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & U' & \longrightarrow & \mathbf{G}_m^2 \\ \downarrow & & \downarrow f & & \downarrow f \\ \mathrm{Spec}(\mathbf{Q}(\zeta_M)) & \xrightarrow{s_M} & U & \longrightarrow & \mathbf{G}_m^2\ . \end{array}$$

It suffices to prove that the image of Y (which we view as a closed subscheme of $\mathbf{G}_m^2$) is contained in U'. This follows from the fact that $f^{-1}(1, \zeta_M) \subset U'$.

To conclude the proof of Lemma 5.2, note that there is a commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \mathbf{G}_m = \mathrm{Spec}(\mathbf{Q}[t, t^{-1}]) & \xrightarrow{z_1=1\ z_2=t} & \mathbf{G}_m^2 = \mathrm{Spec}(\mathbf{Q}[z_1, z_2, z_1^{-1}, z_2^{-1}]) \\ \downarrow & & \downarrow T=t^p & & \downarrow f \\ \mathrm{Spec}(\mathbf{Q}(\zeta_M)) & \longrightarrow & \mathbf{G}_m = \mathrm{Spec}(\mathbf{Q}[T, T^{-1}]) & \xrightarrow{z_1=1\ z_2=T} & \mathbf{G}_m^2 = \mathrm{Spec}(\mathbf{Q}[z_1, z_2, z_1^{-1}, z_2^{-1}]). \end{array}$$

∎

Lemma 5.2 yields a more concrete description of X: we have

$$X \simeq \mathrm{Spec}(\mathbf{Q}(\zeta_M)[t]/(t^p - \zeta_M)) \,. \tag{5.3}$$

This latter isomorphism can be rewritten more simply in a way which depends on whether p divides M or not.

5.1 The case $p \mid M$

Assume first that p divides M. Then $X \simeq \mathrm{Spec}(\mathbf{Q}(\zeta_{Mp}))$ and the map $X \to U'$ comes from the point $(1, \zeta_{Mp}) \in U'$. Applying the functor $H^2(\cdot, 2)$ to (5.2) (*cf.* [9, Lemma 2.1.1]), we get a commutative diagram

$$\begin{array}{ccc} H^2(U',2) & \xrightarrow{s_{Mp}^*} & K_2(\mathbf{Q}(\zeta_{Mp})) \\ \downarrow f_* & & \downarrow \mathrm{Norm} \\ H^2(U,2) & \xrightarrow{s_M^*} & K_2(\mathbf{Q}(\zeta_M)), \end{array}$$

and hence a commutative diagram

$$\begin{array}{ccc} H^2(U',2)/\mathbf{Z}\cdot\{-z_1,-z_2\} & \xrightarrow{s_{Mp}^*} & K_2(\mathbf{Q}(\zeta_{Mp}))/\mathbf{Z}\cdot\{-1,-\zeta_{Mp}\} \\ \downarrow f_* & & \downarrow \mathrm{Norm} \\ H^2(U,2)/\mathbf{Z}\cdot\{-z_1,-z_2\} & \xrightarrow{s_M^*} & K_2(\mathbf{Q}(\zeta_M))/\mathbf{Z}\cdot\{-1,-\zeta_M\} \,. \end{array} \tag{5.4}$$

As explained in Section 3, we have

$$\Theta(\gamma') \in H^2(U_{\gamma'},2)/\mathbf{Z}\cdot\{-z_1,-z_2\}$$

and

$$\Theta(\gamma) \in H^2(U_{\gamma},2)/\mathbf{Z}\cdot\{-z_1,-z_2\} \,.$$

Since U' is an open subset of $U_{\gamma'}$, we have a functorial embedding

$$H^2(U_{\gamma'},2)/\mathbf{Z}\cdot\{-z_1,-z_2\} \hookrightarrow H^2(U',2)/\mathbf{Z}\cdot\{-z_1,-z_2\} \,.$$

Thus, we have

$$\Theta(\gamma') \in H^2(U',2)/\mathbf{Z}\cdot\{-z_1,-z_2\} \,.$$

By Lemma 5.1, we have

$$f_*\Theta(\gamma') = \Theta(\gamma) \,.$$

Therefore, using (5.4), we get $s_M^*(\Theta(\gamma)) = \mathrm{Norm}(s_{Mp}^*(\Theta(\gamma')))$, *i.e.*,

$$\mathrm{Norm}(\Theta_{Mp}(\gamma')) = \Theta_M(\gamma) \,. \tag{5.5}$$

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By Proposition 4.1, equation (5.5) yields the following commutative diagram:

$$\begin{array}{ccc} H_1(X_1(Mp), C_0', \mathbf{Z}) & \xrightarrow{\tilde{\Theta}_{Mp}} & \mathcal{K}_{Mp} \\ \downarrow \pi_2 & & \downarrow \mathrm{Norm} \\ H_1(X_1(M), C_0, \mathbf{Z}) & \xrightarrow{\tilde{\Theta}_M} & \mathcal{K}_M, \end{array} \tag{5.6}$$

where C_0' (resp. C_0) is the set of cusps of $X_1(Mp)$ (resp. $X_1(M)$) in the same diamond orbit as 0. The top horizontal map (resp. bottom horizontal map) is equivariant for the action of $(\mathbf{Z}/Mp\mathbf{Z})^\times/\pm 1$ (resp. $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$).

After applying the Atkin–Lehner involution W_{Mp} and W_M to the two lines of (5.6), we get a commutative diagram

$$\begin{array}{ccc} H_1(X_1(Mp), C_\infty', \mathbf{Z}) & \xrightarrow{\varpi_{Mp}} & \mathcal{K}_{Mp} \\ \downarrow \pi_1 & & \downarrow \mathrm{Norm} \\ H_1(X_1(M), C_\infty, \mathbf{Z}) & \xrightarrow{\varpi_M} & \mathcal{K}_M, \end{array} \tag{5.7}$$

where C_∞' (resp. C_∞) is the set of cusps of $X_1(Mp)$ (resp. $X_1(M)$) in the same orbit as ∞. We have used the facts that $\varpi_{Mp} = \tilde{\Theta}_{Mp} \circ W_{Mp}$ and $\varpi_M = \tilde{\Theta}_M \circ W_M$. This follows from [9, Proposition 4.3.3], where the authors use usual Manin symbols (whereas our map ϖ_M uses Manin symbols twisted by the Atkin–Lehner involution).

Note that ϖ_{Mp} and ϖ_M are *anti-equivariant* for the actions of $(\mathbf{Z}/Mp\mathbf{Z})^\times/\pm 1$ and $(\mathbf{Z}/M\mathbf{Z})^\times/\pm 1$, respectively. This means that for any $x \in H_1(X_1(Mp), C_\infty', \mathbf{Z})$ and $g \in (\mathbf{Z}/Mp\mathbf{Z})^\times/\pm 1$, we have

$$\varpi_{Mp}(g \cdot x) = g^{-1} \cdot \varpi_{Mp}(x) \tag{5.8}$$

(and similarly for ϖ_M). Indeed, we have $W_{Mp} \circ \langle g \rangle = \langle g^{-1} \rangle \circ W_{Mp}$. This could also have been checked easily directly on the definition of ϖ_{Mp} and ϖ_M in terms of dual Manin symbols. Let us note that (5.8) is true independently on whether p divides M or not.

Now, let C be a subset of cusps of $X_1(Mp)$ as in Theorem 1.4. If $C \subset C_\infty'$, then Theorem 1.4 follows from (5.7) (we just restrict ϖ_{Mp} to $H_1(X_1(Mp), C, \mathbf{Z}) \subset H_1(X_1(Mp), C_\infty', \mathbf{Z})$). Let us explain how to deduce the general case from this special case.

Fix $c \in \mathbf{P}^1(\mathbf{Q})$ such that $\Gamma_1(Mp) \cdot c \in C$. An element of $H_1(X_1(Mp), C, \mathbf{Z})$ is of the form $\{c, \gamma c\}$ for some $\gamma \in \Gamma_0(Mp)$. The assumption that all the elements of C are in the same diamond orbit under $\mathrm{Ker}((\mathbf{Z}/Mp\mathbf{Z})^\times \to (\mathbf{Z}/M\mathbf{Z})^\times)$ means that we can actually choose γ in $\Gamma_0(Mp) \cap \Gamma_1(M)$.

We have

$$\begin{aligned} \{c, \gamma c\} &= \{c, \infty\} + \{\infty, \gamma\infty\} + \{\gamma\infty, \gamma c\} \\ &= \{\infty, \gamma\infty\} + (\langle\gamma\rangle - 1) \cdot \{\infty, c\}, \end{aligned}$$

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where $\langle\gamma\rangle$ is the diamond operator associated with γ. Since $\gamma \in \Gamma_0(Mp) \cap \Gamma_1(M)$, we have $\pi_2\left((\langle\gamma\rangle - 1)\cdot\{\infty, c\}\right) = 0$. Thus, we have

$$\pi_2(\{c, \gamma c\}) = \pi_2(\{\infty, \gamma\infty\})\,.$$

We also have

$$\mathrm{Norm}\left(\varpi_{Mp}((\langle\gamma\rangle - 1)\{\infty, c\})\right) = 0$$

in $\mathcal{K}_M$ since by (5.8) we have

$$\varpi_{Mp}((\langle\gamma\rangle - 1)\{\infty, c\}) \in (\langle\gamma\rangle^{-1} - 1)\cdot\mathcal{K}_{Mp}\,.$$

Thus, we have

$$\begin{aligned}\mathrm{Norm}(\varpi_{Mp}(\{c, \gamma c\})) &= \mathrm{Norm}(\varpi_{Mp}(\{\infty, \gamma\infty\}))\\ &= \varpi_M(\pi_2(\{\infty, \gamma\infty\}))\\ &= \varpi_M(\pi_2(\{c, \gamma c\}))\,.\end{aligned}$$

This concludes the proof of Theorem 1.4 in the case $p \mid M$.

5.2 The case $p \nmid M$

Assume now that p does not divide M. Note that in this case we have $(1, \zeta_M) \in U'$. By (5.3), there is an isomorphism

$$X \simeq \mathrm{Spec}(\mathbf{Q}(\zeta_{Mp})) \sqcup \mathrm{Spec}(\mathbf{Q}(\zeta_M))$$

such that:

- The map $X \to U'$ is given by the two inclusions $(1, \zeta_M^{p^*}) \in U'$ and $(1, \zeta_{Mp}) \in U'$, where $p^* \in (\mathbf{Z}/M\mathbf{Z})^\times$ is the inverse of p modulo M.
- The map $X \to \mathrm{Spec}(\mathbf{Q}(\zeta_M))$ is given by the canonical map $\mathrm{Spec}(\mathbf{Q}(\zeta_{Mp})) \to \mathrm{Spec}(\mathbf{Q}(\zeta_M))$ and the identity map $\mathrm{Spec}(\mathbf{Q}(\zeta_M)) \to \mathrm{Spec}(\mathbf{Q}(\zeta_M))$.

Applying the functor $H^2(\cdot, 2)$ to (5.2) (*cf.* [9, Lemma 2.1.1]), we get the following commutative diagram:

(5.9)

$$\begin{array}{ccc} H^2(U',2)/\mathbf{Z}\cdot\{-z_1,-z_2\} & \xrightarrow{s_{Mp}^*\oplus((\sigma_p^{-1})^*\circ s_M^*)} & K_2(\mathbf{Q}(\zeta_{Mp}))/\mathbf{Z}\cdot\{-1,-\zeta_{Mp}\}\oplus K_2(\mathbf{Q}(\zeta_M))/\mathbf{Z}\cdot\{-1,-\zeta_M\} \\ \downarrow f_* & & \downarrow \mathrm{Norm}\oplus\mathrm{id} \\ H^2(U,2)/\mathbf{Z}\cdot\{-z_1,-z_2\} & \xrightarrow{s_M^*} & K_2(\mathbf{Q}(\zeta_M))/\mathbf{Z}\cdot\{-1,-\zeta_M\}\,. \end{array}$$

Combining Lemma 5.1 and (5.9), we get

$$\mathrm{Norm}(\Theta_{Mp}(\gamma')) + (\sigma_p^{-1})^*(\Theta_M(\gamma')) = \Theta_M(\gamma)\,. \tag{5.10}$$

By (5.8), we have

$$\varpi_{Mp}\circ\langle p\rangle = (\sigma_p^{-1})^*\circ\varpi_{Mp}\,.$$

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As in (5.7), we then get a commutative diagram

$$\begin{array}{ccc} H_1(X_1(Mp), C'_\infty, \mathbf{Z}) & \xrightarrow{\varpi_{Mp}} & \mathcal{K}_{Mp} \\ \downarrow{\scriptstyle \pi_1 - \langle p \rangle \pi_2} & & \downarrow{\scriptstyle \mathrm{Norm}} \\ H_1(X_1(M), C_\infty, \mathbf{Z}) & \xrightarrow{\varpi_M} & \mathcal{K}_M \end{array} \tag{5.11}$$

(note that both π_1 and $\langle p \rangle \pi_2$ send C'_∞ to C_∞, so that this diagram makes sense). An argument identical to the one when $p \mid M$ shows that the diagram of Theorem 1.4 commutes. This concludes the proof of Theorem 1.4.

5.3 Proof of Theorem 1.6

Let us now prove Theorem 1.6. Let $M \geq 4$ and $p \geq 5$ be a prime. One needs to prove that

$$\varpi_M \otimes \mathbf{Z}_p : H_1(X_1(M), \mathbf{Z}_p) \to \mathcal{K}_M \otimes \mathbf{Z}_p$$

is annihilated by the operator $U_\ell - 1$ for any prime $\ell \mid M$. If $p \mid M$, then this is a result of Fukaya and Kato (*cf.* Theorem 1.3). Therefore, we shall assume in what follows that p does not divide M.

By Theorem 1.4(ii), we have a commutative diagram

$$\begin{array}{ccc} H_1(X_1(Mp), \mathbf{Z}_p) & \xrightarrow{\varpi_{Mp} \otimes \mathbf{Z}_p} & \mathcal{K}_{Mp} \otimes \mathbf{Z}_p \\ \downarrow{\scriptstyle \pi_1 - \langle p \rangle \pi_2} & & \downarrow{\scriptstyle \mathrm{Norm}} \\ H_1(X_1(M), \mathbf{Z}_p) & \xrightarrow{\varpi_M \otimes \mathbf{Z}_p} & \mathcal{K}_M \otimes \mathbf{Z}_p \end{array} . \tag{5.12}$$

By the result of Fukaya and Kato, one knows that $\varpi_{Mp} \otimes \mathbf{Z}_p$ is annihilated by the Hecke operator $U_\ell - 1$. Since $\pi_1 - \langle p \rangle \pi_2$ commutes with the action of $U_\ell - 1$ on both sides, it suffices to prove that

$$\pi_1 - \langle p \rangle \pi_2 : H_1(X_1(Mp), \mathbf{Z}_p) \to H_1(X_1(M), \mathbf{Z}_p)$$

is surjective. Note that

$$H_1(X_1(M), \mathbf{Z}_p) \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z} \simeq H_1(X_1(M), \mathbf{Z}/p\mathbf{Z})$$

by the Universal Coefficient Theorem (as $H_0(X, \mathbf{Z})$ is torsion-free).

By the Nakayama lemma, the surjectivity of our map $H_1(X_1(Mp), \mathbf{Z}_p) \to H_1(X_1(M), \mathbf{Z}_p)$ is equivalent to the surjectivity of the map

$$\pi_1 - \langle p \rangle \pi_2 : H_1(X_1(Mp), \mathbf{Z}/p\mathbf{Z}) \to H_1(X_1(M), \mathbf{Z}/p\mathbf{Z}) .$$

By the Poincaré duality, it suffices to prove that

$$\pi_1^* - \pi_2^* \circ \langle p \rangle^{-1} : H_1(X_1(M), \mathbf{Z}/p\mathbf{Z}) \to H_1(X_1(Mp), \mathbf{Z}/p\mathbf{Z})$$

is injective. Note that $H_1(X_1(M), \mathbf{Z}/p\mathbf{Z})$ is canonically isomorphic to the parabolic cohomology $H^1_p(\Gamma_1(M), \mathbf{Z}/p\mathbf{Z})$, *i.e.*, the subgroup of $H^1(\Gamma_1(M), \mathbf{Z}/p\mathbf{Z})$ consisting of classes of cocycles which are coboundaries when restricted to stabilizers of cusps.

Thus, it is enough for us to prove that the map

$$\pi_1^* - \pi_2^* \circ \langle p \rangle^{-1} : H^1(\Gamma_1(M), \mathbf{Z}/p\mathbf{Z}) \to H^1(\Gamma_1(Mp), \mathbf{Z}/p\mathbf{Z}) \tag{5.13}$$

is injective. By [2, Lemma 1], the map

$$H^1(\Gamma_1(M), \mathbf{Z}/p\mathbf{Z})^2 \xrightarrow{\pi_1^* + \pi_2^*} H^1(\Gamma_1(M) \cap \Gamma_0(p), \mathbf{Z}/p\mathbf{Z})$$

is injective.

By the inflation-restriction exact sequence, we have an exact sequence

$$0 \to H^1((\mathbf{Z}/p\mathbf{Z})^\times, \mathbf{Z}/p\mathbf{Z}) \to H^1(\Gamma_1(M) \cap \Gamma_0(p), \mathbf{Z}/p\mathbf{Z}) \to H^1(\Gamma_1(Mp), \mathbf{Z}/p\mathbf{Z}).$$

Since $(\mathbf{Z}/p\mathbf{Z})^\times$ has order prime to p, we have $H^1((\mathbf{Z}/p\mathbf{Z})^\times, \mathbf{Z}/p\mathbf{Z}) = 0$, so the map

$$H^1(\Gamma_1(M) \cap \Gamma_0(p), \mathbf{Z}/p\mathbf{Z}) \to H^1(\Gamma_1(Mp), \mathbf{Z}/p\mathbf{Z})$$

is injective. We then conclude that the map

$$H^1(\Gamma_1(M), \mathbf{Z}/p\mathbf{Z})^2 \xrightarrow{\pi_1^* + \pi_2^*} H^1(\Gamma_1(Mp), \mathbf{Z}/p\mathbf{Z})$$

is injective. This proves the injectivity of (5.13), and thus concludes the proof of Theorem 1.6.

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References

[1] F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, 228, Springer, New York, 2005.

[2] B. Edixhoven and C. Khare, *Hasse invariant and group cohomology*. Doc. Math. **8**(2003), 43–50.

[3] T. Fukaya, K. Kato, and R. Sharifi, *Modular symbols in Iwasawa theory*. In: Iwasawa theory 2012, Contributions in Mathematical and Computational Sciences, 7, Springer, Heidelberg, 2014, pp. 177–219.

[4] T. Fukaya and K. Kazuya, *On conjectures of Sharifi*. To appear in Kyoto J. Math.

[5] E. Lecouturier and J. Wang, *On a conjecture of Sharifi and Mazur's Eisenstein ideal*. Int. Math. Res. Not. IMRN **1**(2022), 391–421.

[6] E. Lecouturier and J. Wang, *On the Birch and Swinnerton–Dyer conjecture for certain families of abelian varieties with a rational isogeny*. In preparation.

[7] Ju. I. Manin, *Parabolic points and zeta functions of modular curves*. Izv. Akad. Nauk SSSR Ser. Mat. **36**(1972), 19–66.

[8] R. Sharifi, *A reciprocity map and the two-variable p-adic L-function*. Ann. Math. **173**(2011), no. 1, 251–300.

[9] R. Sharifi and A. Venkatesh, *Eisenstein cocycles in motivic cohomology*. Preprint, 2020. arxiv:2011.07241

[10] J. Tate, *Relations between K_2 and Galois cohomology*. Invent. Math. **36**(1976), 257–274.

[11] R. S. Williams, *Level compatibility in the passage from modular symbols to cup products*. Res. Number Theory 7(2021), no. 1, Article no. 9, 26 pp.

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