A GLOBAL EXISTENCE THEOREM

M. Eisen

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The Cauchy-Peano existence theorem (1) does not allow us to decide from the form of a given system of equations whether or not its solution can be continued for the infinite interval $-\infty < t < \infty$. Several sufficient conditions for such a continuation were given by A. Wintner in (2). The main result of his paper is the following theorem which is not proven.

THEOREM. (a) Let $f_i \in C$ in an $n+1$ - dimensional domain $(i = 1, 2, \ldots, n)$.

(b) There exists a function $L(r)$ continuous for $0 < \Gamma < \infty$ and such that $\int_\delta^\infty \frac{dr}{L(r)} = \infty$ for some $\delta > 0$.

(c) $|f_i(x_1, \ldots, x_n, t)| < L(r)$, where $r^2 = x_1^2 + \ldots + x_n^2$.

Then all solutions of the system

$$\frac{dx_i}{dt} = f_i \tag{1}$$

may be continued over the entire real axis.

Although a proof was suggested it is not obvious that the outlined method would work - the following is a simpler proof.

Multiplying the i-th equation in (1) by $2x_i$ and summing with respect to i we have $\left|\frac{dr}{dt}\right|^2 = 2 \sum_{i=1}^{n} x_i f_i$. It follows that $\frac{dr}{dt} < L(r)$ upon noting that $|2 \Sigma x_i f_i| < 2r\, L(r)$; hence $\int_\delta^r \frac{dr}{L(r)} < t - t_0$.

Suppose the solution can only be continued up to t_1. Then

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r must be unbounded in a neighborhood of t_1; otherwise the solution could be continued beyond t_1 since the hypothesis of the Cauchy-Peano existence theorem would be fulfilled. We have a contradiction since $\int_\delta^r \frac{dr}{L(r)} \to \infty$ as $r \to \infty$ and $\int_\delta^r \frac{dr}{L(r)} < t_1 - t_0$.

REFERENCES

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University of Pittsburgh