MORE PROBLEMS CONNECTED WITH CONVEXITY

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(received April 5, 1968)

This is a continuation of the author's article [3], and it contains further problems connected with the theory of convex sets in E^n. To the list of general references in [3] may be added the recent book [2] on convex polyhedra.

1) Let A and B be two convex bodies in E^2 and let a packing $P = \{B_1, B_2, \ldots\}$ be an infinite sequence of homothetic images of B such that:

a) each B_n is a subset of A,

b) no two of them share interior points,

c) $\text{Area}(A) = \sum_1^\infty \text{Area}(B_n)$.

The existence of such packings is guaranteed by Vitali's Theorem. Let $D(X)$ be the diameter of the set X and put $M_a(P) = \sum_1^\infty D^a(B_n)$. Is it true that there exists a constant $c = c(B)$ such that $c > 1$ and $M_a(P)$ diverges for every P if $a < c$ while it converges for some P if $a > c$? Is it true that $\max_B c(B) = \lg 3/\lg 2$ (attained when B is a triangle) and $\min_B c(B) = 1.306951$ (attained when B is a circular disk)? In what sense is $c(B)$ a measure of the roundness of B? Is the Hausdorff dimension of the residual set $A - \bigcup_{n=1}^\infty B_n$ always at least c? What happens if the requirement that B_n be a homothet of B is changed so that B_n can be a rotated homothet? An affine image? How does the situation change in E^n for $n > 2$?

https://doi.org/10.4153/CMB-1968-058-0 Published online by Cambridge University Press

2) Let K be a strictly convex body in E^n $(n \geq 2)$; K is said to be s.i.u. (= simplex inscription universal) if for any simplex X in arbitrary location relative to K, a homothet of K has all its n + 1 vertices in the boundary ∂K of K. Let ϕ be the map from ∂K to the unit sphere S in E^n by means of the parallel support planes. Call a point $s \in S$ regular if $s = \phi(x) = \phi(y)$ implies $x = y$, and let I(K) be the subset of S consisting of all points which are not regular. Let m(A) be any reasonable measure defined for all sufficiently regular subsets A of S. Is it true that K is s.i.u. if and only if $m(I(K)) = 0$? It is easy to show that if K is smooth (i.e. $I(K) = \phi$) then it is s.i.u. But a non - smooth body K can also be s.i.u.; an example in E^3 is provided by taking S, slicing off two spherical caps whose bases are small circles externally tangent at a point p, taking the closed convex hull, and rounding off the two circular rims so that the radius of the rounding approaches 0 toward p. What (strictly) convex bodies K have the property that for any simplex X in arbitrary orientation relative to K one and only one homothetic image of X can be inscribed into K?

3) Let P be a convex polyhedron in E^3. By a dihedral angle at the edge e we mean, as usual, the angle between the two outward-bound normals to the faces of P meeting at e. Let α be given, $0 < \alpha < \pi$; what is the largest (smallest) number of faces of P if every dihedral angle is $\leq \alpha$ $(\geq \alpha)$? Similar problems can be set up for the numbers of vertices and edges.

4) The rigidity of convex polyhedra in E^3 has been proved already by Cauchy; a well known but apparently unsolved problem is that of the rigidity of non-convex polyhedra. Let P be a convex polyhedron in E^3; P is said to be edge-deformable if there is a convex polyhedron Q, distinct from P, into which P can be continuously changed without introducing any new vertices, edges or walls throughout the change, and so that no edges change their length (in brief: P and Q have the same edge-lengths, are non-congruent, and are polyhedrally homotopic). Otherwise P is called edge-rigid. If P is edge-deformable we let f(P) be the dimension of the family of the admissable deformations. For instance, a regular tetrahedron and octahedron are edge-rigid, while a cube C is edge-deformable and has $f(C) = 2$. What are necessary and sufficient conditions on P to be edge-deformable? What is the minimum number of edges, (walls, vertices) in P so that $f(P) = N$? What is f(P) in terms of combinatorial and metrical parameters of P? If P is edge-deformable how small a change will convert it into an edge-rigid polyhedron?

5) Let A and B be two convex bodies in E^n $(n \geq 2)$ of fixed volumes a and b respectively, let $D(A) > D(B)$, and let B have a centre of symmetry. For any point $p \in A$ let B(p) denote the translate of B centred at p. Let F(A, B) be the probability that when p and q are taken at random in A then $B(p) \cap B(q) = \phi$. Is the maximum of

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$F(A, B)$ attained? If so, for what bodies A and B? The same problem may be phrased for other configurations; for instance, $F(m, A, B)$ may be the probability that with m points $p_1, p_2, \ldots, p_m$ at random in A the sets $B(P_i)$ are in some prescribed configuration (pairwise disjoint, union connected, etc.).

6) Let $B_a(p)$ be the ball in E^n $(n \geq 2)$ centred at p and of radius a. Let $0 < s < t$ and let K be a subset of $B_t(p)$ which is star-shaped at every point of $B_s(p)$. Suppose that s is the largest and t the smallest value possible for this K. What can be given by way of good upper and lower bounds on the probability that a plane which cuts K, cuts it in a connected set? A simply connected set?

The next three problems are in the form of questions and the affirmative answer in each case implies the truth of the famous as yet unproved conjecture of Borsuk. While this probably means that the questions are hard to answer, one may consider whether the Borsuk conjecture implies the truth of those expressed below, and one may also weaken (or strengthen) the conditions in various ways.

7) Let K be a convex body in E^n $(n \geq 2)$ of constant width 1. Let $M(K)$ be the set of midpoints of all chords in K of length 1. Is it true that $M(K)$ can be inscribed into a simplex X which lies in the interior of K? The affirmative answer implies the truth of Borsuk's conjecture, for in that case the $N + 1$ planes of the $n + 1$ $(n - 1)$-dimensional faces of X divide ∂K into $n + 1$ sets of diameter < 1. More generally, if $M(K)$ can be inscribed into a convex polyhedron with $\leq f(n)$ walls then K is a union of $\leq f(n)$ sets of diameter < 1. It may be recalled here that the best value known to date of $f(n)$ is still of the exponential order $c^n (c > 1)$.

8) Let K be a convex body in E^n $(n \geq 2)$ of constant width 1. For a point x in the interior of K, let R be a ray emanating from x. R is called a ray of visibility if the subset of ∂K seen from the point $y \in R$, unobstructed by K, has diameter < 1 for all $y \in R$. It is easy to show that whether R is, or is not, a ray of visibility depends only on its direction, not on its origin x. Is it true that every open half-space, containing some but not all of the interior of K, contains a visibility ray? Is the same still true if the half-space is replaced by the interior of a cone of semi-vertical angle α $(\alpha > \pi/3)$? If the answer to the first question is yes, then K can be inscribed into a simplex X whose vertices lie on visibility rays; hence ∂K is a union of the $n + 1$ closures of the visibility sets from the vertices of X, all of them of diameter < 1.

9) Let K be a convex body in E^n $(n \geq 2)$ of constant width 1. Let F be the union of $n + 1$ rays $R_1, R_2, \ldots, R_{n+1}$ emanating from a point p and not all on one side of any plane through p. For $s \in K$

let $F(s)$ be the translate of F to the origin s, and let $F(s, K)$ be the sum of the lengths of those parts of the rays, which lie in K. Is it true that F can be found so that $\min\{F(s,K): s \in \partial K\} > 0$? If the answer is yes, then K lies in the interior of certain $n+1$ translates of K; these can then be shrunk down in the ratio $\lambda : 1$ $(\lambda < 1)$.

10) Proceeding as in problem 8), we let R be a ray in the direction u; for $y \in R$, we let $V(y)$ be the (open) visibility set in ∂K, and we let $H(K,u)$ be the $\bigcup\{V(y) : y \in R\}$. $H(K,u)$ is called an open hemisphere of K in the direction u, and $E(K,u) = \partial K - H(K,u) - H(K,-u)$ is called the equator of K in the direction u (or $-u$). What is the minimum number $g(n)$ such that every K has an equator which can be covered by some $g(n)$ open hemispheres of K? Is the Borsuk conjecture true if $g(n) = n$ for all K?

11) Let E be an ellipsoid in E^3 with semi-axes a, b, c. Let P be a plane at random through the centre of E, and let α and β be the semi-axes of the ellipse $E \cap P$. If the mean values of α and β are known what can one say about a, b, c? What are a, b, c in terms of means and variances of α and β? What are they if the joint distribution of α and β is known? Is known by a sample of size N? What is a good estimate of the volume $4\pi abc/3$ of E? (This problem arose in a biological laboratory where a block of tissue was examined for the presence of certain types of cells, approximating a fixed ellipsoid, by a microscopic examination of slices of that block; it may be assumed that P cuts E centrally because the cell types in question have an easily recognizable centrally located nucleolus).

12) Let $K \subset E^n$ $(n \geq 2)$ be a convex body. Let G be a connected graph with n vertices $v_1, \ldots, v_n$ such that no vertex is connected to itself by an edge, and no pair of vertices can be connected by more than one edge. When can G be realized as an intersection graph for some n translates $K_1, \ldots, K_n$ of K (that is, $K_i \cap K_j \neq \phi$ if and only if v_i and v_j are joined in G by an edge)? When n is large, are most graphs G realizable or not?

13) Let $\phi(A)$ be a real-valued non-negative functional defined for all convex bodies in E^n (n fixed, ≥ 2), and satisfying these conditions: a) $\phi(A) = \phi(B)$ if A and B are congruent, b) a positive constant a exists such that $\phi(\lambda A) = \lambda^a \phi(A)$ for all $\lambda > 0$, c) $\phi(A) \leq \phi(B)$ if $A \subseteq B$, d) $\phi(A)$ is continuous in A, with respect to the Hausdorff proximity metric. An isodiametric problem for $\phi(A)$ is to maximize it subject to the side-condition $D(A) = 1$; it is easy to show that the maximum is attained, and that it is attained for a convex body of constant width 1. If $\phi(A)$ is the volume of A, or its surface area, then the maximum in the isodiametric problem is attained when A is a ball. The same might be true for the Borsuk functional $\phi(A)$ defined to be the infimum of all numbers x such that A is a union

of $n+1$ sets of diameter $\leq x$. Are there some general further conditions on $\phi(A)$ which ensure that the maximum in the isodiametric problem is assumed when A is a ball?

14) All convex bodies and manifolds in this problem are supposed to be sufficiently smooth. A convex body K in E^n $(n \geq 2)$ of constant width d can be defined as follows. For $x \in \partial K$ let $a(x)$ be the point antipodal to x; let $\rho_1(x), \ldots, \rho_{n-1}(x)$ be the $n-1$ principal radii of curvature ∂K at x, arranged in non-decreasing order of magnitude. $R(x) = (\rho_1(x), \ldots, \rho_{n-1}(x))$ is then a row vector of length $n-1$, we define $\bar{R}(x)$ to be $(\rho_{n-1}(x), \ldots, \rho_1(x))$. Let I be the unit vector $(1, \ldots, 1)$ of length $n-1$. Now K is of constant width if and only if $R(x) + \bar{R}(a(x)) = dI$, for all $x \in \partial K$. This suggests the following question: is it possible to define an abstract (i.e. unimbedded) manifold C of constant width d by generalizing the above? Sectional curvatures enable us to define the curvature vector $R(x)$; we suppose that all entries are positive. To define antipodality we let $a(x)$ be a continuous involution on C, possibly satisfying some further conditions (to ensure maximality) and we demand that $R(x) + \bar{R}(a(x)) = dI$ as before, for all $x \in C$.

15) Let K be a star-shaped body in E^3 with a sufficiently smooth boundary (say, analytic). We suppose that the kernel of K, i.e. the set of points at which K is star-shaped, is itself a convex body in E^3. A point $x \in \partial K$ is said to be elliptic (hyperbolic) if some punctured neighbourhood of x in ∂K lies strictly on one (two) side(s) of the tangent plane to ∂K at x; let P be the set of all parabolic points in ∂K (i.e., neither elliptic nor hyperbolic). It is known that the kernel of K is the intersection of half-spaces bounded by tangent planes to ∂K at points of P. Suppose that K is to be cut up into parts $K_1, \ldots, K_n$, each of which can be cast from a mould prepared in the usual fashion. Is it true that a suitable decomposition $K = K_1 \cup \ldots \cup K_n$ can be found so that $\partial(\partial K \cap K_i) \subseteq P$ for every i?

16) Consider any isoperimetric problem for a convex polyhedron P in E^3 (or in E^n), for instance, the problem of maximizing the volume of P while keeping constant the sum of the edge-lengths of P. Does a solution always exist under the additional assumption that P is of a prescribed combinatorial type ? If not, for which types does it exist?

17) Let $S = S^{n-1}$ be the unit sphere in E^n and let $D = \{s_1, \ldots, s_k\}$ be a finite set of points in S. D is said to have the rotation property if for every real-valued continuous function f on S there is a rotation $p = p_f$ of S such that the k values $f(ps_j)$ are all equal. It is known

(1) that any set of three points in S^2 has the rotation property; it may be conjectured that likewise, any set of n points in S^{n-1} has the rotation property. Is it possible to prove this rotation conjecture by transforming the problem into one about convex bodies as follows: a) by using an approximative technique, if necessary, we assume without loss of generality that f satisfies a Lipschitz condition, b) we replace f by $g = af + b$ where a and b are suitable constants, c) we note that g is the support function of a convex body, d) we observe that the rotation conjecture is now equivalent to the following: let K be a convex body with a distinguished point o, let C be a polyhedral cone bounded by some n planes H_i passing through the vertex v of C, and let the ray R inside C, emanating from v, be the locus of points equidistant from all the walls H_i; then, by a rigid motion, K can be placed in C so that each H_i supports it, and in addition o lies on R?

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https://doi.org/10.4153/CMB-1968-058-0 Published online by Cambridge University Press