Can. J. Math. Vol. **46** (3), 1994 pp. 449–473

WHEN IS A DISTRIBUTION OF SIGNS LOCALLY COMPLETABLE?

Dedicated to the memory of our friend Mario Raimondo

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ABSTRACT. Let V be an irreducible nonsingular algebraic surface, $Y \subset V$ be an algebraic curve and P a point of Y. Suppose a sign distribution is given locally in a neighbourhood of P on some connected components of $V - Y$. We give an algorithmic criterion to decide whether this sign distribution is induced by a regular function or not. As an application, this criterion enables one to decide whether two semialgebraic sets can be locally separated or not.

Introduction. Let V be a 2-dimensional, non-singular, compact, real affine algebraic variety and $Y \subset V$ an algebraic curve. To give a partial distribution of signs on $V - Y$ means to fix a sign on some of the connected components of $V - Y$.

We consider the problem to know whether a given partial distribution of signs σ is completable or not, *i.e.* whether it is the restriction of the distribution of signs induced by a polynomial function. In the case $V = \mathbb{R}^2$, in [F] it is shown that σ is completable if and only if no irreducible component of Y is type changing (see Definition 1.1) and σ is locally completable at any point of Y, included "the point at infinity". This result is suitably generalized to the case of a surface V as above.

It is so natural to look for a criterion of local completability: this is the goal of this paper.

As a matter of fact the local obstructions to completability are due to the presence of type changing components, but these components are hidden. More precisely, a distribution of signs σ on $V - Y$ without type changing components is not locally completable at a point P if and only if it is obtained from a non-completable distribution by contracting to P a type changing curve.

By using this characterization, in the second section we give a procedure, taking as an input the Puiseux expansions of the branches of the germ (Y, P), which allows us to decide if σ is locally completable at P without blowing up.

As an application, this result yields a criterion to decide whether two given semialgebraic subsets of V (or of $\mathbb{R}^2$, via a stereographic projection from S^2) can be polynomially separated or not.

The authors are members of G.N.S.A.G.A. of C.N.R. This work is partially supported by M.U.R.S.T.
Received by the editors October 9, 1992.
AMS subject classification: 14P10.

https://doi.org/10.4153/CJM-1994-024-2 Published online by Cambridge University Press

We have to mention that similar problems concerning distributions of signs and signatures have already been studied by other authors, following an algebraic approach based on the use of the real spectrum, fans and valuations. We deal with the subject from a different, more geometrical, point of view, which can be interesting also because of the algorithmic aspect of the answers.

1. **Visible and hidden type changing components.** Let V be a non-singular real affine algebraic surface. Denote by $\mathcal{R}(V)$ (resp. $\mathcal{P}(V)$) the ring of regular (resp. polynomial) functions on V. Let $Y \subset V$ be an algebraic curve.

DEFINITION 1.1. a) A (partial) distribution of signs σ on $V - Y$ is a continuous map

$$\sigma: A_1 \cup \cdots \cup A_t \longrightarrow \{-1, 1\}.$$

where $A_1, \ldots, A_t$ are some of the connected components of $V - Y$. (Sometimes we will denote $A_1 \cup \cdots \cup A_t$ by $\mathcal{D}(\sigma)$).

b) An irreducible component Y^i of Y is called *type changing* with respect to σ if there exist non-empty open sets $\Omega, \Omega' \subset Y^1 - \operatorname{Sing} Y$ such that
   i) $\Omega \subset \overline{\sigma^{-1}(1)} \cap \overline{\sigma^{-1}(-1)}$
   ii) $\Omega' \subset \overline{\overset{\circ}{\sigma^{-1}(1)}}$ or $\Omega \subset \overline{\overset{\circ}{\sigma^{-1}(-1)}}$.

c) We say that σ is *completable* if there exists $f \in \mathcal{R}(V)$ which induces σ.

d) We say that σ is *locally completable* at $y \in Y$ if there exist an open euclidean neighbourhood U of y in V and $f \in \mathcal{R}(V)$ which induces σ on U.

In [F] similar definitions are given using polynomial functions instead of regular functions. However they are clearly equivalent to those given in Definition 1.1, because every $f \in \mathcal{R}(V)$ can be written as $\frac{P}{Q}$ with $P, Q \in \mathcal{P}(V)$, Q never vanishing on V; so $Q^2 \cdot f$ is a polynomial function having the same signs as f.

The completability of a distribution of signs is a property which is invariant under biregular isomorphisms, *i.e.* if $\pi: V' \longrightarrow V$ is a biregular isomorphism, a distribution of signs σ on $V - Y$ is completable if and only if the distribution $\sigma' = \sigma \circ \pi$ on $V' - \pi^{-1}(Y)$ is completable.

As a matter of fact, the same result is true for a wider class of regular maps; in particular we shall be interested in the case of contractions:

PROPOSITION 1.2. *Suppose that V is obtained from a non-singular algebraic surface V' by contracting an algebraic curve $Z \subset V'$ to a point $O \in Y \subset V$ (i.e. there exists a regular surjective map $\pi: V' \longrightarrow V$ such that $\pi(Z) = O$ and $\pi|_{V'-Z}: V' - Z \longrightarrow V - \{0\}$ is a biregular isomorphism). Let σ be a partial distribution of signs on $V - Y$ and $\sigma' = \sigma \circ \pi$ be the lifted distribution of signs on $V' - \pi^{-1}(Y)$. Then σ is completable (resp. locally completable in O) if and only if σ' is completable (resp. completable in a neighbourhood of $Z = \pi^{-1}(0)$.*

PROOF. Clearly if $f \in \mathcal{R}(V)$ induces σ, then $f \circ \pi$ induces σ'. Conversely suppose that $f' \in \mathcal{R}(V')$ induces σ'. The function $p = f' \circ (\pi|_{V'-Z})^{-1}$ induces σ and is regular on $V - \{0\}$. Since $p = h/g$, with $f, g \in \mathcal{P}(V)$, g nowhere vanishing on $V - \{0\}$, then $p \cdot g^2 \in \mathcal{P}(V)$ and induces σ.

The local case is proved in the same way. ∎

[F] contains a result which relates local and global completability when $V = \mathbb{R}^2$; we can state that result as follows:

PROPOSITION 1.3. *Let σ be a distribution of signs on $S^2 - Y$ and assume that no irreducible component of Y is type changing with respect to σ. Then σ is completable if and only if σ is locally completable at any point of Y.*

PROOF. If $Y \cup \mathcal{D}(\sigma) = S^2$, the result is true ([B-T]). Otherwise choose a point P_0 in a connected component of $S^2 - \big(Y \cup \mathcal{D}(\sigma)\big)$. Clearly σ is completable if and only if so is its restriction to $S^2 - \{P_0\}$. But $S^2 - \{P_0\}$ is biregularly isomorphic to $\mathbb{R}^2$ and here the distribution of signs has a compact domain, so the proposition follows from 4.1 in [F]. ∎

Proposition 1.3 does not hold for any compact surface, unless one adds a condition (Proposition 1.4ii)) always satisfied in S^2. Recall that an irreducible component Y^i of Y, which is not type changing, is called a *change component* if there exists an open set $\Omega \neq \emptyset$ in $Y^i - \operatorname{Sing} Y$ such that $\Omega \subset \overline{\sigma^{-1}(1)} \cap \overline{\sigma^{-1}(-1)}$.

PROPOSITION 1.4. *Let V be a compact non-singular real affine algebraic surface, $Y \subset V$ an algebraic curve. Let σ be a partial distribution of signs on $V - Y$. Denote by Y^c the union of the change components of Y with respect to σ. Then σ is completable if and only if the following three conditions are satisfied:*

i) no irreducible component of Y is type changing with respect to σ;

ii) there exists an algebraic set A, contained in the closure of $V - \overline{\mathcal{D}(\sigma)}$, such that $[A \cup Y^c] = 0$ in $H_1(V, \mathbb{Z}_2)$;

iii) σ is locally completable at any point of Y.

PROOF. The conditions are necessary because, if τ completes σ and is induced by $f \in \mathcal{R}(V)$, then with respect to τ there are no type changing components ([A-B2]) and the union of the change components bounds the set $\overline{\{f > 0\}}$.

If the conditions hold, multiply σ by the distribution of signs induced by a generator of the ideal $I(Y^c \cup A)$. The new distribution of signs σ' has neither type changing nor change components and the sets ${\sigma'}^{-1}(1)$ and ${\sigma'}^{-1}(-1)$ are disjoint open semialgebraic sets with compact closures intersecting only in a finite number of points. So we can apply Proposition 3.2 and Remark 3.3 in [F] and separate them by a regular function. ∎

In a sense, Propositions 1.3 and 1.4 reduce the problem of deciding whether a distribution of signs is completable to a local problem in a neighbourhood of finitely many points of Y, namely, the points $y \in Y$ where the germ (Y, y) is not normal crossing. In fact any distribution without type changing components is completable in a neighbourhood of any regular point or where two branches of Y meet transversally.

A necessary and sufficient condition for the local completability can be found by using the following remarks:

- if (Y, y) is not normal crossing, we can resolve the singularity by a finite sequence of blowings-up;

https://doi.org/10.4153/CJM-1994-024-2 Published online by Cambridge University Press

- the property of being completable is not altered by performing a blowing-up, because of Proposition 1.2.

PROPOSITION 1.5. *Let σ be a partial distribution of signs on $S^2 - Y$ such that no irreducible component of Y is type changing with respect to the germ of σ at $P \in Y$. Let $\pi: M \longrightarrow S^2$ be the blowing-up of S^2 at P; define $\sigma' = \sigma \circ \pi$ and $D = \pi^{-1}(P)$. If σ is not locally completable at P, then one of the following conditions is satisfied:*

a) D is type changing with respect to σ',

b) there is at least one point in D where σ' is not locally completable.

PROOF. Let $D(P,\varepsilon)$ be the ball in $\mathbb{R}^3$ centered at P with radius ε. Choose $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the boundary of $B(P,\varepsilon) = D(P,\varepsilon) \cap S^2$ is transversal to Y. Let $B = B(P,\varepsilon)$ for a fixed $\varepsilon < \varepsilon_0$.

Denote by σ^B the partial distribution of signs on $S^2 - (Y \cup \partial B)$ which does not fix any sign outside B and such that $\sigma^B|_B = \sigma|_B$. Clearly no irreducible component of $Y \cup \partial B$ is type changing with respect to σ^B and, by construction, σ^B is locally completable at any point different from P. So, by Proposition 1.3, σ^B is completable if and only if σ is locally completable at P.

Then, because of the hypothesis, σ^B is not completable and, by Proposition 1.2, also $(\sigma^B)' = \sigma^B \circ \pi$ is not completable on $M - \pi^{-1}(Y \cup \partial B)$. Since M is biregularly isomorphic to $\mathbb{P}_2$, we can assume $M = \mathbb{P}_2(\mathbb{R})$.

We can choose an algebraic curve A in S^2 passing through P, transversal to each branch of (Y,P) and such that the strict transform A' of A is a line in $\mathbb{P}_2(\mathbb{R})$. In particular A' intersects D in a point which does not belong to the strict transform Y' of Y; we can suppose, by shrinking B if necessary, that $A' \cap \pi^{-1}(B) \cap Y' = \emptyset$. Let ω be the contraction of $A' \subset \mathbb{P}_2(\mathbb{R})$ to a point Q; the image $\omega\big(\mathbb{P}_2(\mathbb{R})\big) = \Sigma^2$ is biregularly isomorphic to S^2 because A' is the image of D through a linear change of coordinates in $\mathbb{P}_2(\mathbb{R})$.

Consider the distribution of signs σ'' induced by $(\sigma^B)'$ on $\Sigma^2 - Z$, where $Z = \omega\big(\pi^{-1}(Y \cup \partial B)\big)$. It is easy to see that:

i) no irreducible component of Z different from $\omega(D)$ is type changing with respect to σ'';

ii) σ'' is locally completable at any point of $Z - \omega(D)$;

iii) σ'' is locally completable at the point Q.

The distribution σ'', by Proposition 1.2, is not completable; therefore, by Proposition 1.3, it must happen that either $\omega(D)$ is type changing or σ'' is not locally completable at some point of $\omega(D) \cap \omega(Y')$. This implies immediately the thesis. ∎

The next two theorems are the main results of this section.

THEOREM 1.6. *Let σ be a partial distribution of signs on $S^2 - Y$ such that no irreducible component of Y through $P \in Y$ is type changing with respect to the germ of σ at P.*

Then σ is not locally completable at P if and only if the following conditions are satisfied:

1) *there exist a non-singular algebraic surface V' and a regular surjective map $\pi\colon V' \longrightarrow S^2$ such that π is the contraction of an algebraic curve $E \subset V'$ to P;*
2) *at least one irreducible component of E is type changing with respect to $\sigma' = \sigma \circ \pi$.*

PROOF. The "if" part is a consequence of Proposition 1.2.

Conversely assume that σ is not locally completable at P. Consider the distribution of signs σ^B defined in the proof of Proposition 1.5; replace σ by σ^B but, by an abuse of language, again call it σ. In particular we have that σ is locally completable at any point different from P and that no irreducible component of Y is type changing with respect to σ.

Denote by $\pi\colon V' \longrightarrow S^2$ the composition of the blowings-up of the standard resolution of the singularity $(Y, P) \subset (S^2, P)$ ([B-K]), by Y' the strict transform of Y and by E the exceptional curve. We want to show that π and E satisfy the conditions 1) and 2).

Let us start by considering the first blowing-up $\pi_1\colon M \longrightarrow S^2$, that is, the blowing-up of S^2 at P, and let D be the exceptional curve. If D is type changing, we are done because its strict transform in V' is a type changing irreducible component of E. Otherwise, because of Proposition 1.5, D must contain at least one point P_1 where $\sigma_1 = \sigma \circ \pi_1$ is not locally completable.

In order to reduce ourselves to work again in S^2, arguing as in the proof of Proposition 1.5 and with the same notations, we can consider the regular map $\omega\colon M \longrightarrow S^2$, which contracts the line A' to a point Q. The distribution of signs σ'' induced by σ_1 on $S^2 - \omega\big(\pi^{-1}(Y \cup \partial B)\big)$ is not locally completable at the point $\omega(P_1)$, and there are no irreducible components type changing with respect to it. So we can start again, localizing σ'' at $\omega(P_1)$ and blowing up S^2 at that point.

It is necessary to make the following remarks.

i) If the exceptional divisor obtained by blowing up S^2 at $\omega(P_1)$ is type changing, the same is true for the exceptional divisor obtained by blowing up M at P_1. The reason is that the property of being biregularly isomorphic varieties is preserved by blowing up corresponding points.

ii) The blowing-up of M at P_1 is actually one among the blowings-up of the standard resolution of (Y, P); in fact either the strict transform Y_1 of Y is singular in P_1 or Y_1 and D are not normal crossings in P_1, otherwise σ_1 would be locally completable at P_1.

iii) The recursive process described so far stops if one finds a type changing component of the exceptional curve (in that case the theorem is proved); otherwise it produces a point to blow up. So the process stops at least when, after a finite number of steps, the strict transform of Y becomes non-singular and normal crossings with the exceptional curve. In fact at this moment the lifted distribution is locally completable everywhere; since it is not completable by Proposition 1.2, one irreducible component of E must be type changing by Proposition 1.5. ∎

THEOREM 1.7. *Let σ be a partial distribution of signs on $V - Y$, where V is a non-singular, compact real algebraic surface, $V \subset \mathbb{R}^n \subset \mathbb{P}_n(\mathbb{R})$. Suppose that no irreducible* *component of Y through $P \in Y$ is type changing with respect to the germ of σ at P. Then σ is not locally completable at P if and only if the following conditions are satisfied:*

1) *there exist a non-singular algebraic surface V' and a regular surjective map $\pi\colon V' \longrightarrow V$ such that π is the contraction of an algebraic curve $E \subset V'$ to P;*
2) *at least one irreducible component of E is type changing with respect to $\sigma' = \sigma \circ \pi$.*

PROOF. We start by projecting the surface V in $\mathbb{P}_2(\mathbb{R})$. To do that, if $n > 3$, we can choose a linear subspace L in $\mathbb{P}_n(\mathbb{R})$, $\dim L = n - 4$, such that the projection with center L, $\pi_1\colon \mathbb{P}_n(\mathbb{R}) - L \longrightarrow \mathbb{P}_3(\mathbb{R})$ has the following properties:

i) π_1 induces a biregular isomorphism between a Zariski open set $\Omega \subseteq V$ containing P and a Zariski open set in the algebraic surface $V_1 = \overline{\pi_1(V)}^Z$ (where $\overline{}^Z$ denotes the Zariski closure);

ii) π_1 induces a biregular isomorphism between Y and its image Y_1, which is an algebraic curve in $\mathbb{P}_3(\mathbb{R})$.

In fact, by the theorem of minimal embedding of algebraic singularities (see, for instance, 8.6.15 in [B-C-R]), we can choose L in such a way that i) is satisfied. Moreover, by a transversality argument, L can be taken not intersecting any chord of Y, even the real chords joining two complex conjugate points of Y.

We have, in particular, that V_1 is non-singular at $P_1 = \pi_1(P)$ and the dimension of the Zariski tangent space to Y_1 at P_1 is 2; so we can choose a point $Q \in \mathbb{P}_3(\mathbb{R}) - V_1$ such that

1) Q does not belong to any chord of Y_1 through P_1,
2) at most finitely many lines through Q are chords of Y_1, including the real chords with complex conjugate intersections with Y_1,
3) the line QP_1 is not tangent to V_1.

Therefore the projection π_2 with center Q to $\mathbb{P}_2(\mathbb{R})$ has the following properties:

i) $\pi_{2|V_1}$ is a finite morphism such that $\pi_2(P_1) = P_2$ is not a critical value for π_2 and does not belong to the image of $\operatorname{Sing} V_1$; in particular π_2 induces an analytic isomorphism between a neighbourhood $U_1 \subset V_1$ of P_1 and a neighbourhood U_2 of $P_2 \in \mathbb{P}_2(\mathbb{R})$.

ii) There exists a Zariski open set Ω_1 of Y_1 such that $\pi_{2|\Omega_1}$ is a biregular isomorphism with its image.

Define $Z = \overline{\pi_2(Y_1)}^Z = \overline{\pi_2\big(\pi_1(Y)\big)}^Z$. Z is an algebraic plane projective curve, $\pi_2 \circ \pi_1$ induces a 1-1 correspondence between the irreducible components of Y and those ones of Z, and the germ of Z at P_2 is isomorphic to the germ of Y at P.

Consider the partial distribution of signs σ_2 in the neighbourhood U_2 of P_2, which is induced by σ through $\pi_2 \circ \pi_1$. We can think $U_2 \subset S^2$, by contracting a suitable line in $\mathbb{P}_2(\mathbb{R})$.

If σ is not locally completable, then so is σ_2. By Theorem 1.6, we know that there exists a type changing component of the exceptional curve E' of the standard resolution of (Z, P_2). But this resolution is isomorphic to the standard resolution of (Y, P), since $\pi_2 \circ \pi_1$ induces an isomorphism between the two germs. So there exists a type changing component in the exceptional curve relative to (Y, P) too.

The "if" part of the theorem is a consequence of Proposition 1.2. ■

From the proofs of Theorems 1.6 and 1.7 it follows, in particular, that a distribution of signs σ on $V - Y$ without type changing components is not locally completable at P if and only if the exceptional curve of the standard resolution of (Y, P) contains at least one type changing component. At this point a natural question arises: is it possible to check the existence of such a component without performing the blowings-up of the resolution process?

Note that a component D of E is type changing if and only if it contains two non-empty open sets Ω and Ω' such that, roughly speaking, $\sigma' = \sigma \circ \pi$ changes its sign across Ω and does not change across Ω'. In this case, there is a family $\mathcal{F}_1$ of smooth analytic arcs which meet D transversally at the points of Ω and a family $\mathcal{F}_2$ of arcs of the same kind, meeting D at the points of Ω'.

The projections of the arcs of the families $\mathcal{F}_1$ and $\mathcal{F}_2$ form two families $\mathcal{G}_1$ and $\mathcal{G}_2$ of analytic arcs through $P \in V$ with the following properties:

- every arc in $\mathcal{G}_1$ joins two connected components A_i, A_j of $V - Y$, such that $P \in \overline{A_i} \cap \overline{A_j}$ and $\sigma(A_i) = \sigma(A_j)$,
- every arc in $\mathcal{G}_2$ joins two connected components A_h, A_k of $V - Y$, such that $P \in \overline{A_h} \cap \overline{A_k}$ and $\sigma(A_h) = \sigma(A_k)$.

It is also clear that our problem of investigating the existence of a type changing D will be solved if we can solve the following two problems, which will be made precise later:

PROBLEM 1. For each fixed irreducible component D of the exceptional curve E, find a family $\mathcal{A}$ of analytic arcs through $P \in V$ such that:

1) $\forall \gamma \in \mathcal{A}$, $\pi^{-1}(\gamma)$ is a smooth arc, which meets D transversally;
2) $\{\pi^{-1}(\gamma) \cap D \mid \gamma \in \mathcal{A}\}$ is a dense set in D.

PROBLEM 2. For any connected components A and B of $V - Y$ with $P \in \bar{A} \cap \bar{B}$, find all the analytic arcs through P joining A and B.

In the next section we shall give a solution to these problems, and this solution will be effectively constructed in terms of the Puiseux expansions of the branches of (Y, P). By the proof of Theorem 1.7, we can suppose that (Y, P) is a plane curve germ.

2. **A procedure to test the local completability.** Let us start with some results about a germ (Y, O) of a complex plane curve with an isolated singularity at O; we will come back to the real case later.

Consider the standard resolution of (Y, O) as described in [E-C] or [B-K]; we will use the notation of the latter. It consists of a sequence of maps

$$X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} X_{N-2} \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = U \subset \mathbb{C}^2$$

where U is a neighbourhood of $O \in C$ and, if we denote

- $\phi_i = \pi_1 \circ \cdots \circ \pi_i : X_i \longrightarrow U$, $\phi_0 = \mathrm{id}$
- $E_i = \phi_i^{-1}(0)$ (the exceptional curve in X_i),

- $Y_i = \overline{\phi_i^{-1}(Y - \{0\})}$ (the strict transform of Y in X_i),

π_{i+1} is recursively defined as the blowing-up of X_i in all the points of $E_i \cap Y_i$ where Y_i is singular or does not intersect E_i transversally. Note that every singular point of Y_i belongs to E_i because O is the only singularity of Y in U. In particular, if $Y^1, \ldots, Y^r$ are the analytic irreducible components, or branches, of (Y, O), π_{i+1} blows up the strict transform Y_i^j of Y^j in at most one point.

Each branch Y^j admits a parametrization, called *Puiseux expansion*, of the form

$$\begin{cases} x = t^m \\ y = \sum_{i=1}^{\infty} a_i t^i \end{cases} \tag{2.1}$$

(see, for instance, [B-K] or [W]). If we choose coordinates (x, y) in such a way that no branch of (Y, O) is tangent to the y-axis, we can assume that $y(t)$ has order greater than or equal to m.

From (2.1) we get the characteristic exponents $(m; k_1, \ldots, k_s)$ and the sequence of greatest common divisors

$$d_1 = (m, k_1) \quad \text{and} \quad d_i = (d_{i-1}, k_i) = (d_{i-1}, k_i - k_{i-1}), \quad i = 2, \ldots, s.$$

If $d_s = (m, k_1, \ldots, k_s) = 1$, the parametrization (2.1) is called *irreducible*.

It follows from the definition of characteristic exponents that, by grouping the monomials between two successive characteristic exponents, the Puiseux expansion (2.1) can be written more conveniently in the form

$$\begin{cases} x = t^m \\ y = p_0(t^m) + t^{k_1} p_1(t^{d_1}) + \cdots + t^{k_s} f_s(t^{d_s}) \end{cases} \tag{2.2}$$

where $p_0, \ldots, p_{s-1}$ are polynomials and f_s is a convergent power series of order 0.

In [E-C] it is shown that the standard resolution of (Y^j, O) can be reconstructed starting from (2.2) as follows.

Consider the chain of the euclidean algorithms which calculate $d_1, d_2, \ldots, d_s$, say:

$$\begin{aligned} k_i - k_{i-1} &= \mu_{i,1} d_{i-1} + r_{i,2} \\ d_{i-1} &= \mu_{i,2} r_{i,2} + r_{i,3} \\ &\vdots \\ r_{i,q(i)-2} &= \mu_{i,q(i)-1} r_{i,q(i)-1} + r_{i,q(i)} \\ r_{i,q(i)-1} &= \mu_{i,q(i)} r_{i,q(i)} \end{aligned} \qquad i = 1, \ldots, s \tag{2.3}$$

where $k_0 = 0$, $d_0 = m$. Clearly $r_{i,q(i)} = d_i$.

Then, in order to obtain the Puiseux expansions of the strict transforms of Y^j, we need (after replacing every time the origin $(0,0)$ in the point $\big(x(0), y(0)\big)$)

- to divide $\mu_{1,1}$ times by the variable x
- then to divide $\mu_{1,2}$ times by the variable y
- then to divide $\mu_{1,3}$ times by the variable x

and so on, following successively all the rows of the s euclidean algorithms (2.3).

We are also interested in performing a similar process for a curve C which is not singular. In this case C is parametrized by $\begin{cases} x = t \\ y = a_k t^k + \cdots \end{cases}$ and no characteristic exponent is defined. Blowing up k times, its strict transform is given by $\begin{cases} x = t \\ y = a_k + \cdots \end{cases}$ and this sequence of blowings-up is described by $k = k \cdot 1$. We can therefore take account of this special situation and have it described by the algorithm (2.3) provided that $(1; k)$ play the role of characteristic exponents.

Of course when you have to divide by a variable, it is convenient to have it expressed by a monomial t^h; this is always possible by means of a change of parameter. Then we need to know how the series expressing the other variable changes under a reparametrization. For this we will use the following lemma, the statement of which was kindly suggested to us by Mutsuo Oka.

LEMMA 2.4. *Let $0 < k_1 < \cdots < k_s$ be integers. Let d_1 be a proper divisor of k_1 and consider the greatest common divisors recursively defined by $d_i = (d_{i-1}, k_i)$ $i = 2, \ldots, s$.*
Assume $d_1 > d_2 > \cdots > d_s$ (i.e. d_1 doesn't divide k_{i+1}).
Let $T(t) = \sum_{j=k_1}^{\infty} c_j t^j$ be a complex series with the following properties:
(1) $c_j \neq 0$ $\forall j \in \{k_1, \ldots, k_s\}$
(2) if $k_i < j < k_{i+1}$ and $c_j \neq 0$, then $j \equiv 0 \pmod{d_i}$.
Set $T(t) = s^{k_1}$ and let $t = s \cdot S(s) = s \cdot \sum_{i=0}^{\infty} b_{k_1+i} s^i$ be the change of parameter. Then:
(a) the coefficients $\{b_j\}_{j \geq k_1}$ satisfy the properties (1) and (2);
(b) if $T' = \sum_{j=k_1}^{\infty} c'_j t^j$ satisfies (1) and (2) and if $S' = \sum_{i=0}^{\infty} b'_{k_1+i} s^i$ is obtained from T' in the same way as S is obtained from T, then you have

$$c_j = c'_j \quad \forall j \leq k \iff b_j = b'_j \quad \forall j \leq k, \text{ provided that } b_{k_1} = b'_{k_1}.$$

PROOF. By the properties (1) and (2), $T(t)$ can be written in the following way (compare with (2.2)):

$$T(t) = t^{k_1} p_1(t^{d_1}) + t^{k_2} p_2(t^{d_2}) + \cdots + t^{k_s} f_s(t)$$

where $p_i(t) = \sum_{j=0}^{\mu_i} c_{k_i + j d_i} t^{j d_i}$ and μ_i is the integral part of $\frac{k_{i+1} - k_i}{d_i}$.

Now we must set $T\big(s \cdot S(s)\big) = s^{k_1}$, that is

$$T\Big(s \cdot \sum_{i=0}^{\infty} b_{k_1+i} s^i\Big) = s^{k_1}. \tag{2.4.1}$$

From (2.4.1) we have immediately $c_{k_1} \cdot (b_{k_1})^{k_1} = 1$, hence $b_{k_1} \neq 0$.

Following the classical procedure to invert a series, we shall deduce the properties (1) and (2) for the coefficients of S by calculating the coefficients of s^α in the left member of (2.4.1) and by imposing it to be 0 for $\alpha > k_1$. Note that, in order to calculate such a coefficient, it is enough to truncate T at the order α. This truncation is, for $k_1 \leq \alpha < k_2$, a polynomial in t^{d_1} and in general, for $k_i \leq \alpha < k_{i+1}$, it is a polynomial in t^{d_i}.

Let's start to prove that the coefficients b_α vanish for $k_1 < \alpha < k_1 + d_1$ by induction on α.

For $\alpha = k_1 + 1$, we must calculate the coefficient of s^{k_1+1} in

$$s^{k_1} \cdot \big(S(s)^{k_1}\big)(c_{k_1} + c_{k_1+d_1} s^{d_1} S(s)^{d_1} + \cdots)$$

and therefore, by truncation, in $c_{k_1} \cdot s^{k_1} \cdot S(s)^{k_1}$. The coefficient turns out to be $k_1 c_{k_1} b_{k_1}^{k_1-1} b_{k_1+1}$, so it vanishes if and only if $b_{k_1+1} = 0$.

Assume now $b_\beta = 0$ for each β such that $k_1 < \beta < \alpha < k_1 + d_1$.

Then $S(s)^{k_1} = (b_{k_1} + b_\alpha s^{\alpha-k_1} + \cdots)^{k_1}$, hence the coefficient of the monomial of degree $\alpha - k_1$ in $S(s)^{k_1}$ is $k_1 b_{k_1}^{k_1-1} b_\alpha$. As above, the truncation of T is $c_{k_1} t^{k_1}$, since $\alpha < k_1 + d_1$. Therefore you have

$$k_1 c_{k_1} b_{k_1}^{k_1-1} b_\alpha = 0 \Longleftrightarrow b_\alpha = 0.$$

We have thus proved the thesis on the b_α's, for $\alpha < k_1 + d_1$.

Assume now that the b_α's have the properties (1) and (2) for $\alpha < k_1 + ld_1$, and let us prove that $b_\alpha = 0$ for each α such that $k_1 + ld_1 < \alpha < k_1 + (l+1)d_1$, if $l + 1 \leq \mu_1$.

By induction

$$S(s) = b_{k_1} + b_{k_1+d_1} s^{d_1} + \cdots + b_{k_1+ld_1} s^{ld_1} + \sum_{k_1+ld_1+1}^{\infty} b_\alpha s^{\alpha-k_1}.$$

Note that the truncation of S at the order ld_1 is a polynomial in s^{d_1}; therefore, for every k, each monomial of S^k of low enough degree has a degree divisible by d_1. The monomial of S^k of lowest degree not divisible by d_1 is obtained as a product by taking $k - 1$ times the term b_{k_1} and once the term $b_{k_1+ld_1+1} s^{ld_1+1}$.

Let $\alpha = k_1 + ld_1 + 1$ (which is not divisible by d_1).

To calculate the coefficient of $s^{k_1+ld_1+1}$ in $T\big(s \cdot S(s)\big)$, truncate T at the order $k_1 + ld_1 + 1$ (or equivalently at the order $k_1 + ld_1$). By the above remark, only the term $c_{k_1} t^{k_1}$ of T can give a contribution to $s^{k_1+ld_1+1}$, since the successive terms $c_{k_1+d_1} t^{k_1+d_1}$, $c_{k_1+2d_1} t^{k_1+2d_1}, \ldots$ yield by the substitution $t = s \cdot S(s)$ monomials with degree nondivisible by d_1 only if this degree is at least $k_1 + d_1 + ld_1 + 1$. Consequently the coefficient of $s^{k_1+ld_1+1}$ is $k_1 c_{k_1} b_{k_1}^{k_1-1} b_{k_1+ld_1+1}$; since it must vanish, one has $b_{k_1+ld_1+1} = 0$.

Now the first term of S having a degree not divisible by d_1 is $b_{k_1+ld_1+2} s^{ld_1+2}$. Arguing as before, one proves that $b_{k_1+ld_1+2} = 0$ and iteratively $b_\alpha = 0$ for $k_1 + ld_1 < \alpha < k_1(l+1)d_1$.

Before going on with the proof of the part (a), let us see how to calculate $b_{k_1+ld_1}$.

The terms of T which can give a contribution to $s^{k_1+ld_1}$ are only:

$$c_{k_1} t^{k_1} + c_{k_1+d_1} t^{k_1+d_1} + \cdots + c_{k_1+ld_1} t^{k_1+ld_1}.$$

The substitution in $c_{k_1} t^{k_1}$ yields, for the coefficient of $s^{k_1+ld_1}$, the sum of $k_1 c_{k_1} b_{k_1}^{k_1-1} b_{k_1+ld_1}$ and other terms involving c_{k_1} and some b_α's with $\alpha < k_1 + ld_1$. The last term contributes by $c_{k_1+ld_1} b_{k_1}^{k_1+ld_1}$. In the intermediate terms $c_{k_1+\mathrm{rd}_1} s^{k_1+\mathrm{rd}_1} \big(S(s)\big)^{k_1+\mathrm{rd}_1}$, $r < 1$, one must take

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in $S(s)^{k_1+\mathrm{rd}_1}$ the monomials of degree $(l-r)d_1$. Therefore the coefficient of $s^{k_1+ld_1}$ is given by

$$(2.4.2) \qquad k_1 c_{k_1} b_{k_1}^{k_1-1} b_{k_1+ld_1} + \text{ intermediate terms } + c_{k_1+ld_1} b_{k_1}^{k_1+ld_1}$$

where each of the intermediate terms contains one $c_{k_1+\mathrm{rd}_1}$, with $r < 1$, and some b_α's, with $\alpha < k_1 + ld_1$. So $b_{k_1+ld_1}$ is determined.

Coming back to the proof of (a), we have so far proved the thesis for $k_1 \le \alpha < k_2$. We can suppose, by induction, the thesis holds for $k_1 \le \alpha < k_i$. We have to prove:

- $b_{k_i} \neq 0$
- $b_\alpha = 0$ if $k_i < \alpha < k_{i+1}$ and $\alpha \not\equiv 0 \pmod{d_i}$.

Again, by the inductive hypothesis, we have

$$S(s) = q_1(s^{d_1}) + s^{k_2-k_1} q_2(s^{d_2}) + \cdots + s^{k_{i-1}-k_1} q_{i-1}(s^{d_{i-1}}) + \sum_{\alpha=k_i}^{\infty} b_\alpha s^{\alpha-k_1}.$$

Arguing as above, we note that the truncation of $S(s)$ at the order $k_i - k_1 - 1$ is a polynomial in s^{d_i-1} and therefore the monomial of S^k of lowest degree not divisible by d_{i-1} is $kb_{k_1}^{k-1} b_{k_i} s^{k_i-k_1}$. This implies that, if we truncate T at the order k_i and substitute $t = s \cdot S(s)$, obtaining

$$(2.4.3) \quad c_{k_1} s^{k_1} \big(S(s)\big)^{k_1} + c_{k_1+d_1} s^{k_1+d_1} \big(S(s)\big)^{k_1+d_1} + \cdots + c_{k_2} s^{k_2} \big(S(s)\big)^{k_2} + \cdots + c_{k_i} s^{k_i} \big(S(s)\big)^{k_i},$$

the monomial of lowest degree not divisible by d_{i-1} has coefficient $k_1 c_{k_1} b_{k_1}^{k_1-1} b_{k_i} + c_{k_i} b_{k_1}^{k_i}$.

In fact the intermediate terms of (2.4.3) yield monomials with a degree not divisible by d_{i-1} only when such a degree is greater than k_i.

From the fact that $k_1 c_{k_1} b_{k_1}^{k_1-1} b_{k_i} + c_{k_i} b_{k_1}^{k_i} = 0$, we get $b_{k_i} \neq 0$.

The proof that $b_\alpha = 0$ for $k_i < \alpha < k_{i+1}$ and $\alpha \not\equiv 0 \pmod{d_i}$ is analogous to the proof given when $k_1 < \alpha < k_2$, noting that the truncation of S at the order $k_i - k_1 + ld_i < k_{i+1} - k_1$ is a polynomial in s^d. Moreover the coefficient of s^{k_i+ld} is given by an expression, analogous to (2.4.2), of the type:

$$(2.4.4) \qquad k_1 c_{k_1} b_{k_1}^{k_1-1} b_{k_i+ld_i} + \text{ intermediate terms } + c_{k_i+ld_i} b_{k_1}^{k_i+ld_i} = 0$$

where each of the intermediate terms contains one c_β and some b_α's, with $\alpha, \beta < k_i + ld_i$. (2.4.4) determines $b_{k_i+ld_i}$.

The part (a) is completely proved.

To prove (b), it is enough to note that (2.4.4) can be written in the form

$$F \cdot b_\alpha + G + H \cdot c_\alpha = 0$$

where $F \neq 0$, $H \neq 0$ and G contains only coefficients of T and S with indices lower than α. That allows the recursive calculation of the coefficients b_α of S (respectively c_α of T) in terms of the coefficients of T (resp. of S) of index lower or equal to α and of its own coefficients of index lower than α.

Finally note that all the b_α's are uniquely determined by T and by the choice of $b_{k_1} = (\frac{1}{c_{k-1}})^{\frac{1}{k_1}}$.

This explains why (b) holds only if one chooses $b_{k_1} = b'_{k_1}$. ∎

DEFINITION 2.5. Let $k_1, \ldots, k_s, d_1$ be integers as in Lemma 2.4. We say that $(k_1, \ldots, k_s; d_1)$ are *characteristic numbers* for the series $T(t)$ if the conditions (1) and (2) of Lemma 2.4 are satisfied. This definition makes sense also when $k_1 = 0$; in this case d_1 will be supposed to be any integer not dividing k_2.

REMARK 2.6. The part (b) of Lemma 2.4 can be used also under a weakened hypothesis. In fact if $T(t)$ has characteristic numbers $(k_1, \ldots, k_s; d_1)$ and $T'(t)$ has characteristic numbers $(k'_1, \ldots, k'_s; d'_1)$ and if there exist integer numbers p, q such that

$$pk_i = qk'_i \quad i = 1, \ldots, s \text{ and } pd_1 = qd'_1,$$

then the series $T(t^p)$ and $T'(t^q)$ have the same characteristic numbers; hence Lemma 2.4 (b) holds true for them.

REMARK 2.7. If C is a branch of an analytic curve germ in $(\mathbb{C}^2, 0)$ with a Puiseux expansion (2.2) and characteristic exponents $(m; k_1, \ldots, k_s)$, then $(k_1, \ldots, k_s; d_1)$, where $d_1 = \text{g.c.d.}(m, k_1)$, are characteristic numbers for the series $y(t) - p_0(t^m)$ in (2.2).

In the following lemma, starting from the Puiseux expansion of a branch C, we reconstruct the expansion of the strict transform of C at any stage of the process of standard resolution.

LEMMA 2.8. *Let C be an irreducible germ of analytic curve in $(\mathbb{C}^2, O)$ with a Puiseux expansion (2.2). Let C_ρ be the strict transform of C after ρ blowings-up of its standard resolution.*

If ρ corresponds to the end of the j-th row of the i-th block (i.e. of the i-th euclidean algorithm)

$$r_{i,j-1} = \mu_{i,j} r_{i,j} + r_{i,j+1}$$

(where $r_{i,0} = k_i - k_{i-1}$ and $r_{i,1} = d_{i-1}$), then, up to exchange the variables x and y, the Puiseux expansion of C_ρ is:

$$\begin{cases} x = t^{r_{i,j}} \\ y = t^{r_{i,j+1}} \cdot [\tilde{p}_i(t^{d_i}) + t^{k_{i+1}-k_i} \cdot \tilde{p}_{i+1}(t^{d_{i+1}}) + \cdots] \end{cases}$$

with characteristic exponents

$$(r_{i,j}; r_{i,j+1}, k_{i+1} - k_i + r_{i,j+1}, \ldots, k_s - k_i + r_{i,j+1}).$$

In particular, if ρ corresponds to the end of the i-th block, the expansion of C_ρ is:

$$\begin{cases} x = t^{d_i} \\ y = \tilde{p}_i(t^{d_i}) + t^{k_{i+1}-k_i} \cdot \tilde{p}_{i+1}(t^{d_{i+1}}) + \cdots \end{cases}$$

with characteristic exponents $(d_i; k_{i+1} - k_i, \ldots, k_s - k_i)$.

PROOF. We can assume that at the end of each block, say for instance the i-th block, the variable x is expressed by a monomial and y by a series. In fact the last division to be made in the block is

$$r_{i,q(i)-1} = \mu_{i,q(i)} \cdot d_i, \tag{2.8.1}$$

which means that one of the two variables is given by t^{d_i} and the other one by a series of order multiple of d_i. If $x = t^{d_i}$, we are done. If, on the contrary, $y = t^{d_i}$, before performing the last blowing-up (*i.e.* when x is expressed by a series of order d_i), one can reparametrize once more.

Such a procedure does not change the algorithm, except for replacing, if necessary, the last row (2.8.1) by the rows

$$r_{i,q(i)-1} = (\mu_{i,q(i)} - 1) \cdot d_i + d_i$$
$$d_i = 1 \cdot d_i.$$

In this way we can always suppose that each block consists of an odd number of rows. This explains the lack of ambiguity in the representation of C_ρ at the end of a block.

Let us now prove the lemma by induction on i and j.

Let $i = 1$ and $j = 1$. At the end of the first row, that is after $\mu_{1,1}$ blowings-up, $C_{\mu_{1,1}}$ is given by

$$\begin{cases} x = t^m \\ y = t^{k_1 - \mu_{1,1} m} p_1(t^{d_1}) + t^{k_2 - \mu_{1,1} m} p_2(t^{d_2}) + \cdots \end{cases}.$$

That is precisely what we wanted, since $k_1 = \mu_{1,1} m + r_{1,2}$.

For $i = 1, j = 2$, before performing the blowings-up of the second row, we have to reparametrize, by setting $y = T(t) = s^{r_{1,2}}$. With the notation of Lemma 2.4, we shall have $x = s^m \cdot \big(S(s)\big)^m$. The series T has characteristic numbers $(r_{1,2}, k_2 - k_1 + r_{1,2}, \ldots,$ $k_s - k_1 + r_{1,2}; d_1)$, with $d_1 = (r_{1,2}, m)$. Since $(d_1, k_2 - k_1 + r_{1,2}) = (d_1, k_2 - k_1)$, the g. c. d. sequence is again $(d_1, d_2, \ldots, d_s)$. By Lemma 2.4, the series $S(s)$ has characteristic numbers $(0, k_2 - k_1, \ldots, k_s - k_1; d_1)$. One can easily convince oneself that the same thing is true for $S(s)^m$ and therefore the series $s^m \cdot S(s)^m$ has characteristic numbers $(m,$ $k_2 - k_1 + m, \ldots, k_s - k_1 + m; d_1)$.

Perform now the second row of blowings-up, dividing $\mu_{1,2}$ times by $s^{r_{1,2}}$. At the end of the row, we get

$$\begin{cases} x = s^{m - \mu_{1,2} r_{1,2}} \big(S(s)\big)^m \\ y = s^{r_{1,2}} \end{cases}$$

where the series expressing x has characteristic numbers

$$(r_{1,3} = m - \mu_{1,2} r_{1,2}, k_2 - k_1 + r_{1,3}, \ldots, k_s - k_1 + r_{1,3}; d_1).$$

That proves the case $i = 1, j = 2$; clearly, arguing in the same way, one can inductively prove all the cases $i = 1, j = 1, \ldots, q(1)$.

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Assume now, by induction, that the thesis is true for $i \leq i_0$. In particular at the end of the i_0-th block we have:

$$\begin{cases} x = t^{d_{i_0}} \\ y = \tilde{p}_{i_0}(t^{d_{i_0}}) + t^{k_{i_0+1}-k_{i_0}} \cdot \tilde{p}_{i_0+1}(t^{d_{i_0+1}}) + \cdots \end{cases}$$

with characteristic exponents $(d_{i_0}; k_{i_0+1} - k_{i_0}, \ldots, k_s - k_{i_0})$.

But then the situation is completely similar to the initial one, with d_{i_0} in the place of m and $k_i - k_{i_0}$ in the place of k_{i-i_0}. The same argument as above proves the thesis for $i = i_0 + 1$ and $1 \leq j \leq q(i_0 + 1)$. ∎

Keeping in mind that we are trying to solve Problem 1 of the first section, we want now to look for (complex) irreducible arcs γ through $(0,0) \in \mathbb{C}$ of the special kind defined below.

DEFINITION 2.9. Let D_i be the exceptional divisor of the blowing-up π_i, *i.e.* the irreducible component of the exceptional curve E_i of the standard resolution of C, which appears for the first time when performing the i-th blowing-up. Let γ be an analytic arc through $(0,0) \in \mathbb{C}^2$. We say that γ has the property $*(\rho)$ if:

i) the strict transform $\gamma_{\rho-1}$ of γ intersects $C_{\rho-1}$ in the point $O = C_{\rho-1} \cap D_{\rho-1}$;

ii) $\gamma_{\rho-1}$ admits a parametrization of the type $\begin{cases} x = t \\ y = at + \cdots \end{cases}$

iii) the tangent line to $\gamma_{\rho-1}$ in O is distinct from the tangent line to $C_{\rho-1}$.

Note that, if γ is like that, γ_ρ is a smooth arc which meets D_ρ transversally in a point different from $C_\rho \cap D_\rho$ and ρ is the first level in the resolution process at which γ_ρ and C_ρ separate.

In the following lemma we calculate the characteristic exponents of such a γ for any fixed ρ. Let us denote by $M_1 = \sum_{j=1}^{q(i)} \mu_{i,j}$ the number of blowings-up of the i-th block, and $M = M_1 + \cdots + M_s$ the number of blowings-up of the standard resolution of C. We shall consider also the case $\rho > M$: in this case we extend the resolution process by blowing up the (smooth) curve C_M in the point $C_M \cap D_M$ and so on.

LEMMA 2.10. *Let C be a branch of analytic curve in $(\mathbb{C}^2, O)$ with irreducible Puiseux expansion (2.2), characteristic exponents $(m; k_1, \ldots, k_s)$ and* g. c. d. *sequence $(d_1, d_2, \ldots, d_s = 1)$. Fix a ρ and let γ be an analytic arc through $(0,0)$ which has the property $*(\rho)$ defined in Definition 2.9. Then*

a) γ admits an irreducible Puiseux expansion $\begin{cases} x = t^n \\ y = T(t) \end{cases}$ with characteristic exponents $(n; h_1, \ldots, h_l)$, $l \leq s$, and sequence of g. c. d. *$(d'_1, \ldots, d'_l)$ uniquely determined by ρ (see also Remark 2.11);*

b) if $\rho \geq M = M_1 + \cdots + M_s$, then $l = s$, $n = m$ and $h_i = k_i$, $i = 1, \ldots, s$;

c) if $\rho < M = M_1 + \cdots + M_s$, one has $m = rn$, $k_i = rh_i$, $i = 1, \ldots, l-1$ with $r = \frac{d_{l-1}}{d'_{l-1}}$.

PROOF. Set $M_0 = 0$.

Let us start by supposing $\rho \leq M_1 + \cdots + M_s$ and let i_0 be the least integer such that $\rho \leq M_1 + \cdots + M_{i_0}$. Since γ_j has to pass through $P_j = C_j \cap D_j$ for each $j = 1, \ldots, \rho - 1$,

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one can deduce that the sequence of euclidean algorithms associated to γ must have the same $q(i)$'s and the same $\mu_{i,j}$'s as C for $i = 1, \ldots, i_0 - 1$ and $1 \leq j \leq q(i)$.

γ_j and C_j have to pass through the same point also for $M_1 + \cdots + M_{i_0-1} < j < \rho$, so the next $\rho - (M_1 + \cdots + M_{i_0-1})$ blowings-up must come from only one block of γ since this happens for C. But we want (property $*(\rho)$) that $\gamma_{\rho-1}$ is expressed by $\begin{cases} x = t \\ y = at + \cdots \end{cases}$, so the i_0-th block must be the last one for γ, that is $l = i_0$.

Moreover we can set $d'_l = 1$, in order to obtain an irreducible parametrization for γ.

Starting from these data, we want to reconstruct all the euclidean algorithms relative to γ. To do that, it is enough to reconstruct the last block (which calculates d'_{l-1} and $h_l - h_{l-1}$) and then continue backwards, by substituting the value of d'_{l-1} in the place of the last remainder in the $(l-1)$-st block and so on.

The reconstruction of the last block requires distinguishing some cases.

By hypothesis, we have $\rho = M_1 + \cdots + M_{l-1} + \eta$, where either $\eta < \mu_{l,1}$, or $\eta = \sum_{j=1}^{b} \mu_{l,j} + \eta_0$ with $\eta_0 = 0$ if $b = q(l)$, otherwise $\eta_0 < \mu_{l,b+1}$.

Let us start by considering the case $\eta \geq \mu_{l,1}$ and $\eta_0 \neq 0, 1$. One must have $\mu'_{l,b+1} = \eta_0$ and $\mu'_{l,j} = \mu_{l,j}, j = 1, \ldots, b$. Therefore the last block of γ must have the form

$$\begin{aligned} h_l - h_{l-1} &= \mu_{l,1} d'_{l-1} + r'_{l,2} \\ d'_{l-1} &= \mu_{l,2} r'_{l,2} + r'_{l,3} \\ &\vdots \\ r'_{l,b-1} &= \mu_{l,b} r'_{l,b} + 1 \\ r'_{l,b} &= \eta_0 \cdot 1 \qquad \text{(we have set } d'_l = 1 \text{ and } \mu'_{l,b+1} = \eta_0) \end{aligned}$$

This block calculates d'_{l-1} and $h_l - h_{l-1}$.

The cases $\eta_0 = 0$ or $1 < \eta < \mu_{l,1}$ are completely analogous.

We still have to investigate the cases $\eta = 1$ or $\eta_0 = 1$.

If $\eta_0 = 1$, following the same procedure as above, the last two rows of the l-th block of γ would have the form

$$\begin{aligned} r'_{l,b-1} &= \mu_{l,b} r'_{l,b} + 1 \\ r'_{l,b} &= 1 \cdot 1. \end{aligned} \tag{2.10.1}$$

Evidently the first one of them is not a euclidean division, as the remainder is not lower than the divisor. The correct division should be

$$r'_{l,b-1} = (\mu_{l,b} + 1) \cdot 1. \tag{2.10.2}$$

However, using (2.10.2) instead of (2.10.1), we do not alter the total number of blowings-up of the block, that is $\mu_{l,1} + \mu_{l,2} + \cdots + \mu_{l,b} + 1$, and we get the same result for d'_{l-1} and $h_l - h_{l-1}$.

In the same way, one can deal with the case $\eta = 1$.

What was said above shows that, for our purpose, one can use the algorithm of the general case in these cases too, even if it is not a true euclidean algorithm.

Going on backwards, as explained before, one calculates n, $h_1, \ldots, h_l$.

Let us now consider the case $\rho > M = M_1 + \cdots + M_s$.

By Lemma 2.8, the curve C_M has a parametrization of the type

$$\begin{cases} x = t \\ y = \sum_{i=0}^{\infty} \alpha_i t^i \end{cases}.$$

A priori the curve γ_M is not yet completely resolved; however, by hypothesis, the first s blocks of γ have the same $q(i)$'s and the same $\mu_{i,j}$'s as the blocks of C. Therefore, again by Lemma 2.8, γ_M is represented by

$$\begin{cases} x = t^{d'_s} \\ y = q_s(t^{d'_s}) + t^{h_{s+1}-h_s}(\cdots) \end{cases}.$$

During the following $\rho - M$ blowings-up, the strict transform of C_M is obtained by dividing each time by the variable x. Then, the same thing must happen for γ; in particular the exponent d'_s cannot decrease. Since we want γ_ρ to have exponent equal to 1, it must be $d'_s = 1 = d_s$.

As a consequence, γ has the same number s of blocks as C and, arguing as above, one deduces from $d'_s = d_s = 1$ that $h_i = k_i$ for $i = 1, \ldots, s$ and $n = m$.

We have thus proved a), b).

To prove c), it is enough to note that if the euclidean algorithms for the two pairs of integers (h, k) and (h', k') have the same number of rows and the same quotients, then the two pairs are one multiple of the other by a rational number given by the quotient of the two g. c. d.'s. ∎

REMARKS 2.11. i) As for Lemma 2.10 a), we need to note that, if $\rho \leq \mu_{1,1}$, the last block to be reconstructed is also the first one and gives $h_1 = \rho \cdot 1$. So we get $d_1 = n = 1$ and $h_1 = \rho$, even if in this case h_1 is not really a characteristic exponent according to the usual definition.

ii) The result of Lemma 2.10 is clearly algorithmic. We emphasize that, in order to calculate $h_1, \ldots, h_l, n$, it is not necessary to reconstruct all the blocks as in the proof of Lemma 2.10, but only the last one, because of Lemma 2.10 c).

PROPOSITION 2.12. *Let C be an analytic branch in $(\mathbb{C}^2, 0)$ with irreducible Puiseux expansion (2.2),*

$$\begin{cases} x = t^m \\ y = p_0(t^m) + \sum_{i=1}^{s-1} t^{k_i} p_i(t^{d_i}) + t^{k_s} f(t) \end{cases}.$$

Let α be an integer, $0 \leq \alpha \leq s$, and let γ be an arc having the property $*(\rho)$ with $\rho \geq M_0 + \cdots + M_\alpha + 1$.

Then γ admits an irreducible parametrization of the type (setting $h_0 = 0$ and $d'_0 = n$)

$$\begin{cases} x = t_n \\ y = \sum_{i=0}^{\alpha-1} t^{h_i} q_i(t^{d'_i}) + t^{h_\alpha} g(t) \end{cases} \tag{2.12.1}$$

where $q_0 = p_0, \ldots, q_{\alpha-1} = p_{\alpha-1}$ as elements of the ring $\mathbb{C}[X]$, and $g(t)$ is a convergent series.

Moreover the constant term of g is equal to the constant term of p_α if $\alpha < s$, or to the constant term of f if $\alpha = s$.

PROOF. By Lemma 2.10 we already know that γ admits a parametrization of the type (2.12.1) with characteristic exponents $(n; h_1, \ldots, h_l)$, $1 \geq \alpha$, uniquely determined by ρ; in (2.12.1) all the monomials of order greater or equal to h_α have been collected in $t^\alpha g(t)$.

The proof consists of several steps.

STEP 1. The case $\alpha = 0$. In this case we have only to show that p_0 and g have the same constant terms. This is obvious, since both γ and C pass through $O \in \mathbb{C}$ for $t = 0$.

STEP 2. If $\alpha = 1$, one easily sees that $p_0 = q_0$. In fact it is enough to impose that the two curves pass through the same point during the first $\mu_{1,1}$ blowings-up.

STEP 3. If $\rho > M_1$, the constant term b of q_1 and the constant term a of p_1 are tied by the relation $b^{n/d'_1} = a^{m/d_1}$.

PROOF OF STEP 3. First of all $\frac{n}{d'_1} = \frac{m}{d_1}$ by Lemma 2.10.
After $\mu_{1,1}$ blowings-up, by Lemma 2.8, $C_{\mu_{1,1}}$ and $\gamma_{\mu_{1,1}}$ are respectively represented by:

$$\begin{cases} x = t^m \\ y = t^{r_{1,2}} p_1(t^{d_1}) + \cdots \end{cases} \qquad \begin{cases} x = t^n \\ y = t^{r'_{1,2}} q_1(t^{d'_1}) + \cdots \end{cases}.$$

We must reparametrize in order to have $y = s^{r_{1,2}}$ (resp. $y = s^{r'_{1,2}}$). To make this reparametrization, we choose arbitrarily one $r_{1,2}$-th root c_0 of a^{-1} (resp. one $r'_{1,2}$-th root c'_0 of b^{-1}). We get x expressed by a series with leading coefficient c_0^m (resp. $c_0'^n$). This implies that, at the following change of parameter, when the order of x has become $m - \mu_{1,2}r_{1,2} = r_{1,3}$ (resp. $r'_{1,3}$), we shall have to choose an $r_{1,3}$-th root of c_0^{-m} (resp. an $r'_{1,3}$-th root of $c_0'^{-n}$). After that reparametrization y is expressed by a series of order $r_{1,2}$ starting with a determination of $a^{m/r_{1,3}}$ (resp. of order $r'_{1,2}$ starting with a determination of $b^{n/r'_{1,3}}$).

Going on like that, at the end of the first block, that is after an even number of reparametrizations, y will be expressed (Lemma 2.8) by a series of order 0 starting with a^{m/d_1} (resp.of order 0 starting with b^{n/d'_1}). Since $\rho > M_1$, the curves C_{M_1} and γ_{M_1} have to pass through the same point $(0, a^{m/d_1}) = (0, b^{n/d'_1})$. This implies $a^{m/d_1} = b^{n/d'_1}$.

STEP 4. If $q_0 = p_0, \ldots, q_i = p_i$ and the constant terms b and a of q_{i+1} and p_{i+1} satisfy the relation $b^{d'_i/d'_{i+1}} = a^{d_i/d_{i+1}}$, then there exists a root of unity ω such that, setting $t = \omega \cdot \vartheta$, the new parametrization of γ by means of ϑ satisfies $q_0 = p_0, \ldots, q_i = p_i$ and $b = a$.

PROOF OF STEP 4. By hypothesis, $a = \varepsilon \cdot b$ with ε a root of unity of order a divisor of d'_i/d'_{i+1}. The number ω we look for must satisfy, in particular, $\omega^{h_{i+1}} \cdot b = a = \varepsilon \cdot b$, so ω must be a h_{i+1}-st root of ε. But in the subgroup of $\mathbb{C}^*$ generated by ε, each element is a h_{i+1}-st power, since g. c. d. $(h_{i+1}, d'_i/d'_{i+1}) = 1$. Therefore $\varepsilon = \omega_0^{h_{i+1}}$ with ω_0 a root of unity, the order of which divides d'_i/d'_{i+1}, and so divides $d'_i, d'_i, \ldots, d'_1, n$. So ω_0 is the root we looked for.

STEP 5. Proof of the proposition recursively on α.

For $\alpha = 0$ and $\alpha = 1$, the proposition is proved by the Steps 1, 2, 3, 4.

Suppose now $\rho \geq M_1 + \cdots + M_\alpha + 1$. In particular $\rho \geq M_1 + \cdots + M_{\alpha-1} + 1$, so, by induction, we know that $q_0 = p_0, \ldots, q_{\alpha-2} = p_{\alpha-2}$. After $M_1 + \cdots + M_{\alpha-1}$ blowings-up, $C_{M_1+\cdots+M_{\alpha-1}}$ and $\gamma_{M_1+\cdots+M_{\alpha-1}}$ are respectively given by

$$\begin{cases} x = t^{d_{\alpha-1}} \\ y = p_{\alpha-1}^{(\alpha-1)}(t^{d_{\alpha-1}}) + \cdots \end{cases} \qquad \begin{cases} x = t^{d'_{\alpha-1}} \\ y = q_{\alpha-1}^{(\alpha-1)}(t^{d'_{\alpha-1}}) + \cdots \end{cases}$$

where the upper index (λ) indicates that λ blocks of blowings-up have been performed.

From $\rho > M_1 + \cdots + M_\alpha \geq M_1 + \cdots + M_{\alpha-1} + \mu_{\alpha,1}$ we get, arguing as is Step 2, $q_{\alpha-1}^{(\alpha-1)} = p_{\alpha-1}^{(\alpha-1)}$. Apply now Lemma 2.4, using its part (b) in both directions. Since $q_0 = p_0, \ldots, q_{\alpha-2} = p_{\alpha-2}$, each time we have to reparametrize during the first $\alpha - 2$ blocks of C and γ, we can choose (see proof of Step 3) the same determination of the rational power of the least order coefficients of the series involved. So, by Lemma 2.4, we get $q_\beta^{(i)} = p_\beta^{(i)}$ for $1 \leq \beta \leq \alpha - 2$ and $1 \leq i \leq \beta$.

As for $p_{\alpha-1}$ and $q_{\alpha-1}$, note that, by Step 3 applied to $C_{M_1+\cdots+M_{\alpha-2}}$ and $\gamma_{M_1+\cdots+M_{\alpha-2}}$, the constant terms b and a of $q_{\alpha-1}^{(\alpha-2)}$ and $p_{\alpha-1}^{(\alpha-2)}$ satisfy the relation $b^{d'_{\alpha-2}/d'_{\alpha-1}} = a^{d_{\alpha-2}/d_{\alpha-1}}$. Hence (Step 4) there exists a $(d'_{\alpha-2}/d'_{\alpha-1}$-st root of unity, ω, such that, substituting $t = \omega \cdot \vartheta$ (which does not alter t^d for $d = n, d'_1, d'_2, \ldots, d'_{\alpha-2})$ one can assume $b = a$.

This implies that one can choose the same determination of the successive rational powers of $b = a$ also when performing the blowings-up of the $(\alpha - 1)$-st block. So, by Lemma 2.4, from $q_{\alpha-1}^{(\alpha-1)} = p_{\alpha-1}^{(\alpha-1)}$ we deduce $q_{\alpha-1}^{(\alpha-2)} = p_{\alpha-1}^{(\alpha-2)}$. For the first $\alpha - 2$ blocks we have already chosen the determinations which allow the use of Lemma 2.4, so we deduce successively $q_{\alpha-1}^{(\alpha-3)} = p_{\alpha-1}^{(\alpha-3)}, \ldots, q_{\alpha-1}^{(1)} = p_{\alpha-1}^{(1)}, q_{\alpha-1} = p_{\alpha-1}$.

It remains to prove that g and p_α have the same constant term (if $\alpha = s$, replace p_α by f).

Since $C_{M_1+\cdots+M_\alpha}$ and $\gamma_{M_1+\cdots+M_\alpha}$ pass through the same point for $t = 0$, $g^{(\alpha)}$ and $p_\alpha^{(\alpha)}$ have the same constant term. Arguing as before, we deduce that the same fact is true for $g^{(\beta)}$ and $p_\alpha^{(\beta)}$ $\forall \beta < \alpha$. In particular, at the end of the procedure, we get that g and p_α have the same constant term. ∎

REMARK 2.13. By Lemma 2.10 and Proposition 2.12 we have proved that any arc γ having the property $*(\rho)$ admits a Puiseux expansion in which the coefficients of all the monomials of order lower or equal to h_{l-1} (h_l is the last characteristic exponent of γ) are equal to the corresponding coefficients of the expansion of C (that is, up to the term $a_{k_{l-1}} t^{k_{l-1}}$). As a matter of fact, if ρ is strictly greater than $M_1 + \cdots + M_{l-1} + 1$, more coefficients of γ are determined, as the following theorem proves.

THEOREM 2.14. *Assume that γ has the property $*(\rho)$ with respect to C. Then γ admits an irreducible parametrization* $\begin{cases} x = t^n \\ y = \sum_{i=1}^{\infty} b_i t^i \end{cases}$ *where*

i) if $M_1 + \cdots + M_{l-1} < \rho \leq M_1 + \cdots + M_l$, all the coefficients b_i are determined for $i = 0, \ldots, h_{l-1}, \ldots, h_{l-1} + \rho'$, with $\rho = \min\big(\mu_{l,1}, \rho - (M_1 + \cdots + M_{l-1} + 1)\big)$

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ii) if $\rho > M_1 + \cdots + M_s$, all the coefficients b_i are determined for $i = 0, \ldots, h_s + \rho'$ with $\rho' = \rho - (M_1 + \cdots + M_s + 1)$

iii) the first nondetermined coefficient for γ must be different from the corresponding one in C.

PROOF i) AND ii). By Remark 2.13, we already know the thesis for the b_i's with $i \leq h_{l-1}$. As for the following terms, it is enough to write down the expressions of γ and C after $M_1 + \cdots + M_{l-1}$ blowings-up (case i)), or after $M_1 + \cdots + M_s$ blowings-up (case ii)), and to impose that the further strict transforms of C and γ pass through the same point for $t = 0$.

iii). The first nondetermined coefficient for γ characterizes (by means of the process described in Lemma 2.4 and Proposition 2.12) the tangent to $\gamma_{\rho-1}$, which must be different from that one of $C_{\rho-1}$. ∎

THE REAL CASE. Theorem 2.14 gives a complete answer to Problem 1 in the case of a complex curve. Now we come again to the real case.

Let C be a branch of a real analytic curve in $(\mathbb{R}^2, O)$. It is known (see, for instance, [M]) that C admits a parametrization of the type

$$\begin{cases} x = \varepsilon \cdot t^m \\ y = \sum_{i=1}^{\infty} a_i t^i \quad a_i \in \mathbb{R} \end{cases}$$

where $\varepsilon = \pm 1$; working, if necessary, with the coordinates $(-x, y)$ instead of (x, y), we can always assume $\varepsilon = 1$.

From now on, we will therefore assume C is given by:

$$\begin{cases} x = t^m \\ y = \sum_{i=1}^{\infty} a_i t^i \quad a_i \in \mathbb{R} \end{cases} \tag{2.15}$$

In order to solve Problem 1 in the real case, let γ be a real analytic arc in $(\mathbb{R}^2, O)$ having the property $*(\rho)$ with respect to C. Let

$$\begin{cases} x = \varepsilon \cdot t^n \\ y = \sum_{i=1}^{\infty} b_i t^i = q_0(t^n) + t^{h_1} q_1(t^{d_1}) + \cdots \end{cases} \tag{2.16}$$

be an irreducible parametrization of γ, with $\varepsilon = \pm 1$ (in general, it is not possible to require that $\varepsilon = 1$ both for C and for γ, and we have already made the choice $\varepsilon = 1$ for C).

REMARK 2.17. What was said at the beginning of the section about the standard resolution of C holds true also in the real case. In particular the sequence of blowings-up involved in the resolution is again described by the chain of euclidean algorithms (2.3). Moreover, it is easy to convince oneself that Lemma 2.10 holds true also in the real case (which involves reparametrizations of the type $y = T(t) = \varepsilon \cdot s^k$ with k equal to order of T and $\varepsilon = \pm 1$, according to the sign of the least order coefficient of T). In fact the euclidean algorithms used in the proof of Lemma 2.10 work on the exponents, not on the coefficients. We have also proved that the characteristic exponents $(n; h_1, \ldots, h_l)$ of γ are determined by ρ, as explained in Lemma 2.10.

If we allow the parameter t to vary in $\mathbb{C}$, we can regard both C and γ as complex curves (with real coefficients). In particular, by a suitable complex change of parameter

$$t = \sum_{i=1}^{\infty} c_i s^i, \tag{2.18}$$

we can express γ in the form (2.2). Substituting (2.18) in (2.16), we find that it must be

$$\varepsilon \cdot \Big(\sum_{i=1}^{\infty} c_i s^i\Big)^n = s^n.$$

This implies that $c_i = 0\ \forall i > 1$ and that $c = c_1$ satisfies the relation $c^n = \frac{1}{\varepsilon} = \varepsilon$, and consequently $c^{2n} = 1$.

We are now in the position to apply to C and γ the results found so far, *i.e* Proposition 2.12, Remark 2.13 and Theorem 2.14: after reparametrizing γ by means of $t = c \cdot s$, the coefficients of the series $\sum b_i (cs)^i$ must be equal to those of the series $\sum a_i t^i$ of C up to the order determined in Theorem 2.14. In particular, for the coefficients of the first $(l-1)$ characteristic terms we must have $b_{h_i} c^{h_i} = a_{k_i}$ for $i = 1, \ldots, l-1$.

Hence, c^{h_i} turns out to be a real number for $i = 1, \ldots, l-1$. Since c is a root of 1, c^{h_i} can only be 1 or -1. This implies that $c^{2h_i} = 1$ for $i = 1, \ldots, l-1$ and therefore, since $c^{2n} = 1$, it is $c^{2d'_i} = 1$ for $i = 1, \ldots, l-1$.

Hence if we set $\delta = c^{d'_{l-1}}$, we have only two possibilities: $\delta = 1$ or $\delta = -1$. Note that the value of δ determines the value of $c^{d'_i}$, $i = 1, \ldots l-2$, since $c^{d'_i} = (c^{d'_{l-1}})^{d'_i/d'_{l-1}} = (\delta)^{d'_i/d'_{l-1}}$ and d'_{l-1} divides d'_i.

In particular this implies that the relations found above $q_i(c^{d'_i} s^{d'_i}) = p_i(s^{d'_i})$ are of a very special type: either $q_i(X) = p_i(X)$ or $q_i(-X) = p_i(X)$, according to the value of $c^{d'_i}$.

We have thus proved the following result:

THEOREM 2.19 (SOLUTION OF PROBLEM 1). *Let C be a branch of analytic curve in $(\mathbb{R}^2, O)$ with irreducible Puiseux expansion*

$$\begin{cases} x = t^m \\ y = f(t) = p_0(t^m) + t^{k_1} p_1(t^{d_1}) + \cdots \end{cases}$$

and let γ be a real analytic arc having the property $*(\rho)$ with respect to C. Then γ admits an irreducible parametrization

$$\begin{cases} x = \varepsilon \cdot t^n \\ y = g(t) = q_0(t^n) + t^{h_1} q_1(t^{d'_1}) + \cdots \end{cases}$$

where:

1) *the characteristic exponents $(n; h_1, \ldots, h_1)$ of γ are determined by Lemma 2.10;*
2) *ε and g satisfy one of the two following conditions:*
    a) *$\varepsilon = 1$ and the truncation of $g(t)$ to the order $h_{l-1} + \rho'$ (see Theorem 2.14) coincides with the corresponding truncation of $f(t)$;*

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b) $\varepsilon = (-1)^{n/d'_{l-1}}$ *and the truncation of* $g(t)$ *to the order* $h_{l-1} + \rho'$ *can be obtained from that one of* $f(t)$ *according to the rule:*

$$\begin{cases} q_i(X) = p_i(X) & \text{if } d'_i/d'_{l-1} \text{ is even} \\ q_i(-X) = p_i(X) & \text{if } d'_i/d'_{l-1} \text{ is odd.} \end{cases}$$

SOLUTION OF PROBLEM 2. Each region with a sign in $(\mathbb{R}^2, O) - (Y, O)$ is bounded by two analytic half-branches, possibly belonging to the same branch of (Y, O). We have assumed that no branch of (Y, O) is tangent to $\{x = 0\}$, so that each of the half-branches mentioned above is expressed by

$$\begin{cases} x = \pm t \\ y = f(t) = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \cdots \end{cases}$$

where

i) $t \in [0, \varepsilon)$

ii) $x = t$, or $x = -t$ according to whether the half-branch lies in the half-plane $\{x \geq 0\}$ or in $\{x \leq 0\}$.

iii) $f(t)$ is a Puiseux series.

Our aim is to know the analytic arcs $\gamma\colon [0, \varepsilon) \longrightarrow \mathbb{R}^2$, which have their images contained in a region of the plane bounded by two fixed analytic half-branches γ_1 and γ_2 through $(0, 0)$. Two cases may arise:

a) γ_1 and γ_2 lie in the same quadrant bounded by the y-axis and by the x-axis,

b) γ_1 and γ_2 do not lie in the same quadrant.

One can easily check that the arcs γ we looked for are the arcs satisfying the following conditions:

CASE a). Assume, for instance, that both γ_1 and γ_2 lie in $\{x \geq 0, y \geq 0\}$. Therefore they are expressed by

$$\gamma_1 : \begin{cases} x = t \\ y = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \cdots \end{cases} \qquad \gamma_2 : \begin{cases} x = t \\ y = b_1 t^{\beta_1} + b_2 t^{\beta_2} \cdots \end{cases}$$

with $a_1 > 0, b_1 > 0$.

If $a_1 t^{\alpha_1} \neq b_1 t^{\beta_1}$ (precisely $\alpha_1 \leq \beta_1$, and if $\alpha_1 = \beta_1$ then $a_1 \geq b_1$), the arcs g lying in the region of $\{x \geq 0\}$ bounded by γ_1 and γ_2 must be of the form

$$\begin{cases} x = t \\ y = c_1 t^{\delta_1} + c_2 t^{\delta_2} + \cdots \end{cases}$$

where $c_1 > 0, \delta_1 \in [\alpha_1, \beta_1]$ and, moreover,

- if $\delta_1 = \alpha_1 \Rightarrow c_1 \leq a_1$ and if $c_1 = a_1 \Rightarrow \delta_2 \geq \alpha_2$ and if $\delta_2 = \alpha_2 \Rightarrow c_2 \leq a_2$ and so on.
- if $\delta_1 = \beta_1 \Rightarrow c_1 \geq b_1$ and if $c_1 = b_1 \Rightarrow \delta_2 \leq \beta_2$ and if $\delta_2 = \beta_2 \Rightarrow c_2 \geq b_2$ and so on.

When the expressions of y in γ_1 and γ_2 begin with a common part $y = g(t)+a_1t^{\alpha_1}+\cdots$, $y = g(t)+b_1t^{\beta_1}+\cdots$ with $a_1t^{\alpha_1} \neq b_1t^{\beta_1}$, the condition is the same, except that γ is given by

$$\begin{cases} x = t \\ y = g(t) + c_1t^{\delta_1} + \cdots \end{cases}.$$

For the analytic arcs through $(0,0)$ and lying in the other region bounded by γ_1 and γ_2, one finds all the arcs lying in the three quadrants and the arcs γ which satisfy a condition analogous to that one found above, but with "reversed" inequalities.

CASE b). In this case we have

$$\gamma_1 : \begin{cases} x = \pm t \\ y = a_1t^{\alpha_1} + a_2t^{\alpha_2} + \cdots \end{cases} \qquad \gamma_2 : \begin{cases} x = \pm t \\ y = b_1t^{\beta_1} + b_2t^{\beta_2} + \cdots \end{cases}.$$

It is easy to write down conditions, analogous to the ones found before, assuring that an arc γ, not tangent to $\{x = 0\}$, lies "above or below γ_1" and "above or below γ_2".

Problem 2 is thus completely solved.

THE PROCEDURE. Assuming we know the Puiseux expansions of the branches $Y^1, \ldots, Y^r$ of (Y, O), we are now ready to give a procedure to decide whether a given distribution of signs is locally completable or not. Recall this will be done checking if the exceptional curve E_n in the standard resolution of (Y, O) contains at least one type changing component.

For each component D of E_N there is a lowest index ρ such that D is the strict transform of an irreducible component D_ρ of E_ρ through $\pi_{\rho+1} \circ \pi_{\rho+2} \circ \cdots \circ \pi_N$. Since we know that D is type changing (in X_N) if and only if D_ρ is type changing (in X_ρ), we will test the nature of each D "the first time it appears", *i.e.* by testing D_ρ.

D_ρ was produced by the blowing-up of $X_{\rho-1}$ in a point $P_{\rho-1}$ belonging to an irreducible component $Y^\alpha_{\rho-1}$ of $Y_{\rho-1}$, so we can investigate D_ρ by using the resolution process of Y^α. However note that more components of $Y_{\rho-1}$ may pass through $P_{\rho-1}$ and the resolution process of each of them produces, at the ρ-th step, the same divisor D_ρ.

This explains why, in order to economize on the number of tests, it is helpful to consider, for each i, $0 \leq i < N$, the partition $\mathcal{P}_i$ of the index set $\{1, \ldots, r\}$ into subset $\mathcal{P}_{i,j}$ such that two indexes α and β belong to the same subset if and only if Y^α_i and Y^β_i pass through the same point in X_i. This can be concretely made by using Theorem 2.14.

It is easy to see that:

i) the partition is increasingly refined;

ii) there is a 1-1 correspondence between the elements $\mathcal{P}_{\rho-1,j}$ of the partition $\mathcal{P}_{\rho-1}$ and the irreducible components of E_ρ which do not belong to the strict transform of $E_{\rho-1}$.

Therefore we can perform our test as follows.

- Fix a ρ, $1 \leq \rho \leq N$ (starting from $\rho = 1$).
- Choose an index α_j in $\mathcal{P}_{\rho-1,j}$ for each j.
- Find a family $\mathcal{A}$ of arcs having the property $*(\rho)$, by applying Theorem 2.14 to the curve Y^α, for each j.

- Test if the family $\mathcal{A}$ contains open subfamilies $\mathcal{G}_1$, $\mathcal{G}_2$ of arcs joining regions respectively with the same or the opposite signs, by using the solution of Problem 2.

- If the answer is YES, the test stops (because we have found a type changing component in E_ρ). Otherwise repeat the test at the successive level $\rho + 1$.

3. **An application to a separation problem.** Let V be a compact, non-singular, real algebraic surface; let A, B be two semialgebraic subsets of V such that their interior parts $\overset{\circ}{A}$ and $\overset{\circ}{B}$ do not intersect; denote by $Z = \overline{\bar{A} \cap \bar{B}}^Z$ the Zariski closure of $\bar{A} \cap \bar{B}$.

DEFINITION 3.1. We say that A and B are (polynomially) separated if there exists a polynomial function f on V which is positive on $A - Z$ and negative on $B - Z$ (we will write $f(A - Z) > 0$ and $f(B - Z) < 0$).

Some results about the separation of semialgebraic sets by polynomials can be found in [B2], [R], [F-G] and are used in [F] in order to reduce the completability of a given partial distribution of signs to a local problem.

Now, following the opposite direction, we will apply the results of the first two sections to the problem of deciding whether two given semialgebraic sets in V can be separated by polynomials. In order to avoid trivial cases, we will always suppose both $A - Z$ and $B - Z$ not empty. Adapting classical techniques of separation, we get the following criterion:

PROPOSITION 3.2. *Let A and B be semialgebraic subsets of V such that $\overset{\circ}{A} \cap \overset{\circ}{B} = \emptyset$ and $A - Z \neq \emptyset$ and $B - Z \neq \emptyset$ (with $Z = \overline{\bar{A} \cap \bar{B}}^Z$). If ∂A denotes the boundary of A, define $Y = \overline{\partial A \cup \partial B}^Z$. Let σ be the partial distribution of signs on $V - Y$ defined by $\sigma(A - Y)$ and $\sigma(B - Y) = -1$.*

Then A and B are polynomially separated if and only if σ is completable.

PROOF. First of all note that we can work using regular functions on V; in fact if $f = \frac{P}{Q}$ is a regular function which separates A and B, then $f \cdot Q^2 = P \cdot Q$ is a polynomial separating function.

Moreover, the thesis of Proposition 3.2 is trivially true if at least one of the semialgebraic sets A, B is 0-dimensional; in fact in this case it is easy to see that A and B are separated and σ is completable. So we can assume that $\dim A \geq 1$ and $\dim B \geq 1$, which implies $\dim \partial A = \dim \partial B = 1$.

Since $Y \supset Z$, if A and B are separated, then σ is completable.

Conversely, suppose that σ is completable, *i.e.* there exists $f \in \mathcal{R}(V)$ such that $f(A - Y) > 0, f(B - Y) < 0$ and $f|_Y \equiv 0$.

We will prove that A and B are separated by modifying f in order to extend its sign on $(Y - Z) \cap (\bar{A} \cup \bar{B})$. The procedure consists of two steps: first we extend the sign on the 1-dimensional part of $(Y - Z) \cap (\bar{A} \cup \bar{B})$, and if it is not empty, afterwards on the remaining 0-dimensional part.

STEP 1. Assume that $\dim\big((Y - Z) \cap (\bar{A} \cup \bar{B})\big) = 1$; otherwise go directly to Step 2. We can suppose, for instance, that $\dim\big((Y - Z) \cap \bar{A}\big) = 1$.

Denote $F = \mathrm{Reg}\big((Y - Z) \cap \bar{A}\big)$ and $G = \mathrm{Sing}\big((Y - Z) \cap \bar{A}\big)$. Since F is an open semialgebraic set of dimension 1 in Y, it is known ([R]) that there exists $h \in \mathcal{R}(Y)$ such that $h(F) > 0$ and $\bar{F} = \{y \in Y : h(y) \geq 0\}$.

Since $(Y - Z) \cap \bar{B} \cap \bar{F} = \emptyset$, we have $h\big((Y - Z) \cap \bar{B}\big) < 0$.

Extend h to a regular function on V and denote it again by h. By slightly modifying h, if necessary, we may assume that h is positive on the isolated points of $(Y - Z) \cap \bar{A}$ and negative on the isolated points of $(Y - Z) \cap \bar{B}$.

Consider now the closed semialgebraic set

$$S = (\bar{A} \cap \{h \leq 0\}) \cup (\bar{B} \cap \{h \geq 0\}).$$

By a consequence of Łojasiewicz inequality ([B1]), there exists $\psi \in \mathcal{R}(V)$, $\psi \geq 0$, such that $f + \psi \cdot h$ and f have the same sign on S and, for the zero-set $V(\psi)$, one has $V(\psi) = \overline{V(f) \cap S}^Z$. So, if we set $R = f + \psi \cdot h$, we have that $R\big((A - Y) \cap S\big) > 0$. Moreover $R\big((A - Y) - S\big) > 0$, because there R is the sum of two positive functions. Hence we have $R(A - Y) > 0$. In the same way we see that $R(B - Y) < 0$.

So R separates $A - Y$ and $B - Y$ as f did, but we claim that R does not vanish anymore on $(Y - Z) \cap (\bar{A} \cup \bar{B})$, except on a finite number of points. More precisely we claim that:

$$(3.2.1) \quad R(\bar{A} - Z) \geq 0, \quad R(\bar{B} - Z) \leq 0 \quad \text{and} \quad V(R) \cap \bar{A} \subseteq G \cup Z, \quad V(R) \cap \bar{B} \subseteq Z.$$

PROOF OF (3.2.1). The first two inequalities follow easily from the fact that $f|_Y \equiv 0$, so that on Y we have $R = \psi \cdot h$.

Since $V(R) \cap (\bar{A} \cup \bar{B}) \subseteq Y$, it is enough to study $V(\psi \cdot h)$. Note that: $V(\psi) = \overline{V(f) \cap S}^Z \subseteq \overline{Y \cap \bar{A} \cap \{h \leq 0\}}^Z \cup \overline{Y \cap \bar{B} \cap \{h \geq 0\}}^Z \subseteq \overline{\bar{A} \cap (Z \cup G)}^Z \cup Z$, because $h(F) > 0$ and $h\big((Y - Z) \cap \bar{B}\big) < 0$.

Then $V(\psi) \cap \bar{A} \subseteq G \cup Z$ and $V(\psi) \cap \bar{B} \subseteq Z$. Recalling the properties of h, we get $R(F) > 0$ and $R\big((Y - Z) \cap \bar{B}\big) < 0$, which imply (3.2.1).

STEP 2. In order to extend the sign on the 0-dimensional set G, we use again the Łojasiewicz inequality recalled above.

If we apply that argument to the regular functions R and 1, relatively to the closed semialgebraic set $\bar{B}$, we get a regular function $\eta \geq 0$ such that $R + \eta$ and R have the same sign on $\bar{B}$ and $V(\eta) = \overline{V(R) \cap \bar{B}}^Z$. It easy to see that the function $Q = R + \eta$ is such that $Q(A - Z) > 0$ and $Q(B - Z) < 0$, *i.e.* A and B are separated. ∎

REMARK 3.3. It is important to note that the Proposition 3.2 assures that, in order to decide whether two given semialgebraic sets are polynomially separated, it is equivalent to check whether the associated distribution of signs σ, defined in Proposition 3.2, is completable. In particular, because of the results of §1 and §2, the local completability at the singular points of Y can be tested by using the criterion described in the second section.

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