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PROJECTIVE APPROXIMATIONS

K. VARADARAJAN

Introduction. Let R be an associative ring with $1 \neq 0$. Throughout we will be considering unitary left R-modules. Given a chain complex C over R, a free approximation of C is defined to be a free chain complex F over R together with an epimorphism $\tau:F \rightarrow C$ of chain complexes with the property that $H(\tau):H(F) \simeq H(C)$. In Chapter 5, Section 2 of [**3**] it is proved that any chain complex C over $\mathbf{Z}$ has a free approximation $\tau:F \rightarrow C$. Moreover given a free approximation $\tau:F \rightarrow C$ of C and any chain map $f:F' \rightarrow C$ with F' a free chain complex over $\mathbf{Z}$, there exists a chain map $\varphi:F' \rightarrow F$ with $\tau \circ \varphi = f$. Any two chain maps φ, ψ of F' in F with $\tau \circ \varphi = \tau \circ \psi$ are chain homotopic. The proof given in [**3**, pp. 225-226] is valid word for word when $\mathbf{Z}$ is replaced by a principal ideal domain R. A projective approximation of C could be defined as a projective complex C together with an epimorphism $\tau:P \rightarrow C$ with $H(\tau):H(P) \simeq H(C)$. Observing that any submodule of a projective module is projective whenever R is a Hereditary ring, the proof on pages 225-226 of [**3**] yields the result that any chain complex C over a Hereditary ring admits a projective approximation $\tau:P \rightarrow C$. Moreover, given a chain map $f:P' \rightarrow C$ with P' projective, there exists a lift $\varphi:P' \rightarrow P$ of f (i.e., $\tau \circ \varphi = f$). If φ, ψ are any two lifts of f then φ and ψ are chain homotopic. In [**2**] A. Dold proves the existence of a projective approximation $\tau:P \rightarrow C$ of C under any one of the following conditions:

(1) R is an arbitrary ring and C is a positive chain complex over R.

(2) R is a ring of finite global dimension and C is an arbitrary chain complex over R. Actually he deduces this from a decomposition result (Hilfssatz 3.7 in [**2**]) which asserts the following: Let $f:P' \rightarrow C$ be a chain map with P' projective. Assume either P' and C are both positive or that R has finite global dimension. Then there exists a factorization $f = g \circ i$ where $i:P' \rightarrow P$ is an injective chain map with $P/i(P')$ projective (hence P also projective) and $g:P \rightarrow C$ an epimorphism with $H(g):H(P) \simeq H(C)$.

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In case P' and C are positive, P could be chosen to be positive. His proof uses techniques from double complexes.

Given a projective approximation $\tau:P \rightarrow C$ of C and a chain map $f:P' \rightarrow C$ with P' projective, questions about the existence of a lift $\varphi:P' \rightarrow P$ of f and homotopy uniqueness of lifts are not dealt with in [**2**]. Under one of the restrictions that either the chain complexes to be considered should all be positive or the ring R has to have finite global dimension Dold states the following result (Korollar 3.2 in [**2**]). Let $[X, Y]$ denote the additive group of chain homotopy classes of chain maps from X to Y, where X, Y are chain complexes over R. Let $\varphi:X \rightarrow Y$ be a chain map with $H(\varphi):H(X) \simeq H(Y)$ and P a projective chain complex over R. Then the map $[f] \rightarrow [\varphi \circ f]$ yields an isomorphism $[P, X] \rightarrow [P, Y]$. This is again deduced as a consequence of Satz 3.1 in [**2**] which itself follows immediately from 4.3, Chapter XVII of [**1**]. The proof of this result in [**1**] depends heavily on powerful "Hyperhomology" techniques dealing with double complexes and spectral sequences. As an immediate consequence of Korollar 3.2 of [**2**] we see that if $\tau:P \rightarrow C$ is a projective approximation of C and $f:P' \rightarrow C$ is any chain map with P' projective, then there exists a chain map $\varphi:P' \rightarrow P$ with $\tau \circ \varphi \sim f$ ($\sim$ means chain homotopic). Korollar 3.2 does not imply the existence of an actual lift of f.

The main results proved in our present paper could be stated as follows. For any module M we denote the projective dimension of M by h.d M.

THEOREM 1. *Let R be any ring and C a chain complex over R. Assume either that C is positive or that there exists a fixed integer r with* h.d $H_i(C) \leqq r$ *for all i. Then there exists a projective approximation $P \xrightarrow{\tau} C$ of C. In case C is positive there exists a free approximation $F \xrightarrow{\tau} C$ of C.*

THEOREM 2. *Let C be a positive chain complex over a ring R and $P \xrightarrow{\tau} C$ a positive, projective approximation of C. Let $f:P' \rightarrow C$ be any chain map with P' positive and projective. Then there exists a lift $\varphi:P' \rightarrow P$ of f.*

If f, g are maps of P' into C which are chain homotopic and φ, ψ are arbitrary lifts of f, g then $\varphi \sim \psi$.

In case R has finite global dimension, say r then h.d $M \leqq r$ for any R-module M. As a particular case of Theorem 1, we get the result that any chain complex C over a ring R of finite global dimension has a projective approximation. Thus Theorem 1 strengthens the result of Dold stated earlier in the introduction. Our method of proof is quite direct and simple and avoids complicated hyperhomology arguments involving double complexes and spectral sequences.

1. Positive chain complexes. As usual, for any chain complex C we denote the module of i-cycles of C by $Z_i(C)$ and the module of i-boundaries by $B_i(C)$.

LEMMA 1.1. *Let $f:C \to C'$ be a chain map and j a given integer. Suppose*

$$f_{j+1}:C_{j+1} \to C'_{j+1} \quad \textit{and} \quad H_j(C) \xrightarrow{H_j(f)} H_j(C')$$

are onto. Then

$$f_j|Z_j(C):Z_j(C) \to Z_j(C') \textit{ and } f_j|B_j(C):B_j(C) \to B_j(C')$$

are onto maps.

Proof. Writing δ' for the boundary map in C' and δ for the boundary map in C, from the assumption that $f_{j+1}:C_{j+1} \to C'_{j+1}$ is onto, we immediately see that

$$\delta'_{j+1} \circ f_{j+1}:C_{j+1} \to B_j(C')$$

is onto. But $\delta'_{j+1} \circ f_{j+1} = f_j \circ \delta_{j+1}$. Hence any element in $B_j(C')$ can be written as $f_j(\delta_{j+1}c)$ for some $c \in C_{j+1}$. This proves that

$$f_j|B_j(C):B_j(C) \to B_j(C')$$

is onto.

Let $x' \in Z_j(C')$. Writing $[x']$ for the homology class of x', the assumption that $H_j(f):H_j(C) \to H_j(C')$ is onto yields an element $x \in Z_j(C)$ with $[f_j(x)] = [x']$. This means

$$x' - f_j(x) \in B_j(C').$$

Hence, there exists a $b \in B_j(C)$ with $x' - f_j(x) = f_j(b)$. This yields $x' = f_j(x + b)$ and $x + b \in Z_j(C)$.

Definition 1.2. A projective complex P together with an epimorphism $\tau:P \to C$ will be called a *projective approximation of* C if

$$H(\tau):H(P) \simeq H(C).$$

In case P is free, it will be called a *free approximation of* C.

COROLLARY 1.3. *If $\tau:P \to C$ is any projective approximation of C then*

$$\tau|B(P):B(P) \to B(C) \quad \textit{and} \quad \tau|Z(P):Z(P) \to Z(C)$$

are both epimorphisms.

Proof. This is immediate from Lemma 1.1.

From now onwards in Section 1, we will be dealing only with positive chain complexes. A chain complex C will be said to be positive if $C_j = 0$ for $j < 0$. Thus the word chain complex will mean a positive chain complex for the rest of Section 1. For any complex C and any integer $k \geqq 0$, $C^{(k)}$ will denote the k-selection of C, namely $C_i^{(k)} = C_i$ for $i \leqq k$ and $C_i^{(k)} = 0$ for $i > k$. The boundary map $\delta:C_i^{(k)} \to C_{i-1}^{(k)}$ is the same as $\delta:C_i \to C_{i-1}$ for $i \leqq k$. A complex C will be said to be of dimension $\leqq k$ if $C_i = 0$ for $i > k$ or equivalently if $C = C^{(k)}$.

PROPOSITION 1.4. *Let C be a chain complex and k an integer* $\geqq 0$. *Let P be a projective complex of dimension* $\leqq k$ *and* $f:P \to C$ *a chain map satisfying the following conditions*:

(i) $f_i:P_i \to C_i$ *is onto for* $i \leqq k$

(ii) $f_k|Z_k(P):Z_k(P):Z_k(P) \to Z_k(C)$ *is onto, and*

(iii) $H_i(f):H_i(P) \to H_i(C)$ *is an isomorphism for* $i < k$.

Then there exists a projective complex P' *with* dim $P' \leqq k + 1$, $P'^{(k)} = P$ *and a chain map* $f':P' \to C$ *extending* f *and satisfying*

(a) $f'_{k+1}:P'_{k+1} \to C_{k+1}$ *is onto*

(b) $f'_{k+1}|Z_{k+1}(P'):Z_{k+1}(P') \to Z_{k+1}(C)$ *is onto, and*

(c) $H_k(f'):H_k(P') \simeq H_k(C)$.

Proof. Write g_k for $f_k|Z_k(P)$. By assumption $g_k:Z_k(P) \to Z_k(C)$ is onto. Let

$$K_k = g_k^{-1}(B_k(C)).$$

Choose an epimorphism $\alpha:S \to K_k$ with S projective. Writing h_k for $g_k|K_k = f_k|K_k$ we know that $h_k:K_k \to B_k(C)$ is onto. Since

$$C_{k+1} \xrightarrow{\delta_{k+1}} B_k(C) \to 0$$

is exact and S is projective, there exists a map $\beta:S \to C_{k+1}$ with

(1) $\quad \delta_{k+1} \circ \beta = h_k \circ \alpha.$

Clearly $h_k \circ \alpha:S \to B_k(C)$ is onto.

Choose an epimorphism $\gamma:T \to Z_{k+1}(C)$ with T projective. Define P' as follows:

$$P'^{(k)} = P;\ P'_{k+1} = S \oplus T,\ P'_i = 0 \text{ for } i > k + 1$$

and let

$$\delta_{k+1}^{P'}: S \oplus T \to P'_k = P_k$$

be given by $\delta_{k+1}^{P'}(s, t) = \alpha(s)$. Observe that $\alpha(s) \in K_k \subset Z_k(P) \subset P_k$. Now

$$\delta_k^{P'} \circ \delta_{k+1}^{P'}(s, t) = \delta_k^{P}(\alpha(s)) = 0$$

since $\alpha(s) \in Z_k(P)$. Thus P' is a projective complex with dim $P' \leqq k + 1$.

Define $f':P' \to C$ by

$$f'_i = f_i \quad \text{for } i \leqq k$$

and $f'_{k+1}:S \oplus T \to C$ by

$$f'_{k+1}(s, t) = \beta(s) + \gamma(t).$$

We claim that $f':P' \to C$ is a chain map. We have only to check that

$$\delta_{k+1}f'_{k+1}(s, t) = f'_k\,\delta_{k+1}^{P'}(s, t) \quad \text{for any } (s, t) \in S \oplus T.$$

Now,

$$\delta_{k+1}f'_{k+1}(s, t) = \delta_{k+1}\beta(s) + \delta_{k+1}\gamma(t) = \delta_{k+1}\beta(s)$$

(since $\gamma(t) \in Z_{k+1}(C)$) and

$$f'_k\,\delta_{k+1}^{P'}(s, t) = f_k\alpha(s) = h_k\alpha(s).$$

From (1) we see that

$$\delta_{k+1}f'_{k+1}(s, t) = f'_k\,\delta_{k+1}^{P'}(s, t).$$

Clearly $(0, t) \in Z_{k+1}(P')$ for any $t \in T$. Since $f'_{k+1}(0, t) = \gamma(t)$ and $\gamma:T \to Z_{k+1}(C)$ is onto, it follows that

(2) $f'_{k+1}|Z'_{k+1}(P'):Z_{k+1}(P') \to Z_{k+1}(C)$ is onto.

Let $c \in C_{k+1}$. Then, since $\delta_{k+1} \circ \beta:S \to B_k(C)$ is onto, we get an $s \in S$ with $\delta_{k+1}\beta(s) = \delta_{k+1}c$. Hence

$$c - \beta(s) \in Z_{k+1}(C).$$

Since $\gamma:T \to Z_{k+1}(C)$ is onto, there exists a $t \in T$ with $c - \beta(s) = \gamma(t)$. Hence

$$c = \beta(s) + \gamma(t) = f'_{k+1}(s, t).$$

This shows that

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(3) $f'_{k+1}:P'_{k+1} = S \oplus T \to C_{k+1}$ is onto.

Also, $Z_k(P') = Z_k(P)$ and $B_k(P') = \delta^{P'}_{k+1}(S \oplus T) = \alpha(S) = K_k$. Thus

$$f'_k|Z_k(P') = f_k|Z_k(P) = g_k:Z_k(P) \to Z_k(C)$$

is onto and $B_k(P') = K_k = g_k^{-1}(B_k(C))$.

It follows that f'_k induces an isomorphism of

$$H_k(P') = Z_k(P')/B_k(P') = Z_k(P)/K_k$$

onto $H_k(C)$. This completes the proof of Proposition 1.4.

PROPOSITION 1.5. *Let C be any positive chain complex. Then there exists a positive, projective approximation $f:P \to C$ of C.*

Proof. Choose an epimorphism $f_0:P_0 \to C_0$ with P_0 projective. Assume $k \geqq 0$ and that we have constructed projective complexes ${}^{(i)}P$ and chain maps ${}^{(i)}f:{}^{(i)}P \to C$ for $0 \leqq i \leqq k$ satisfying the following conditions.
(a) ${}^{(i)}P =$ the i-th skeleton of ${}^{(i+1)}P$ for $0 \leqq i \leqq k - 1$
(b) ${}^{(i+1)}f|{}^{(i)}P = {}^{(i)}f$
(c) ${}^{(i)}f_j:{}^{(i)}P_j \to C_j$ is onto for $0 \leqq j \leqq i$
(d) ${}^{(i)}f_i|Z_i({}^{(i)}P):Z_i({}^{(i)}P) \to Z_i(C)$ is onto and
(e) $H({}^{(i)}f):H_j({}^{(i)}P) \to H_j(C)$ is an isomorphism of $j < i$.
The construction of the epimorphism $f_0:P_0 \to C_0$ starts the inductive step at $k = 0$. Applying Proposition 1.4, we get a projective complex ${}^{(k+1)}P$ with ${}^{(k)}P =$ the k-th skeleton of ${}^{(k+1)}P$ and a chain map ${}^{(k+1)}f:{}^{(k+1)}P \to C$ extending ${}^{(k)}f:{}^{(k)}P \to C$ such that

$${}^{(k+1)}f_{k+1}:{}^{(k+1)}P_{k+1} \to C_{k+1} \quad \text{and}$$

$${}^{(k+1)}f_{k+1}|Z_{k+1}({}^{(k+1)}P):Z_{k+1}({}^{(k+1)}P) \to Z_{k+1}(C)$$

are onto and

$$H_k({}^{(k+1)}f):H_k({}^{(k+1)}P) \simeq H_k(C).$$

Then $P \xrightarrow{f} C$ defined by $P^{(k)} = {}^{(k)}P$ and $f|P^{(k)} = {}^{(k)}f$ satisfies the requirements of Proposition 1.5.

PROPOSITION 1.6. *Let C be a positive chain complex and $P \xrightarrow{T} C$ a positive, projective approximation of C. Then there exists a positive, free approxima-*

tion $F \xrightarrow{f} C$ *with* P *a subcomplex of* F, $f|P = \tau$ *and* P_i *a direct summand of* F_i *for each* $i \geqq 0$.

Proof. Set $Q_{-1} = 0$. Choose a projective module Q_0 such that $P_0 \oplus Q_0 = F_0$ is free. Assume $k > 1$ and that we have chosen projective modules $Q_0, \ldots, Q_{k-1}$ with $P_i \oplus Q_{i-1} \oplus Q_i = F_i$ is free for $0 \leqq i < k$. We can choose a projective module Q_k with $P_k \oplus Q_{k-1} \oplus Q_k = F_k$ is free. Define

$$\delta_i^F : F_i = P_i \oplus Q_{i-1} \oplus Q_i \to F_{i-1} = P_{i-1} \oplus Q_{i-2} \oplus Q_{i-1}$$

by

$$\delta_i^F(x_i, q_{i-1}, q_i) = (\delta_i^P x_i, 0, q_{i-1})$$

for any $x_i \in P_i$, $q_i \in Q_i$, $q_{i-1} \in Q_{i-1}$. Then it is clear that (F, δ^F) is a chain complex. Moreover

$$\text{Ker } \delta_i^F = Z_i(P) \oplus 0 \oplus Q_i \quad \text{and} \quad \text{Im } \delta_{i+1}^F = B_i(P) \oplus 0 \oplus Q_i.$$

It follows that the obvious inclusion map $j:P \to F$ given by $j(x) = (x, 0, 0)$ for any $x \in P_i$ is a chain map with $H(j):H(P) \simeq H(F)$. The map $f:F \to C$ given by

$$f_i(x_i, q_{i-1}, q_i) = \tau(x_i)$$

for any $(x_i, q_{i-1}, q_i) \in F_i$ is a chain map satisfying the requirements of Proposition 1.6.

We end this section by remarking that Lemma 1.1 and Corollary 1.3 are valid for all chain complexes C. We started assuming C to be positive from Proposition 1.4 onwards.

2. Chain complexes C **with h.d** $H_i(C) \leqq r$ **for all** i**.** We now deal with chain complexes which are not necessarily positive.

PROPOSITION 2.1. *Let* C *be a chain complex which satisfies the condition that* h.d $H_i(C) \leqq r$ *where* r *is a fixed integer. Then there exists an exact sequence* $0 \to A \to P \xrightarrow{f} C \to 0$ *with* P *projective,* $H(f):H(P) \to H(C)$ *onto and* h.d $H_i(A) \leqq \text{Max}(0, r-2)$ *for all* i.

Proof. Let $\eta_i:Z_i(C) \to H_i(C)$ denote the canonical quotient map. Let $\alpha_i:S_i \to Z_i(C)$ be an epimorphism with S_i projective. Write K_i for $\alpha_i^{-1}(B_i(C))$. We then have a commutative diagram of exact rows with vertical maps epimorphisms:

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & S_i & \xrightarrow{\eta_i \circ \alpha_i} & H_i(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_i|K_i} & & \downarrow{\scriptstyle \alpha_i} & & \| & & \\ 0 & \longrightarrow & B_i(C) & \longrightarrow & Z_i(C) & \xrightarrow[\eta_i]{} & H_i(C) & \longrightarrow & 0 \end{array}$$

DIAGRAM 1

Let $T_{i+1} \xrightarrow{\beta_i} K_i$ be an epimorphism with T_{i+1} projective. Then clearly

(4) $\alpha_i \circ \beta_i : T_{i+1} \to B_i(C)$ is an epimorphism.

Since

$$C_{i+1} \xrightarrow{\delta_{i+1}} B_i(C) \to 0$$

is exact and T_{i+1} is projective, there exists a map $g_{i+1} : T_{i+1} \to C_{i+1}$ with

(5) $\delta_{i+1} g_{i+1} = \alpha_i \circ \beta_i$.

Let $P_i = S_i \oplus T_i$ and define

$$\delta_i^P : P_i = S_i \oplus T_i \to P_{i-1} = S_{i-1} \oplus T_{i-1}$$

by

$$\delta_i^P(x_i, y_i) = (\beta_{i-1}(y_i), 0).$$

Observe that

$$\beta_{i-1}(y_i) \in K_{i-1} \subset S_{i-1}.$$

Clearly $\delta_{i-1}^P \circ \delta_i^P = 0$. Hence $\{P, \delta^P\}$ is a chain complex. Clearly each P_i is projective. Define $f_i : P_i \to C_i$ by

$$f_i(x_i, y_i) = \alpha_i(x_i) + g_i(y_i).$$

Then

$$\delta_{i+1} f_{i+1}(x_{i+1}, y_{i+1}) = \delta_{i+1} g_{i+1}(y_{i+1})$$

(since $\alpha_{i+1}(x_{i+1}) \in Z_{i+1}(C)$) and

$$\delta_{i+1} g_{i+1}(y_{i+1}) = \alpha_i \circ \beta_i(y_{i+1})$$

by (5). Also

$$f_i\,\delta^P_{i+1}(x_{i+1}, y_{i+1}) = f_i(\beta_i(y_{i+1}), 0) = \alpha_i\beta_i(y_{i+1}).$$

This shows that $\delta_{i+1}f_{i+1} = f_i\delta^P_{i+1}$ proving that $f:P \to C$ is a chain map. Given $a \in C_{i+1}$, since $\alpha_i \circ \beta_i:T_{i+1} \to B_i(C)$ is onto, we find a $y_{i+1} \in T_{i+1}$ with

$$\delta a = \alpha_i \circ \beta_i(y_{i+1}) = \delta g_{i+1}(y_{i+1}).$$

Hence $a - g_{i+1}(y_{i+1}) \in Z_{i+1}(C)$. Since $\alpha_{i+1}:S_{i+1} \to Z_{i+1}(C)$ is onto, we get

$$a - g_{i+1}(y_{i+1}) = \alpha_{i+1}(x_{i+1})$$

for some $x_{i+1} \in S_{i+1}$. This means

$$a = \alpha_{i+1}(x_{i+1}) + g_{i+1}(y_{i+1}) = f_{i+1}(x_{i+1}, y_{i+1}).$$

Then $f_{i+1}:P_{i+1} \to C_{i+1}$ is onto for each i.

Also

$$Z_i(P) = \text{Ker } \delta^P_i = S_i \oplus \text{Ker } \beta_{i-1} \quad \text{and}$$

$$B_i(P) = \text{Im } \delta^P_{i+1} = \beta_i(T_{i+1}) \oplus 0 = K_i \oplus 0.$$

Hence

$$H_i(P) = (S_i/K_i) \oplus \text{Ker } \beta_{i-1}.$$

Also the restriction of the map $H_i(f):H_i(P) \to H_i(C)$ to S_i/K_i is the isomorphism of S_i/K_i onto $H_i(C)$ induced by $\eta_i \circ \alpha_i$. Hence $H_i(f):H_i(P) \to H_i(C)$ is a split epimorphism, with the inverse of the isomorphism $S_i/K_i \simeq H_i(C)$ as a splitting. It follows that Ker $H_i(f)$ and Ker β_{i-1} are two direct summands of S_i/K_i in $H_i(P)$. Hence

$$\text{Ker } H_i(f) \overset{\simeq}{\to} \text{Ker } \beta_{i-1}$$

under some isomorphism. The exact sequences

$$0 \to \text{Ker } \beta_{i-1} \to T_i \to K_{i-1} \to 0 \quad \text{and}$$

$$0 \to K_{i-1} \to S_{i-1} \to H_{i-1}(C) \to 0,$$

with T_i and S_{i-1} projective yield h.d Ker $\beta_{i-1} \leqq \text{Max}(0, r - 2)$. This implies

$$\text{h.d}(\text{Ker } H_i(f)) \leqq \text{Max}(0, r - 2).$$

Let $A = \text{Ker } f$. Then in the exact homology sequence

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$$\ldots \to H_i(A) \to H_i(P) \xrightarrow[H_i(f)]{} H_i(C) \underset{\partial}{\to} H_{i-1}(A) \to H_{i-1}(P)$$

$$\xrightarrow[H_{i-1}(f)]{} H_{i-1}(C) \to$$

Since $H_i(f):H_i(P) \to H_i(C)$ is an epimorphism for each i, it follows that

$$H_i(A) \simeq \text{Ker } H_i(f).$$

Hence h.d $H_i(A) \leqq \text{Max}(0, r - 2)$. This completes the proof of Proposition 2.1.

Remark 2.2. It is perhaps worthwhile observing that in case C reduces to a single module $C_0 = M$ with h.d $M \leqq r$, if $0 \to K \to P_1 \xrightarrow{\delta} P_0 \xrightarrow{\epsilon} M \to 0$ is exact with P_1, P_0 projective, then h.d $K \leqq \text{Max}(0, r - 2)$. If P is the complex

$$\to 0 \to P_1 \xrightarrow{\delta_1} P_0 \to 0 \to \ldots$$

then $\epsilon:P \to M$ is a map with

$$H_0(\epsilon):H_0(P) \simeq M,$$

$$H_1(P) \simeq K = \text{Ker } H_1(\epsilon) \quad \text{and}$$

$$H_j(P) = 0 \quad \text{for } j \neq 0, 1; \text{ and}$$

$$\text{h.d } K \leqq \text{Max } (0, r - 2).$$

PROPOSITION 2.3. *Let C be a chain complex which satisfies* h.d $H_i(C) \leqq 1$ *for all i. Then there exists a projective approximation $f:P \to C$ of C.*

Proof. The proof is similar to that of Proposition 2.1. If $\alpha_i:S_i \to Z_i(C)$ is an epimorphism with S_i projective, whenever h.d $H_i(C) \leqq 1$, the module $K_i = \text{Ker } \eta_i \circ \alpha_i$ is automatically projective. Hence we could take $T_{i+1} = K_i$ and $\beta_i = 1d_{K_i}$. Then for the complex P defined as in the proof of Proposition 2.2, we would have $H_i(P) = S_i/K_i$ (since Ker $\beta_{i-1} = 0$, β_{i-1} being the identity map of K_{i-1}). The map $f:P \to C$ is an epimorphism with $H(f):H(P) \simeq H(C)$.

3. Proof of theorem 1. Now, we take up the proof of Theorem 1. Let C be a positive chain complex. Then from Propositions 1.5 and 1.6 we see that there exists a positive free approximation $F \xrightarrow{f} C$ of C. Suppose on the other hand C is not necessarily a positive chain complex but satisfies the restriction h.d $H_i(C) \leqq r$ for all i, where r is a fixed integer $\geqq 0$. If $r \leqq 1$, Proposition 2.3 guarantees the existence of a projective approximation $f:P \to C$ of C. Suppose $r \geqq 2$. We then use induction on r. By Proposition 2.1 there exists an epimorphism $f:P \to C$ with P projective,

$$\text{h.d }(\text{Ker } H_i(f)) \leqq r - 2 \quad \text{and } H_i(f):H_i(P) \to H_i(C)$$

a split epimorphism for all i. Let C_f denote the mapping cone of f and $j:C \to C_f$ the inclusion of C in C_f. Then we have an exact sequence

$$0 \to C \xrightarrow{j} C_f \to \Sigma P \to 0.$$

In the associated homology exact sequence

$$\begin{array}{ccccccccccc}
\ldots \to H_{i+1}(\Sigma P) & \xrightarrow{\delta} & H_i(C) & \xrightarrow{j_*} & H_i(C_f) & \to & H_i(\Sigma P) & \xrightarrow{\delta} & H_{i-1}(C) & \xrightarrow{j_*} & H_{i-1}(C_f) \\
\Vert & \nearrow_{H_i(f)} & & & & & \Vert & \nearrow_{H_{i-1}(f)} & & & \\
H_i(P) & & & & & & H_{i-1}(P) & & & &
\end{array}$$

each of the maps $\delta:H_{i+1}(\Sigma P) \to H_i(C)$ is onto. Hence

$$H_i(C_f) \simeq \text{Ker } H_{i-1}(f).$$

It follows that h.d $H_i(C_f) \leqq r - 2$. By the inductive assumption there exists a projective approximation $Q \xrightarrow{\tau} C_f$ of C_f. Denoting the natural epimorphism $C_f \to \Sigma P$ by η we see that $\eta \circ \tau:Q \to \Sigma P$ is an epimorphism. If $L = \text{Ker } \eta \circ \tau$ then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & Q & \xrightarrow{\eta\tau} & \Sigma P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau/L} & & \downarrow{\scriptstyle \tau} & & \Vert & & \\
0 & \longrightarrow & C & \xrightarrow[j]{} & C_f & \xrightarrow[\eta]{} & \Sigma P & \longrightarrow & 0
\end{array}$$

DIAGRAM 2

where the two rows are exact sequences of chain complexes. Since $\tau:Q \to C_f$ is an epimorphism, it follows easily from the above commutative diagram that $\tau/L:L \to C$ is an epimorphism. Since

$$H(\tau):H(Q) \simeq H(C_f),$$

from the exact homology sequences and the five lemma we immediately see that

$$H(\tau/L):H(L) \simeq H(C).$$

Since Q_i and $(\Sigma P)_i$ are projective for each i, the top exact sequence splits for each i and yields projectivity of L_i for each i. Thus $\tau/L:L \to C$ is a projective approximation to C. This completes the proof of Theorem 1.

4. Proof of theorem 2. Since $\tau_0:P_0 \to C_0$ is onto and P'_0 is projective, there exists a map $\varphi_0:P'_0 \to P_0$ with $\tau_0 \circ \varphi_0 = f_0$. Let $k \geqq 0$ and assume we have constructed maps $\varphi_i:P'_i \to P_i$ for $0 \leqq i \leqq k$ satisfying

$$\left.\begin{array}{ll} \text{(i)} & \tau_i\varphi_i = f_i \text{ and} \\ \text{(ii)} & \varphi_{i-1}\delta_i^{P'} = \delta_i^P \circ \varphi_i \end{array}\right\} \text{for } 0 \leqq i \leqq k.$$

Since $\tau_{k+1}:P_{k+1} \to C_{k+1}$ is onto and P'_{k+1} is projective, there exists a map $\theta_{k+1}:P'_{k+1} \to P_{k+1}$ with $\tau_{k+1} \circ \theta_{k+1} = f_{k+1}$. Consider the map

$$\beta_k:\delta_{k+1}^P\theta_{k+1} - \varphi_k\delta_{k+1}^{P'}:P'_{k+1} \to P_k.$$

Then

$$\begin{aligned} \tau_k \circ \beta_k &= \tau_k\delta_{k+1}^P\theta_{k+1} - \tau_k\varphi_k\delta_{k+1}^{P'} \\ &= \delta_{k+1}^C\tau_{k+1}\theta_{k+1} - f_k\delta_{k+1}^{P'} \\ &= \delta_{k+1}^C f_{k+1} - f_k\delta_{k+1}^{P'} \end{aligned}$$

$$(6) \qquad = 0 \text{ since } f:P' \to C \text{ is a chain map.}$$

Let $L = \text{Ker } \tau$. Since $H(\tau):H(P) \simeq H(C)$ we immediately get $H(L) = 0$. Since L is a positive chain complex, there exists a positive projective approximation, say $Q \xrightarrow{\epsilon} L$ of L. Then Q is a positive, projective complex with $H(Q) = 0$. Hence Q is chain contractible. Let $S_i:Q_i \to Q_{i+1}$ yield a contraction of Q.

From $\tau_k \circ \beta_k = 0$ (from (6)) we see that $\beta_k(P'_{k+1}) \subset L_k$. Also

$$\begin{aligned} \delta_k^P\beta_k &= \delta_k^P(\delta_{k+1}^P\theta_{k+1} - \varphi_k\delta_{k+1}^{P'}) \\ &= -\delta_k^P\varphi_k\delta_{k+1}^{P'} \\ &= -\varphi_{k-1}\delta_k^{P'}\delta_{k+1}^{P'} \\ &= 0. \end{aligned}$$

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Thus $\beta_k(P'_{k+1}) \subset Z_k(L)$. Since $\epsilon:Q \to L$ is a projective approximation to L, from Lemma 1.1 it follows that

$$\epsilon_k|Z_k(Q):Z_k(Q) \to Z_k(L)$$

is onto. The projective nature of P'_{k+1} now yields a map

$$\alpha_k:P'_{k+1} \to Z_k(Q) \quad \text{with } \epsilon_k \circ \alpha_k = \beta_k.$$

Consider the map $\varphi_{k+1}:P'_{k+1} \to P_{k+1}$ defined by

$$\varphi_{k+1} = \theta_{k+1} - \epsilon_{k+1}S_k\alpha_k.$$

We then have

$$\delta^P_{k+1}\varphi_{k+1} = \delta^P_{k+1}\theta_{k+1} - \delta^P_{k+1}\epsilon_{k+1}S_k\alpha_k$$
$$= \delta^P_{k+1}\theta_{k+1} - \delta^L_{k+1}\epsilon_{k+1}S_k\alpha_k$$

(since L is a subcomplex of P)

$$= \delta^P_{k+1}\theta_{k+1} - \epsilon_k\delta^Q_{k+1}S_k\alpha_k$$

(since $\epsilon:Q \to L$ is a chain map)

$$= \delta^P_{k+1}\theta_{k+1} - \epsilon_k(\mathrm{Id}_{Q_k} - S_{k-1}\delta^Q_k)\alpha_k$$
$$= \delta^P_{k+1}\theta_{k+1} - \epsilon_k\alpha_k + \epsilon_kS_{k-1}\delta^Q_k\alpha_k$$
$$= \delta^P_{k+1}\theta_{k+1} - \beta_k \quad (\text{because } \delta^Q_k\alpha_k = 0)$$
$$= \delta^P_{k+1}\theta_{k+1} - \{\delta^P_{k+1}\theta_{k+1} - \varphi_k\delta^{P'}_{k+1}\}$$
$$= \varphi_k\delta^{P'}_{k+1}.$$

Thus $\delta^P_{k+1}\theta_{k+1} = \varphi_k\delta^{P'}_{k+1}$. Moreover,

$$\tau_{k+1}\varphi_{k+1} = \tau_{k+1}\theta_{k+1} - \tau_{k+1}\epsilon_{k+1}S_k\alpha_k.$$

From $\epsilon_{k+1}(Q_{k+1}) = L_{k+1} = \operatorname{Ker} \tau_{k+1}$ we see that

$$\tau_{k+1}\varphi_{k+1} = \tau_{k+1}\theta_{k+1} = f_{k+1}.$$

This completes the proof of the inductive step in the construction of the chain map $\varphi:P' \to P$ satisfying $\tau\varphi = f$.

Suppose φ, $\bar{\varphi}$ are any two lifts of f. Then $\tau(\varphi - \bar{\varphi}) = 0$. Hence

$$(\varphi - \bar{\varphi})(P') \subset L.$$

Thus $\varphi - \bar{\varphi}:P' \to L$ is a chain map. Since $\epsilon:Q \to L$ is a positive, projective approximation of L, by what we have proved already there exists a chain map $\gamma:P' \to Q$ with $\varphi - \bar{\varphi} = \epsilon\gamma$. Now Q is chain contractible. Hence $\gamma \sim 0$. It follows that $\varphi - \bar{\varphi} \sim 0$ or $\varphi \sim \bar{\varphi}$. This shows that any two lifts $\varphi, \bar{\varphi}$ of the same map f are chain homotopic.

Now, let $f \sim g:P' \to C$ and let $D_i:P_i' \to C_{i+1}$ yield a chain homotopy between f and g. Since

$$P_{i+1} \xrightarrow{\tau_{i+1}} C_{i+1} \to 0$$

is exact, and P_i' projective there exist maps $E_i:P_i' \to P_{i+1}$ with $\tau_{i+1}E_i = D_i$. Then

$$\begin{aligned} \tau_i(\delta_{k+1}^P E_i + E_{i-1}\delta_i^{P'}) &= \delta_{i+1}^C \tau_{i+1} E_i + \tau_i E_{i-1}\delta_i^{P'} \\ &= \delta_{i+1}^C D_i + D_{i-1}\delta_i^{P'} \qquad (7) \\ &= g_i - f_i. \end{aligned}$$

Let $\lambda_i:P_i' \to P_i$ be given by

$$\lambda_i = \delta_{i+1}^P E_i + E_{i-1}\delta_i^{P'}.$$

Then

$$\delta_i^P \lambda_i = \delta_i^P \delta_{i+1}^P E_i + \delta_i^P E_{i-1}\delta_i^{P'} = \delta_i^P E_{i-1}\delta P_i'$$

and

$$\lambda_{i-1}\delta_i^{P'} = (\delta_i^P E_{i-1} + E_{i-2}\delta_{i-1}^{P'})\delta_i^{P'} = \delta_i^P E_{i-1}\delta_i^{P'}.$$

Hence the λ_i's yield a chain map $\lambda:P' \to P$. From (7) we get $\tau\lambda = g - f$. However, we know that $\psi - \varphi$ is also a lift of $g - f$. Since λ and $\psi - \varphi$ are lifts of the same map $g - f$, by what we have proved already we see that λ and $\psi - \varphi$ are chain homotopic. Let $G_i:P_i' \to P_{i+1}$ satisfy

$$\delta_{i+1}^P G_i + G_{i-1}\delta_i^{P'} = \psi_i - \varphi_i - \lambda_i.$$

Thus we get

$$\begin{aligned} \psi_i - \varphi_i &= \delta_{i+1}^P G_i + G_{i-1}\delta_i^{P'} + \lambda \\ &= \delta_{i+1}^P G_i + G_{i-1}\delta_i^{P'} + \delta_{i+1}^P E_i + E_{i-1}\,\delta_i^{P'}. \end{aligned}$$

Hence $J_i = G_i + E_i:P_i' \to P_{i+1}$ satisfy the condition that

$$\delta_{i+1}^P J_i + J_{i-1}\delta_i^{P'} = \psi_i - \varphi_i.$$

This shows that ψ and φ are chain homotopic. This completes the proof of Theorem 2.

COROLLARY 4.1. *Let* $P \xrightarrow{\tau} C$ *and* $P' \xrightarrow{\tau'} C$ *be any two positive, projective approximations of a positive chain complex C. Then there exists a chain map* $\varphi:P \to P'$ *with* $\tau' \circ \varphi = \tau$. *Any such chain map* $\varphi:P \to P'$ *is a chain equivalence.*

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University of Calgary,
Calgary, Alberta

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