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DEFICIENCIES OF LATTICES IN CONNECTED LIE GROUPS

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We complete the determination of the groups of positive deficiency which occur as lattices in connected Lie groups. The torsion free groups among them are 3-manifold groups. We show that any other torsion free 3-manifold group which is such a lattice is the group of an aspherical closed geometric 3-manifold.

If G is a finitely presentable group its *deficiency* $\operatorname{def}(G)$ is the maximum over all finite presentations for G of the number of generators minus the number of relators. Lott showed in [**7**] that if Γ is a lattice in a connected Lie group G and $\operatorname{def}(\Gamma) > 0$ then either

(i) Γ has a finite normal subgroup N such that Γ/N is a lattice in $\mathrm{PSL}(2,\mathbb{R})$; or
(ii) $\operatorname{def}(\Gamma) = 1$ and Γ is isomorphic to a torsion-free nonuniform lattice in $\mathbb{R} \times \mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{C})$; or
(iii) Γ is free Abelian of rank 1 or 2 or is the fundamental group of the Klein bottle.

This was an improvement upon earlier work of Lubotzky, who assumed G simple and either of rank ≥ 2 or $G = \mathrm{Sp}(n,1)$ or $F_{4(-20)}$, in which cases $\operatorname{def}(\Gamma) \leq 0$, or $G = SO(n,1)$ (for $n \geq 3$) or $SU(n,1)$ (for $n \geq 2$), in which cases $\operatorname{def}(\Gamma) \leq 1$ [**8**]. We shall show that in case (i) the subgroup N must be trivial, and exclude the Klein bottle group. Excepting the lattices in $\mathrm{PSL}(2,\mathbb{R})$ with finite Abelianisation, all the remaining possibilities have positive deficiency.

THEOREM 1. *Let Γ be a finitely presentable group with a nontrivial finite normal subgroup N such that Γ/N is a lattice in $\mathrm{PSL}(2,\mathbb{R})$. Then $\operatorname{def}(\Gamma)$ is nonpositive.*

PROOF: A group has a presentation of positive deficiency if and only if it is the fundamental group of a finite 2-complex with nonpositive Euler characteristic. The latter property is clearly inherited by subgroups of finite index.

Let P be a cyclic subgroup of N of prime order p, and let $A = C_N(P)$ and $G = C_\Gamma(P)$. Then G/A is again a lattice in $\mathrm{PSL}(2,\mathbb{R})$, since $[\Gamma : G] < \infty$, and so is either a nontrivial free group or is the fundamental group of an aspherical closed surface. If

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G/A is free then $G \cong (G/A) \times A$. In general, G has a subgroup H of index $|A|$, and the class in $H^2(G/A; A)$ corresponding to the central extension $1 \to A \to G \to G/A \to 1$ has image 0 in $H^2(H; A)$. Therefore the preimage of H in G splits as a direct product $H \times A$, and so Γ has a subgroup $D \cong H \times P$ of finite index. The deficiency of D is at most $\beta_1(D; \mathbb{F}_p) - \beta_2(D; \mathbb{F}_p) = -\beta_2(H; \mathbb{F}_p)$, and so $\operatorname{def}(D) \leq 0$. Therefore $\operatorname{def}(\Gamma) \leq 0$, by the observation in the first paragraph of this proof. ∎

The estimate is best possible, in general. Let $F(r)$ be the free group of rank r. If $H \cong F(r)$ for some r greater than 1 or is the fundamental group of an aspherical closed orientable surface and C is a nontrivial finite cyclic group then $H \times C$ is a lattice in $\mathrm{PSL}(2, \mathbb{R}) \times SO(2)$, and has deficiency 0 or -1, respectively. (When C has order 2 there are such lattices in $SL(2, \mathbb{R})$.)

THEOREM 2. *Let G be a connected Lie group with a virtually Abelian lattice Γ. Then G/Γ is compact.*

PROOF: Let $p : G \to G/\operatorname{Rad}(G)$ be the natural epimorphism, where $\operatorname{Rad}(G)$ is the radical of G. Since Γ is amenable and G/Γ has finite volume, G is amenable, by [**13,** Proposition 4.1.11]. (This follows easily from the definition of amenability.) Hence $G/\operatorname{Rad}(G)$ is compact, by [**13,** Corollary 4.1.9]. The closure of $p(\Gamma)$ in $G/\operatorname{Rad}(G)$ is a compact Lie subgroup, and so has finitely many components. On replacing Γ by a subgroup of finite index, if necessary, we may assume that $p(\Gamma)$ is Abelian, and hence that $\overline{p(\Gamma)}$ is Abelian. Let H_o be the component of the identity in $H = p^{-1}\left(\overline{p(\Gamma)}\right)$ and let $\Gamma_o = H_o \cap \Gamma$. Then Γ_o is a lattice in the connected solvable Lie group H_o, and therefore is cocompact, by [**10,** Theorem 3.1]. It follows easily that Γ is cocompact. ∎

A group Γ is an *n-dimensional crystallographic group* if it is a lattice in $\operatorname{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes O(n)$, the isometry group of Euclidean n-space $\mathbb{E}^n$. The intersection of Γ with the translation subgroup $\mathbb{R}^n$ is free Abelian of rank n, has finite index in Γ and is the maximal Abelian normal subgroup of Γ. An n-dimensional crystallographic group is *orientable* if it is a subgroup of $\mathbb{R}^n \rtimes SO(n)$.

COROLLARY. *Let Γ be a crystallographic group. Then Γ is a lattice in a connected Lie group if and only if it is orientable.*

PROOF: Suppose that Γ is an n-dimensional crystallographic group which is a lattice in the connected Lie group G. Let K be a maximal compact subgroup of G. Then Γ acts discretely and cocompactly by left multiplication on the symmetric space $X = G/K$, which is diffeomorphic to $\mathbb{R}^d$ for some d. Let A be the maximal Abelian normal subgroup of Γ. Then $A \cong Z^n$, and A acts cocompactly on $\mathbb{R}^d$, since $[G : A]$ is finite. As A also acts discretely we must have $d = n$. Since G is connected every element of G must preserve the orientation of X. The converse is clear. ∎

In particular, the Klein bottle group is not isomorphic to such a lattice.

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Lott's list can be revised in the light of these results as follows. Either

(i) Γ is a lattice in $\mathrm{PSL}(2,\mathbb{R})$; or

(ii) $\Gamma \cong Z \times F(r)$ for some nonnegative integer r; or

(iii) Γ is isomorphic to a torsion-free nonuniform lattice in $\mathrm{PSL}(2,\mathbb{C})$.

The only lattices in $\mathrm{PSL}(2,\mathbb{R})$ which do *not* have positive deficiency are those with signature $(0; e_1, \ldots e_k; t)$ where $t = 0$ or 1 (see [**6**, p. 99]).

The torsion free groups of type (i) are the finitely generated nonabelian free groups and the fundamental groups of aspherical closed orientable surfaces other than the torus, and have deficiency greater than 1, while the groups of types (ii) and (iii) have deficiency 1, as observed in [**7**]. All these groups are also fundamental groups of compact orientable 3-manifolds with nonempty boundary. (A torsion free nonuniform lattice in $\mathrm{PSL}(2,\mathbb{C})$ is the fundamental group of a compact orientable $\mathbb{H}^3$-manifold whose boundary is a nonempty union of tori. Every such manifold is homotopy equivalent to a finite aspherical 2-complex with Euler characteristic 0.)

The multiplicativity of the Euler characteristic in finite coverings appears to be of little use when the deficiency is not positive, and the range of examples is much greater. There are already many examples in dimension 3. If M is a compact 3-manifold then $\pi = \pi_1(M)$ has a presentation of deficiency 0, and has positive deficiency if and only if either ∂M has an aspherical component or M has $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ as a summand. The cocompact lattices in connected 3-dimensional Lie groups were determined in [**11**]. As $\pi_2(G) = 0$ for any Lie group, 3-dimensional coset spaces G/Γ have no $S^2 \times S^1$ summands, and so such lattices have deficiency 0. More generally, we have the following result. We shall say that a compact manifold is *geometric* if its interior is homeomorphic to a manifold with a complete geometry of finite volume in the sense of Thurston [**12**].

THEOREM 3. *Let M be a compact 3-manifold with fundamental group π. Then M is aspherical and geometric if and only if all boundary components of M are aspherical, M has no fake 3-cells and π is torsion free but not free and is a lattice in a Lie group with finitely many components.*

PROOF: The conditions are clearly necessary. Suppose that they hold. A 3-manifold with no fake 3-cells is aspherical and Seifert fibred or is an infrasolvmanifold if and only if it is finitely covered by such a manifold [**3**], and is hyperbolic if and only if it is finitely covered by an $\mathbb{H}^3$-manifold [**4**]. Thus, on passing to a subgroup of finite index, if necessary, we may assume that M is orientable and π is a lattice in a connected Lie group G. Let $G_1 = \mathrm{Rad}(G)K$, where K is the maximal compact connected normal subgroup of a Levi subgroup of G, and let $p : G \to G_2 = G/G_1$ be the canonical epimorphism. Then $G_1 \cap \pi$ and $p(\pi)$ are lattices in G_1 and G_2, respectively [**1**].

Suppose first that $G_1 \cap \pi \neq 1$. Since G_1 is an extension of a compact group by

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a solvable group $G_1 \cap \pi$ is amenable, by [**13**, Proposition 4.1.11]. Since the compact quotient $G_1/\operatorname{Rad}(G) \cong K$ is semisimple $G_1 \cap \pi$ is virtually solvable, by Tits' theorem. Hence M is either Seifert fibred, with interior a $\mathbb{H}^2 \times \mathbb{E}^1$- or $\widetilde{SL}$-manifold, or is an infrasolvmanifold [**3**]. In all cases it is geometric.

If π has no nontrivial solvable normal subgroup then we may assume that $G = G_2$, which is a semisimple Lie group whose Lie algebra has no compact factors. The group π therefore acts discretely with finite covolume on the symmetric space X of compact subgroups of G. Moreover π is irreducible. Therefore X must be typewise homogeneous, and has no Euclidean factor. (See [**9**, Section 7 of Chapter IX].)

If X has rank greater than 1 then π is arithmetic and its Abelianisation π/π' is finite, by [**9**, Theorems IX.1.11 and IX.7.14], respectively. Since $H_1(M;\mathbb{Z}) \cong \pi/\pi'$ is finite so is $H_1(\partial M;\mathbb{Z})$. Hence M must be a closed 3-manifold, as it has no spherical boundary components. Moreover π is a duality group, by [**2**, Theorem 11.4.1], and has cohomological dimension at least 2, as it is not free. Hence π is not a nontrivial free product, so M is aspherical and π is a Poincaré duality group. Therefore $\dim(X) = cd(\pi) = 3$, by [**2**, Theorem 11.4.1] again. However there are no 3-dimensional symmetric spaces of rank greater than 1. (See [**5**, Section 6 of Chapter X].) Therefore we may assume that X has rank 1. If now X/π is compact then $cd(\pi) = \dim(X)$. If X/π is not compact then $cd(\pi) = \dim(X) - 1$, and π contains parabolic subgroups of the same cohomological dimension. Since $cd(\pi) \le 3$ we may conclude that $X = \mathbb{H}^2$ or $\mathbb{H}^3$, in either case. If $X = \mathbb{H}^2$ then π is the fundamental group of an aspherical closed orientable surface (since it is not free) and the interior of M is a $\mathbb{H}^2 \times \mathbb{E}^1$-manifold. If $X = \mathbb{H}^3$ then M is homotopy equivalent to an $\mathbb{H}^3$-manifold, and therefore is homeomorphic to an $\mathbb{H}^3$-manifold, by [**4**]. ∎

The fundamental group $\pi = \pi_1(M)$ of any compact 3-manifold M is virtually torsion free. If π has nontrivial torsion then M is not aspherical. Therefore if moreover π is isomorphic to a lattice in a Lie group with finitely many components it must be virtually free, by the theorem. Hence it is a free product of finite 3-manifold groups and a free group. Finite cyclic groups are lattices in $SO(2)$. The group $\pi_1(RP^3 \sharp RP^3) \cong (Z/2Z) * (Z/2Z)$ is a lattice in $\operatorname{Isom}(\mathbb{E}^1)$. However, it is not a lattice in a connected Lie group, by the Corollary to Theorem 2. All other free products of cyclic groups are isomorphic to lattices in $\operatorname{PSL}(2,\mathbb{R})$ with signature of the form $(0; e_1, \dots e_k; t)$, where t is positive.

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