DIRECT-SUM DECOMPOSITION OF ATOMIC AND ORTHOGONALLY COMPLETE RINGS

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In this paper we give a necessary and sufficient condition for decomposition (as a direct sum of fields) of a ring R in which for every $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that

$$x^{n(x)} = x. \tag{1}$$

We would like to emphasize that *in what follows R stands for a ring every element x of which satisfies* (1).

It is well known [1] that R is commutative and that $x^{n(x)-1}$ is an idempotent element of R, i.e., for every $x \in R$

$$(x^{n(x)-1})^2 = x^{n(x)-1} \tag{2}$$

which implies that R has no nonzero nilpotent element, i.e., for every $x \in R$ and every natural number $k \geqq 1$,

$$x^k = 0 \quad \textit{implies} \quad x = 0. \tag{3}$$

LEMMA 1. *The ring R is partially ordered by $\leqq$ where for all elements x and y of R*

$$x \leqq y \quad \textit{if and only if} \quad xy = x^2. \tag{4}$$

PROOF. Since $xx = x^2$ we see that $\leqq$ is reflexive.

Next, let $x \leqq y$ and $y \leqq x$, i.e. $xy = x^2$ and $yx = y^2$. But then

$$x^2 - xy - yx - y^2 = (x-y)^2 = 0$$

which, in view of (3), implies $x - y = 0$, i.e. $x = y$. Hence $\leqq$ is antisymmetric.

Finally, let $x \leqq y$ and $y \leqq z$, i.e., $xy = x^2$ and $yz = y^2$. Thus, $x^2z = xyz = xy^2 = x^2y = x^3$. Consequently, $x^2z^2 = x^3z$ and $x^3z = x^4$. But then

$$x^2z^2 - 2x^3z + x^4 = (xz - x^2)^2 = 0$$

which, in view of (3), implies $xz = x^2$ which in turn, in view of (4), implies $x \leqq z$. Hence $\leqq$ is transitive.

Thus, Lemma 1 is proved.

Clearly, from (4) and (2) it follows that for all elements x, y and z of R

$$y \leqq z \quad implies \quad xy \leqq xz \tag{5}$$

and

$$x^{n(x)-1}y \leqq y. \tag{6}$$

DEFINITION 1. *A nonzero element a of R is called an atom of R provided for every $x \in R$*

$$x \leqq a \quad implies \quad x = a \quad or \quad x = 0. \tag{7}$$

Moreover, R is called atomic provided for every nonzero element r of R there exists an atom a of R such that $a \leqq r$.

LEMMA 2. *Let a be an atom of R. Then*

$$r^{n(r)-1}a = a \quad or \quad ra = 0$$

for every element r of R.

PROOF. By (6) we have $r^{n(r)-1}\, a \leqq a$ and since a is an atom, by (7) we have $r^{n(r)-1}\, a = a$ or $r^{n(r)-1}\, a = 0$. However, $r^{n(r)-1}\, a = 0$ in view of (1) implies $ra = 0$.

DEFINITION 2. *A subset S of R is called orthogonal provided $xy = 0$ for distinct elements x and y of S.*

LEMMA 3. *The set $(e_i)_{i \in I}$ of all idempotent atoms of R is an orthogonal set.*

PROOF. Since each e_i is both an atom and an idempotent, from Lemma 2 it follows that $e_i e_j = e_i = e_j$ or $e_i e_j = 0$.

LEMMA 4. *Let a be an atom of R. Then $a^{n(a)-1}$ is an idempotent atom of R.*

PROOF. From (2) it follows that $a^{n(a)-1}$ is idempotent.

Now, let $x \leqq a^{n(a)-1}$. But then (5) and (1) imply $ax \leqq a$. Since a is an atom (7) implies $ax = a$ or $ax = 0$.

If $ax = a$ then $a^{n(a)-1}x = a^{n(a)-1}$ which by (4) implies $a^{n(a)-1} \leqq x$. Hence $x = a^{n(a)-1}$.

If $ax = 0$ then $a^{n(a)-1}x = 0$; but $a^{n(a)-1}x = x^2$ by definition of $\leqq$, therefore $x^2 = 0$ which by (3) implies $x = 0$.

LEMMA 5. *Let $(e_i)_{i \in I}$ be the set of all idempotent atoms of R. Then for every $i \in I$ the ideal F_i of R given by*

$$F_i = \{re_i | r \in R\} \tag{8}$$

is a subfield of R. Moreover,

$$F_i \cap F_j = \{0\} \quad if \quad i \neq j. \tag{9}$$

PROOF. Since $e_i^2 = e_i$ it follows that e_i is an element of F_i and also the unit of F_i.

Now, let $re_i \neq 0$. We show that re_i has an inverse in F_i. If $n(r) = 2$ then Lemma 2 implies $re_i = e_i$ which shows that re_i is its own inverse in F_i. If $n(r) > 2$ then Lemma 2 implies $(re_i)(r^{n(r)-2}e_i) = e_i$ which shows that $r^{n(r)-2}e_i$ is the inverse of re_i in F_i.

Next, if $i \neq j$ and $re_i = qe_j$ for $r, q \in R$ then Lemma 3 implies $re_ie_j = qe_j = re_i = 0$.

LEMMA 6. *Let R be atomic and let $(e_i)_{i \in I}$ be the set of all idempotent atoms of R. Then for every nonzero element q of R there exists an idempotent atom, say, e_k such that $qe_k \neq 0$. Moreover, for every $r \in R$ the* $\sup_i re_i$ *exists and*

$$r = \sup_i re_i. \tag{10}$$

PROOF. In view of (7) there exists an atom a such that $a \leqq q$, i.e., $aq = a^2 \neq 0$. But then Lemma 4 and (1) imply that $e_k = a^{n(a)-1}$ is an idempotent atom and $a^{n(a)-1}q = a^{n(a)} = a \neq 0$, i.e., $qe_k \neq 0$.

Next, since $rre_i = (re_i)^2$ for every $i \in I$, it follows that r is an upper bound of $(re_i)_{i \in I}$. Let h be any upper bound of $(re_i)_{i \in I}$, i.e. $hre_i = (re_i)^2$ for every $i \in I$. We show that $r \leqq h$. Because otherwise, $hr - r^2 = q \neq 0$ and therefore $hre_k - r^2e_k = qe_k \neq 0$, contradicting that $hre_i = rre_i$ for every $i \in I$.

Thus, Lemma 6 is proved.

Let us observe that if $(e_i)_{i \in I}$ is the set of all idempotent atoms of R then in view of (9) we may consider the direct sum $\oplus_{i \in I} F_i$ of the fields F_i given by (8). In this connection we have the following

LEMMA 7. *Let R be atomic and let $(e_i)_{i \in I}$ be the set of all idempotent atoms of R. Then*

$$\alpha(r) = (re_i)_{i \in I} \tag{11}$$

is an isomorphism from R into the direct sum $\oplus_{i \in I} F_i$ of fields F_i.

PROOF. It is obvious that α is a homomorphism. We show that α is one-to-one. Indeed, if $r \neq q$ then $\alpha(r) \neq \alpha(q)$. Because otherwise, (10) would imply $r = \sup_i re_i = \sup_i qe_i = q$, contradicting $r \neq q$.

Thus, Lemma 7 is proved.

Let us observe that the existence of an isomorphism from R onto a subring of a direct sum of fields is a well known fact and is proved without imposing any special condition (such as atomicity) on R. However, for the proof of our Theorem we need (as seen below) the special isomorphism α described in Lemma 7. In fact the existence of the isomorphism α is crucial for the proof of our Theorem which states that atomicity and orthogonal completeness of R is a necessary and sufficient condition for R to be isomorphic to a direct sum of fields. The proof uses Lemma 8 below.

First however, we observe that if $(r_i)_{i\in I}$ is a subset of R such that $\sup_i r_i$ exists then, since $r_i \leqq \sup_i r_i$, in view of (3) we have

$$r_i \sup_i r_i = r_i^2. \tag{12}$$

LEMMA 8. *Let $(r_i)_{i\in I}$ be a subset of R such that $\sup_i r_i$ exists. Then for every element b of R the $\sup_i br_i$ exists and*

$$b \sup_i r_i = \sup_i br_i. \tag{13}$$

PROOF. In view of (12) we have

$$(br_i)(b \sup_i r_i) = b^2 r_i \sup_i r_i = (br_i)^2$$

which, in view of (4), implies $br_i \leqq b \sup_i r_i$ for every $i \in I$. Thus, $b \sup r_i$ is an upper bound of $(br_i)_{i\in I}$.

Next, let u be any upper bound of $(br_i)_{i\in I}$, i.e., $br_i \leqq u$, which, in view of (4), implies that for every $i \in I$,

$$br_i u - b^2 r_i^2 + r_i^2 = r_i^2.$$

But then from (12) it follows that for every $i \in I$

$$r_i(bu - b^2 \sup_i r_i + \sup_i r_i) = r_i^2$$

and therefore

$$r_i \leqq bu - b^2 \sup_i r_i + \sup_i r_i$$

which implies

$$\sup_i r_i \leqq bu - b^2 \sup_i r_i + \sup_i r_i.$$

But then from (4) it follows that

$$(\sup_i r_i)(bu - b^2 \sup_i r_i + \sup_i r_i) = (\sup_i r_i)^2$$

which yields

$$(b \sup_i r_i)u = (b \sup_i r_i)^2$$

implying by (4) that $b \sup r_i \leqq u$. Hence $(br_i)_{i\in I}$ has a supremum which is equal to $b \sup r_i$.

DEFINITION 3. *The ring R is called orthogonally complete provided* $\sup S$ *of every orthogonal subset S of R exists.*

Finally, we prove:

THEOREM. *The ring R is isomorphic to a direct sum of fields if and only if R is atomic and orthogonally complete.*

PROOF. Let β be an isomorphism from R onto a direct sum $\oplus_{i \in I} K_i$ of fields K_i. Let r be a nonzero element of R and let $\beta(r) = (r_i)_{i \in I}$. Without loss of generality we may assume that $r_1 \neq 0$. Let u_1 be the unit of K_1. But then $a = r\beta^{-1}((k_i)_{i \in I})$ with $k_1 = u_1$ and $k_i = 0$ for $i \neq 1$ is obviously an atom of R such that $a \leqq r$. Thus, R is atomic. Next, let S be an orthogonal subset of R and let $\beta[S] = ((k_i(s))_{i \in I})_{s \in S}$. But then, in view of the orthogonality of S, clearly, $\beta^{-1}((k_i)_{i \in I}) = \sup S$ where $k_i = k_i(s)$ if $k_i(s) \neq 0$ for some $s \in S$, and, otherwise $k_i = 0$. Thus, R is orthogonally complete.

Conversely, we show that if R is atomic and orthogonally complete then R is isomorphic to the direct sum $\oplus_{i \in I} F_i$ of fields F_i mentioned in Lemma 7. To this end we show that the isomorphism α mentioned in Lemma 7 is an onto mapping. Let $(r_i e_i)_{i \in I}$ be an element of $\oplus_{i \in I} F_i$. From Lemma 3 it follows readily that $(r_i e_i)_{i \in I}$ is an orthogonal subset of R. Let $h = \sup_i r_i e_i$. But then from (13) and Lemma 3 it follows that $he_j = e_j \sup_i r_i e_i = r_j e_j$ for every $j \in I$. Hence $(he_j)_{j \in I} = (r_i e_i)_{i \in I}$. However, from (11) it follows that $\alpha(h) = (he_j)_{j \in I} = (r_i e_i)_{i \in I}$. Thus, $(r_i e_i)_{i \in I}$ is in the range of α and therefore α is an onto mapping, as desired.

Reference

[1] N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Coll. Publ. Vol. 37 (1956), p. 217.

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