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SOME ELEMENTARY EXACT CHANNEL FLOWS

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Abstract

New polynomial solutions of the Navier-Stokes equations for steady uni-directional flow of a viscous incompressible fluid, with a free surface, down inclined channels of specialized cross-section are considered. An inverse method is used to obtain the geometrical shape of the channel by equating the polynomial solution to zero (*i.e.* the no-slip condition) and thence determining the boundary shape.

1. Introduction

Although laminar flows in open channels require exceptionally low speeds, small depths and smooth beds, it is useful to study such fluid behaviour because of possible applications to drainage, run-off or overland flow problems. When surface tension is neglected these flows have a no-shear boundary condition on the free surface requiring that the tangential shearing stress and the component of flow velocity normal to the surface both vanish on it. Now it is well known that a no-shear surface may be realized in practice at a plane of symmetry for uni-directional flow under gravity through an inclined pipe; hence pipe flows and channel flows are closely related. Various exact solutions of the pipe-flow equations have been obtained through the analogy with the problem of torsion of an elastic bar having the same cross-section as the pipe (see [1], Chapter V and [2], Section 37). Of these the only exact solutions given in the form of finite polynomials are those for pipes whose cross-sections are an ellipse (including a circle) and an equilateral triangle, and so the corresponding exact channel flows are for the semi-ellipse and $(30°, 60°, 90°)$-triangle. We present here some new polynomial channel-flow solutions and consequently their related pipe flows.

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2. Uniform channel flows

Consider uni-directional steady flow under gravity down a channel at an angle α to the horizontal. Axes $Oxyz$ are chosen with z in the direction of flow, y normal to the free surface, x across the stream at this surface and O on one edge of the free surface (see Figure 1).

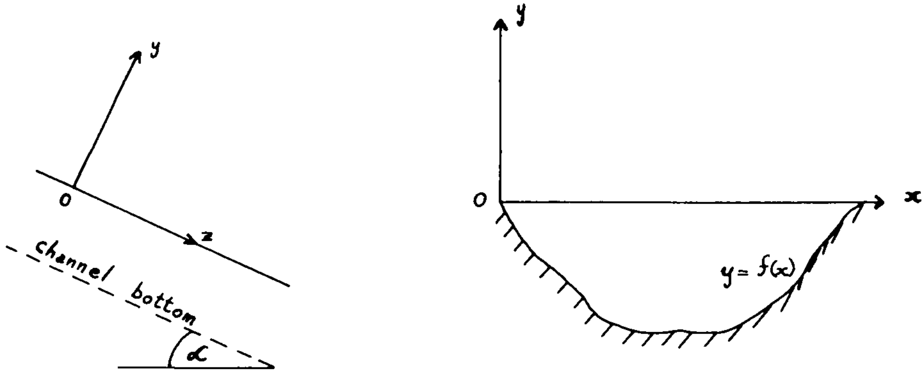

Figure 1. Coordinate system for the channel flows.

The velocity is represented by $(0, 0, w)$ and hence the continuity equation for an incompressible fluid gives $w = w(x, y)$.

The boundary conditions on the free surface, when surface tension is neglected, are

$$[p]_{y=0} = p_0, \qquad [\partial w/\partial y]_{y=0} = 0,$$

where p_0 is the atmospheric pressure. Thus, if ρ is the density and g is the acceleration due to gravity, the momentum equations yield the pressure relation

$$p = p_0 - y\rho g \cos \alpha$$

everywhere. The partial differential system for w is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{g \sin \alpha}{\nu} = -K, \tag{1}$$

where ν is the kinematic viscosity and K is constant, subject to the boundary conditions

$$[\partial w/\partial y]_{y=0} = 0 \tag{2}$$

and the no-slip boundary condition

$$w = 0 \tag{3}$$

on all boundaries of the channel that lie below the free surface. Therefore the solution obviously depends on the geometry of the channel cross-section.

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Two simple channel sections lead to previously-derived polynomial solutions of equations (1) to (3). These are the triangle $x \leqslant a$, $y \leqslant 0$, $x + \sqrt{3}y \geqslant 0$ for which the solution is

$$w_T = (K/4a)(x - a)(3y^2 - x^2)$$

and the semi-ellipse $(2x - a)^2/a^2 + y^2/b^2 \leqslant 1, y \leqslant 0$ for which

$$w_E = \frac{Ka^2b^2}{2(a^2 + 4b^2)}\left\{\frac{4x}{a} - \frac{4x^2}{a^2} - \frac{y^2}{b^2}\right\}.$$

A particular case of the latter, when the channel depth $b = \frac{1}{2}a$, produces the exact solution for flow down a semi-circular channel, is

$$w_C = \tfrac{1}{4}K(ax - x^2 - y^2).$$

These solutions are identical to the corresponding solutions for flow under gravity down elliptical and equilateral-triangular pipes.

3. Generation of new polynomial solutions

The forms for w_E and w_T suggest that we investigate the possibility of further solutions of (1) to (3) which are finite polynomials in x and y. It is soon evident that the only possible 2-variable polynomials are those with the form

$$w = \sum_{k=0}^{M} y^{2(M-k)}P_{2k+1}(x) \tag{4}$$

where M is any finite positive integer and

$$P_j(x) = \sum_{i=0}^{j} A_{ij}x^i. \tag{5}$$

Thus w has $(M + 2)(M + 1)$ unknown coefficients A_{ij}. Note that the free surface boundary condition (2) is automatically satisfied by (4) and (5). When (4) is substituted into (1) and terms with the same power of y are equated then

$$P''_{2M+1}(x) + 2P_{2M-1}(x) = -K,$$

$$P''_{2M+1-2k}(x) + (2k + 2)(2k + 1)P_{2M-1-2k}(x) = 0 \qquad (k = 1,\dots,M - 1).$$

The coefficients of x^i are equated in each of these equations to yield $M(M + 1)$ equations for the unknown coefficients A_{ij}.

New exact solutions of (1) to (3) are thus possible in the form given by (4) and (5) if the remaining $2(M + 1)$ coefficients can be obtained from the no-slip condition (3). Since this is the only condition that incorporates the geometry of the channel cross-section, we propose to generate new solutions by imposing the no-slip condition and investigating the resulting equation to see if it produces physically reasonable channel sections. This will be done for various values of M and for channels with 2, 1 and 0 vertical sides respectively.

4. Channels with two vertical sides

The no-slip condition becomes

$$\begin{aligned} w &= 0 \quad \text{on } x = 0, \qquad y \leqslant 0, \\ w &= 0 \quad \text{on } x = a, \qquad y \leqslant 0, \\ w &= 0 \quad \text{on } y = f(x), \quad 0 \leqslant x \leqslant a, \end{aligned}$$

where a is the width of the channel. The conditions on the vertical walls require

$$\begin{aligned} P_{2k+1}(0) &= 0 \quad (k = 0, 1, \ldots, M), \\ P_{2k+1}(a) &= 0 \quad (k = 0, 1, \ldots, M), \end{aligned}$$

another $2(M + 1)$ equations. We now have an equal number of coefficients and equations, and so the condition on the bottom boundary cannot be satisfied. Essentially this means that for finite-polynomial solutions we must remove this boundary condition and only consider channels of infinite depth. (However there is an infinite series solution for the rectangular cross-section that can be obtained by separation of variables.) Taking $M = 1$ (or any other finite positive integral value) we easily solve the A_{ij} equations to obtain

$$w = \tfrac{1}{2}Kx(a - x),$$

the expected solution for flow vertically downwards under gravity between parallel plates.

5. Channels with one vertical side

The no-slip condition is

$$\begin{aligned} w &= 0 \quad \text{on } x = a, y \leqslant 0, \\ w &= 0 \quad \text{on } y = f(x), 0 \leqslant x \leqslant a, \end{aligned}$$

with $f(0) = 0$ (see Figure 2). The condition on $x = a$ gives $(M + 1)$ equations and the condition $f(0) = 0$ produces another equation. Therefore there are M coefficients left to be determined from the remaining condition. With $M = 1$, (4) and (5) become

$$w = y^2(A_{01} + A_{11}x) + (A_{03} + A_{13}x + A_{23}x^2 + A_{33}x^3). \tag{6}$$

When this form is substituted into equations (1) to (3) they are satisfied completely by

$$w = (x - a)\left[-3A_{33}y^2 + x\left(-\tfrac{1}{4}K - 2aA_{33} + A_{33}x\right)\right]$$

provided that the expression in square brackets, when put equal to zero, represents the sloping side of the channel cross-section. Here A_{33} is our one arbitrary constant. The only physically sensible sections defined by this for $0 \leqslant x \leqslant a$, $y \leqslant 0$ are those with A_{33} values given by

$$-K/4a \leqslant A_{33} \leqslant 0.$$

When $A_{33} = 0$ the solution reduces to that of Section 4; when $A_{33} = -K/4a$ the exact solution w_T for the triangle is obtained; while for $-K/4a < A_{33} < 0$ the channel sloping wall is the lower part of the positive branch of a hyperbola which passes through the origin. The associated solutions w are new exact solutions of the Navier-Stokes equations.

For example, with $A_{33} = -K/8a$ the velocity field is given by

$$w = (K/8a)(a - x)\left[x(x + 2a) - 3y^2\right],$$

and the channel wall is $y = -\sqrt{x(x + 2a)/3}$ together with $x = a$ (see Figure 2). Thus the depth of the channel is a, and for other values of A_{33} the depth is seen to take any value greater than $a/\sqrt{3}$.

With $M = 2$, (4) and (5) become

$$w = y^4(A_{01} + A_{11}x) + y^2(A_{03} + A_{13}x + A_{23}x^2 + A_{33}x^3) + (A_{05} + A_{15}x + A_{25}x^2 + A_{35}x^3 + A_{45}x^4 + A_{55}x^5).$$

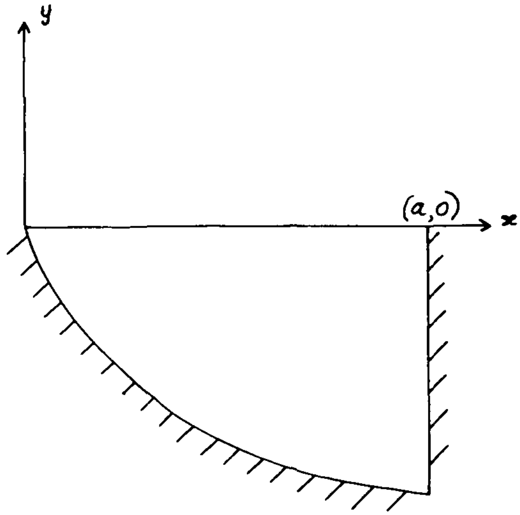

Figure 2. Channel with one vertical side and hyperbolic sloping side.

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When this is substituted into (1) to (3) they are satisfied by

$$w = (x-a)$$
$$\times\left[5A_{55}y^4 + \left\{-3A_{35} + 20a^2A_{55} + 20aA_{55}x - 10A_{55}x^2\right\}y^2\right.$$
$$\left. + x\left\{-\tfrac{1}{2}K - 2aA_{35} + A_{35}x + 16a^3A_{55} - 4a^2A_{55}x - 4aA_{55}x^2 + A_{55}x^3\right\}\right],$$

again provided that the expression in square brackets, when put equal to zero, defines the sloping side of a physically sensible channel. The two arbitrary constants A_{35} and A_{55} must be chosen so that $y < 0$ for all $0 < x < a$. Note that if $A_{55} = 0$ we regain the solutions for $M = 1$.

Other suitable combinations of $A_{55} \neq 0$ and A_{35} produce further exact solutions for flow down channels. For example, with $A_{55} = K/64a^3$, $A_{35} = 0$, the velocity distribution is given by

$$w = (K/64a^3)(x-a)$$
$$\times\left\{5y^4 + 10(2a^2 + 2ax - x^2)y^2 - x(16a^3 + 4a^2x + 4ax^2 - x^3)\right\}$$

and the sloping wall is

$$y = -\sqrt{x^2 - 2ax - 2a^2 + \left(2\sqrt{5}/5\right)\sqrt{x^4 - 4ax^3 + a^2x^2 + 14a^3x + 5a^4}}\ .$$

This shape for $0 \leqslant x \leqslant a$ is very similar to a slightly perturbed hyperbolic curve.

The technique can be extended for larger values of M to produce more exact solutions but the algebra quickly becomes unwieldy.

6. Channels with no vertical side

The no-slip condition is

$$w = 0 \quad \text{on } y = f(x), \qquad 0 \leqslant x \leqslant a,$$

with $f(0) = f(a) = 0$. The conditions at $x = 0$, a produce two equations and so there are $2M$ coefficients left to be determined from the remainder. For $M = 1$ we again use (6), which on substitution into the relevant equations yields

$$w = y^2\left(-\tfrac{1}{2}K - A_{23} - 3A_{33}x\right) + x(x-a)\left(A_{23} + A_{33}a + A_{33}x\right)$$

as the exact solution provided that the expression on the right hand side equal to zero defines a physically reasonable channel bottom.

With $A_{33} = 0$ and $-K/2 < A_{23} < 0$ the semi-elliptical channel solutions are obtained, where the channel depth b is related to A_{23} by

$$A_{23} = -2Kb^2/\left(a^2 + 4b^2\right).$$

In general, the no-slip boundary is

$$y^2 = 2x(x-a)(A_{33}x + A_{33}a + A_{23})/(6A_{33}x + 2A_{23} + K),$$

and we have to determine where the right-hand side is positive for all $0 < x < a$. The appropriate region in the (A_{33}, A_{23})-plane is the inside of the quadrilateral whose vertices are (0,0), $(K/(4a), -K/2)$, $(0, -K/2)$ and $(-K/(4a), K/4)$. The semi-ellipse results coincide with the vertical diagonal of this quadrilateral, and other new exact solutions are associated with the remaining (A_{33}, A_{23})-values inside the quadrilateral.

For example, consider $A_{23} = 0$, $A_{33} = -K/(8a)$, then

$$y^2 = x(x-a)(x+a)/(3x-4a),$$

and the shape of the channel is

$$y = -\sqrt{x(a-x)(a+x)/(4a-3x)} \qquad (0 \leqslant x \leqslant a).$$

This has a turning point at an x-value which is the root of $3x^3 - 6ax^2 + 2a^3 = 0$ giving $x \approx 0.7a$ and hence $y \approx -0.4a$. This indicates that the exact solution

$$\begin{aligned} w &= y^2\left(-\frac{K}{2} + \frac{3Kx}{8a}\right) + \left(\frac{Kax}{8} - \frac{Kx^3}{8a}\right) \\ &= (K/8a)\{x(a^2 - x^2) - y^2(4a - 3x)\} \end{aligned}$$

represents flow under gravity down an asymmetrical channel whose depth is less than half its width (see Figure 3).

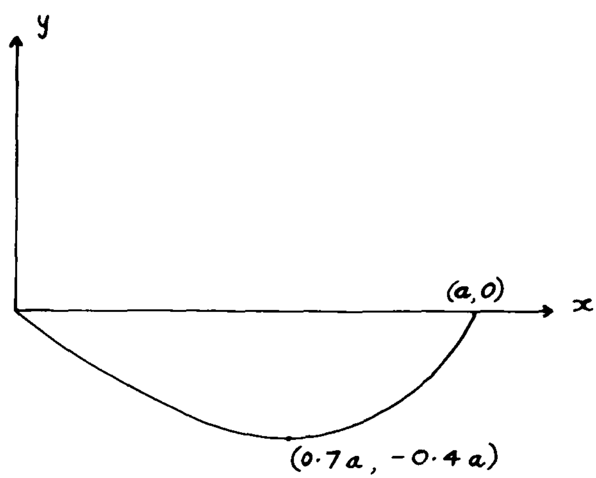

Figure 3. Asymmetrical channel shape.

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When $M = 2$ the channel boundary is a quartic in y containing 4 arbitrary constants. This leads to indentations in the channel wall, which occur with greater frequency as higher values of M are chosen.

7. Conclusion

Some new exact solutions of the Navier-Stokes equations for steady uni-directional viscous flow down sloping channels are presented. These are all finite-polynomial solutions leading to cross-sectional channel shapes that are determined by the no-slip requirement on the wetted boundary of the channel. Discharge rates can be obtained by direct integration. The solutions are applicable to the corresponding pipe flows, and are related to analogous torsion problems in elasticity theory.

References

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