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GENERALIZED BOOLEAN LATTICES

Dedicated to the memory of Hanna Neumann

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1. Introduction

Hashimoto (1952; Theorems 8.3 and 8.5) proved the following theorems:

THEOREM A. *If L is a distributive lattice, then there exists a generalized Boolean algebra L_r and an isomorphism from the lattice of all congruence relations of L onto the lattice of all congruence relations of L_r.*

THEOREM B. *Any distributive lattice L is isomorphic with a sublattice of a relatively comolemented distributive lattice L_r^* such that* (1) *the lattice of congruence relations on L_r^* is isomorphic with that on L and* (2) *the length of the closed interval $[a, b]$ in L is equal to that of $[a, b]$ regarded as an interval in L_r^*.*

It has been noted that Hashimoto's proofs are somewhat difficult to follow and are proved with the apparatus of topology; hence these purely lattice theoretic theorems are not placed in their most natural setting. In 1958 Grätzer and Schmidt (1958; Theorem 1) asserted the following generalization of the Hashimoto theorems:

To any distributive lattice L there exists a generalized Boolean algebra B having the properties:

(1) L is a sublattice of B;

(2) the lattice of all congruence relations of L is isomorphic to the lattice of all congruence relations of B;

(3) if the interval $[a, b]$ of L is of finite length, then $[a, b]$ has the same length as an interval of B.

In this note we give a counterexample to (2). In Section 2 we construct an ideal E of B and prove that the lattice of all congruence relations of L is isomorphic to the lattice of all congruence relations of E (Corollary 2.11). We prove that $E = B$ if and only if $0 \in L$ (Theorem 2.4); otherwise, E is a maximal ideal of B (Corollary 2.5). Our example shows that in general L cannot be embedded into E. The construction of E is algebraic and, moreover, we prove that E is unique up to isomorphism (Corollary 2.12). Thus we strengthen Theorem A.

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Our example also shows that Grätzer (1971; Lemma 5, page 104)) is incorrect (Lemma 5 is in essence (2) above). We prove in Section 3 (also, see Corollary 4.3) that a necessary and sufficient condition that this lemma be true is that the lattice L has a smallest element.

In Section 4 we investigate distributive lattices K and L, where L is a sublattice of K and each congruence of L has a unique extension to K. In this case we prove that there is a generalized Boolean lattice B that is R-generated by L and contains K as a sublattice (Theorem 4.5). From this result we obtain several interesting corollaries; one of which asserts that the lattice L_r^* of Theorem B is unique up to isomorphism.

In Section 5 we give the example.

Throughout this note L will denote a distributive lattice, $\mathscr{C}(L)$ will denote the lattice of all congruence relations on L, $\mathscr{I}(L)$ will denote the lattice of all ideals of L, $\mathbb{N}$ will denote the set of natural numbers, $\mathbb{Z}$ will denote the set integers, $\square$ will denote the empty set, and $X \backslash Y$ will denote the set of elements that belong to the set X but not to the set Y. Unless otherwise stated, isomorphism will mean a homomorphism that is one-to-one (not necessarily onto). For the standard results and definitions concerning lattices, the reader is referred to Grätzer (1971) in particular, to Sections 9 and 10 of Chapter 2.

2. Evenly generated ideals

Throughout this section let B be a generalized Boolean lattice and let L be a sublattice of B that generates B, that is, the smallest subring of B that contains L is B.

LEMMA 2.1 (i) $B = \{a_1 + \cdots + a_n \mid n \in \mathbb{N}$ *and* $a_1, \cdots, a_n \in L\}$.

(ii) *If T is a sublattice of L and $x \in B$ such that $x = a_1 + \cdots + a_n$ ($n \in \mathbb{N}$), where $a_1, \cdots, a_n \in T$, then $x = b_1 + \cdots + b_n$, where $b_1, \cdots, b_n \in T$ and $b_1 \leqq \cdots \leqq b_n$.*

The proof of this lemma is similar to the proof of Grätzer (1971; Lemma 3, page 102) and will be omitted.

COROLLARY 2.2. *If T is a sublattice of L and $x \in B$ such that $x = a_1 + \cdots + a_{2n-1}$ ($n \in \mathbb{N}$), where $a_1, \cdots, a_{2n-1} \in T$, then $x \geqq a$ for some a in T.*

PROOF. By the lemma, $x = b_1 + \cdots + b_{2n-1}$, where $b_1, \cdots, b_{2n-1} \in T$ and $b_1 \leqq \cdots \leqq b_{2n-1}$. Thus $x \geqq x \wedge b_1 = xb_1 = b_1$, since $2n-1$ is odd and $b_1 b_j = b_1$ for each j.

LEMMA 2.3. *Let T be an ideal of L and $E_T = \{x \mid x \in B$ and there exists $a_1, \cdots, a_{2n} \in T$ ($n \in \mathbb{N}$) such that $x = a_1 + \cdots + a_{2n}\}$. Then*

(i) *E_T is an ideal of B;*

(ii) *$E_T = E_L \cap (T]_B$, where $(T]_B$ denotes the ideal of B generated by T;*

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(iii) $E_T = \{x \mid x \in B$ *and there exists* $a_1, \cdots, a_{2n} \in T$ *such that* $a_1 \leqq \cdots \leqq a_{2n}$ *and* $x = a_1 + \cdots + a_{2n}\}$.

The proof of this lemma is straightforward and will be omitted. We shall call the ideal E_T of this lemma the *ideal of B evenly generated by T*. Also for the remainder of this note we shall denote E_L simply by E.

In Corollary 2.2, if $T = L$ and x in B has a representation as a sum of an odd number of elements from L, then x exceeds an element of L. The next theorem shows that if L does not contain the zero of B and x in B has a representation as a sum of an even number of elements of L, then x does not exceed an element of L. Also, if $x = a_1 + \cdots + a_n$, where $a_1, \cdots, a_n \in L$, then $x \in (a_1 \vee \cdots \vee a_n]_B$ and so each element of B is exceeded by an element of L. We say that L *R-generates* B if L generates B and if L has a least element, then it is the zero of B. (The definition of R-generates given in Grätzer (1971; page 102) also required that if L has a largest element, then it must be the largest element of B.) If L does not have a smallest element, then the definitions of R-generates and generates coincide.

THEOREM 2.4. *If L generates B, then the following assertions are equivalent*:

(i) $0 \in L$.

(ii) $L \cap E \neq \square$.

(iii) $L \subseteq E$.

(iv) $E = B$.

PROOF. (i) implies (ii) is immediate as $0 \in L \cap E$.

(ii) implies (iii). Let $a \in L$ and $b \in L \cap E$. Then $a + b \in E$ and hence $a = a + b + b \in E$.

(iii) implies (iv) is obvious as E is an ideal of B.

(iv) implies (i). If $b \in L$, then $b = a_1 + \cdots + a_{2n}$, where $a_1, \cdots, a_{2n} \in L$ and $a_1 \leqq \cdots \leqq a_{2n}$. Then $b \wedge a_1 \in L$ and $b \wedge a_1 = ba_1 = a_1 + \cdots + a_1$, where there are $2n$ summands. Therefore $ba_1 = 0$ and so $0 \in L$.

COROLLARY 2.5. *If $0 \notin L$, then the index of E in B is two and hence E is a maximal ideal of B. Moreover, L is a sublattice of the relatively complemented lattice $B \backslash E$ and if M is an ideal of B such that $M \cap L = \square$, then $M \subseteq E$.*

PROOF. By the theorem, $E \neq B$ and $L \subseteq B \backslash E$. If $x \in B \backslash E$, then $x = a_1 + \cdots + a_{2n-1}$, where $a_1, \cdots, a_{2n-1} \in L$. If $n = 1$, $x = a_1$ and if $n > 1$, $a_1 + \cdots + a_{2n-2} \in E$. Thus $E + x = E + a$ for some $a \in L$. If $a, b \in L$, then $a + b \in E$ and so $E + a = E + b$. Therefore the index of E in B is two.

Since E is an ideal of B and B is a generalized Boolean lattice, $B \backslash E$ is relatively complemented. If $M \in \mathscr{I}(B) \backslash \mathscr{I}(E)$, then $M \cap L \neq \square$ by Corollary 2.2.

LEMMA 2.6. *Let* $\theta \in \mathscr{C}(L)$ *and* I_θ *be the ideal of* B *generated by* $\{a+b \mid a,b \in L$ *and* $a\theta b\}$. *Then* $I_\theta = \{\sum_{i=1}^{n} a_i + b_i \mid n \in \mathbb{N},\ a_i, b_i \in L,\ a_i \leqq b_i,$ *and* $a_i \theta b_i\}$ *and so* $I_\theta \in \mathscr{I}(E)$.

PROOF. For $a, b \in L$, $a\theta b$ if and only if $a \wedge b\ \theta\ a \vee b$. Also, $a+b = ab + a + b + ab = a \wedge b + a \vee b$. Thus, I_θ is the ideal of B generated by $\{a+b \mid a,b \in L,\ a \leqq b,$ and $a\theta b\}$. Let $I = \{\sum_{i=1}^{n} a_i + b_i \mid n \in \mathbb{N},\ a_i, b_i \in L,$ $a_i \leqq b_i$, and $a_i \theta b_i\}$. Then clearly $\{a+b \mid a,b \in L,\ a \leqq b,$ and $a\theta b\} \subseteq I \subseteq I_\theta$ and I is closed with respect to addition. To prove that I is an ideal of B, it suffices to show that for $x \in I$ and $a \in L$, $ax \in I$. Let $x = \sum_{i=1}^{n} a_i + b_i$, where $a_i, b_i \in L$, $a_i \leqq b_i$, and $a_i \theta b_i$. For each i, $aa_i = a \wedge a_i \leqq a \wedge b_i = ab_i$ and $a_i \theta b_i$ implies $a \wedge a_i\ \theta\ a \wedge b_i$. Thus $ax \in I$ and so $I = I_\theta$. Clearly $I_\theta \in \mathscr{I}(E)$.

LEMMA 2.7. *Let* $I \in \mathscr{I}(E)$ *and* $\theta_I = \{(a,b) \mid a,b \in L$ *and* $a+b \in I\}$. *Then* θ_I *is a congruence relation of* L.

PROOF. It is easily verified that θ_I is an equivalence relation of L and if $a\theta b$ and $t \in L$, then $a \wedge t\ \theta\ b \wedge t$. Now $a \vee t + b \vee t = a + t + at + b + t + bt = a + b + (a+b)t \in I$. Therefore $a \vee t\, \theta\, b \vee t$ and so $\theta_I \in \mathscr{C}(L)$.

THEOREM 2.8. *The mapping* g *of* $\mathscr{I}(E)$ *into* $\mathscr{C}(L)$ *given by* $(I)g = \theta_I$ *is an isomorphism of* $\mathscr{I}(E)$ *onto* $\mathscr{C}(L)$. *The inverse of* g *is given by* $(\theta)g^{-1} = I_\theta$.

PROOF. By Lemma 2.7, g is a mapping of $\mathscr{I}(E)$ into $\mathscr{C}(L)$. If $\theta \in \mathscr{C}(L)$, then by Lemma 2.6, $I_\theta \in \mathscr{I}(E)$. We will show that $(I_\theta)g = \theta_{I_\theta} = \theta$. If $a\theta b$, then $a + b \in I_\theta$ and so $a\,\theta_{I_\theta}\, b$. Conversely if $a\theta_{I_\theta} b$, then $a + b \in I_\theta$ and, without loss of generality, we may assume that $a \leqq b$. Now $a + b \in I_\theta$ implies that $a + b = \sum_{i=1}^{n} a_i + b_i$, where $a_i, b_i \in L$, $a_i \leqq b_i$, and $a_i\theta b_i$. We induct on n. If $n = 1$, then $a + b = (a+b)b = a_1 b + b_1 b$, $a_1 b \leqq b_1 b$, and $a_1 b\, \theta\, b_1 b$. Since $b_1 b \leqq b$, we may assume that $a + b = a_1 + b_1$, where $a_1\, \theta\, b_1$ and $a_1 \leqq b_1 \leqq b$. Therefore $0 = a_1 + a_1 = (a_1 + b_1)a_1 = aa_1 + ba_1 = aa_1 + a_1$. Hence $a_1 \leqq a$. Now $a + b = a_1 + b_1 \leqq b_1$, as $a_1 \leqq b_1$, and so $n = a \vee (a+b) \leqq a \vee b_1 \leqq a \vee b = b$. Whence $a \vee b_1 = b$ and $a \vee a_1 = a$. Thus $a_1 \theta b_1$ implies $a \vee a_1\, \theta\, a \vee b_1$ or $a\theta b$. Next assume that $n > 1$. Again $a + b = \sum_{i=1}^{n} a_i b + b_i b$ and so we may assume that $a_i \leqq b_i \leqq b$ for each i. Let $d = \bigvee_{i=1}^{n} b_i$. For $1 \leqq k \leqq n$, $aa_k + a_k = (a+b)a_k = \sum_{i=1, i \neq k}^{n} a_i a_k + b_i a_k$. Now $a\ \theta_{I_\theta} b$ implies that $aa_k\, \theta_{I_\theta}\, a_k$ and hence by the inductive hypothesis $aa_k\, \theta\, a_k$. Since $a_k\, \theta\, b_k$, we have $aa_k\, \theta\, b_k$ and so $a\, \theta\, a \vee b_k$. Also $ad + d = (a+b)d = \sum_{i=1}^{n} a_i d + b_i d = \sum_{i=1}^{n} a_i + b_i = a + b$. Hence $a \vee d = a + d + ad = b$. Thus we have $a\theta \bigvee_{i=1}^{n} (a \vee b_i)$, which is equivalent to $a\theta b$. Therefore g maps $\mathscr{I}(E)$ onto $\mathscr{C}(L)$.

Let $I, J \in \mathscr{I}(E)$ with $I \neq J$ and let $x \in J \backslash I$. Then there exists $a_1, \cdots, a_{2n} \in L$ with $a_1 \leqq \cdots \leqq a_{2n}$ and $x = a_1 + \cdots + a_{2n}$. Multiplying x respectively by $a_2, \cdots, a_{2n-2}$, we obtain $a_1 + a_2, \cdots, a_{2n-1} + a_{2n} \in J$. Hence for some k, $(1 \leqq k \leqq n)$ $a_{2k-1} + a_{2k} \in J \backslash I$. Therefore $(a_{2k-1}, a_{2k}) \in \theta_J \backslash \theta_I$ and so g is one-to-one.

If $I, J \in \mathscr{I}(E)$, then clearly $I \subseteq J$ if and only if $\theta_I \subseteq \theta_J$. Therefore g is an isomorphism of $\mathscr{I}(E)$ onto $\mathscr{C}(L)$. It is now clear that $(\theta)g^{-1} = I_\theta$.

An immediate consequence of this theorem is

COROLLARY 2.9. (i) *If* $\theta \in \mathscr{C}(L)$, *then* $\theta = \theta_{I_\theta}$.

(ii) *If* $I \in \mathscr{I}(E)$, *then* $I = I_{\theta_I}$.

COROLLARY 2.10. (Hashimoto, (1972; Theorem 7.2) $\mathscr{I}(E)$ *is isomorphic to* $\mathscr{C}(E)$.

PROOF. Since E is a generalized Boolean lattice and E generates itself, we have by the theorem that $\mathscr{I}(E)$ is isomorphic to $\mathscr{C}(E)$.

COROLLARY 2.11. $\mathscr{C}(L)$ *is isomorphic to* $\mathscr{C}(E)$.

COROLLARY 2.12. *If D is a generalized Boolean lattice such that* $\mathscr{C}(L)$ *is isomorphic to* $\mathscr{C}(D)$, *then D is isomorphic to E.*

PROOF. As noted above, $\mathscr{C}(D)$ is isomorphic to $\mathscr{I}(D)$. Hence $\mathscr{I}(D)$ and $\mathscr{I}(E)$ are isomorphic. The compact elements of $\mathscr{I}(E)$ are the principal ideals of E, which are isomorphic to E. Since compact elements are preserved under isomorphism, it follows that E is isomorphic to D.

If $T \in \mathscr{I}(L)$, let $\theta_T = \{(a, b) \mid a, b \in L \text{ and } a \vee t = b \vee t \text{ for some } t \in T\}$. It is easily verified that $\theta_T = \{(a, b) \mid a, b \in L \text{ and } a \vee b = (a \wedge b) \vee t \text{ for some } t \in T\}$ and that the mapping that sends T into θ_T is an isomorphism of $\mathscr{I}(L)$ onto a sublattice of $\mathscr{C}(L)$ (Hashimoto (1973; Theorem 5.1)). Moreover, for each $t \in T$, $[t]\theta_T = T$, where $[t]\theta_T$ denotes the congruence class of θ_T containing t.

COROLLARY 2.13. *Let h be the mapping of* $\mathscr{I}(L)$ *into* $\mathscr{I}(E)$ *given by* $(T)h = E_T$. *Then h is an isomorphism of* $\mathscr{I}(L)$ *onto a sublattice of* $\mathscr{I}(E)$.

PROOF. It suffices to show that $E_T = I_{\theta_T}$. If $s, t \in T$, then $s\theta_T t$ and so $s + t \in I_{\theta_T}$. It follows that $E_T = I_{\theta_T}$. Conversely, let $a, b \in L$ with $a\theta_T b$ and $a + b \in I_{\theta_T}$. Then $a \vee b = (a \wedge b) \vee t$ for some t in T and so $a + b + ab = ab + t + abt$. Therefore $a + b = t + abt$ and since t, $abt \in T$, we have $a + b \in E_T$. Again it follows that $I_{\theta_T} \subseteq E_T$.

We have now obtained the Hashimoto theorems mentioned in the introduction. For let L be a distributive lattice and B be a generalized Boolean lattice R-generated by L. Then $E = E_L$ is a generalized Boolean lattice and $\mathscr{C}(E)$ is isomorphic to $\mathscr{C}(L)$ (Corollary 2.11). Moreover, we have proven that, up to isomorphism, E is unique (Corollary 2.12). If L has a smallest element, then $E = B$ (Theorem 2.4). If L does not have a smallest element, then $L \subseteq B \backslash E$ (Theorem 2.4). Let $K = B \backslash L$. Then K is a prime dual ideal of B since E is a prime ideal of B (Corollary 2.5). Thus K is a relatively complemented distributive lattice, L is a sublattice of K, and K does not have a smallest element (Corollary 2.2).

If we let $E_K = \{c_1 + \cdots + c_{2n} \mid c_1, \cdots, c_{2n} \in K\}$, then it is easily verified that $E_K = E$. Since K generates B, $\mathscr{C}(K)$ is isomorphic to $\mathscr{C}(E_K)$ (Corollary 2.11) and so $\mathscr{C}(K)$ is isomorphic to $\mathscr{C}(L)$. We will prove in Section 4 that, up to isomorphism, K is unique (see Corollary 4.9).

COROLLARY 2.14. *Let F be a maximal ideal of B and $K = B\backslash F$. Then $F = \{x \mid x \in B$ and there exists $c_1, \cdots, c_{2n} \in K (n \in \mathbb{N})$ such that $x = c_1 + \cdots + c_{2n}\}$. Hence K generates B and $\mathscr{C}(K)$ is isomorphic to $\mathscr{C}(F)$.*

PROOF. Let $E_K = \{c_1 + \cdots + c_{2n} \mid c_1, \cdots, c_{2n} \in K\}$. Since F is maximal, K is a dual ideal of B and hence a sublattice of B. Let $c, d \in K$. Since $c + d = c \wedge d + c \vee d$, we may assume that $c \leqq d$. Thus $c \wedge (c + d) = 0$. Thus $c + d \in F$, as F is prime, and it follows that $E_K \subseteq F$. If $x \in F$ and $c \in K$, $x + c \in F + c \subseteq K$ and so $x \in K + c \subseteq E_K$. Therefore $F = E_K$ and K generates B. Since E_K is the ideal of B evenly generated by K we have by Corollary 2.11 that $\mathscr{C}(K)$ is isomorphic to $\mathscr{C}(F)$.

Observe that K is a relatively complemented lattice, but in general does not have a smallest element. Hence in general $\mathscr{I}(K)$ and $\mathscr{C}(K)$ are not isomorphic. To this end we prove

THEOREM 2.15. *Let F be a maximal ideal of B and $K = B\backslash F$. Then the following are equivalent:*

(i) *K is a generalized Boolean lattice.*
(ii) *K has a smallest element.*
(iii) *K is isomorphic to F.*
(iv) *There exists $c \in K$ such that $cF = \{0\}$.*
(v) *There exists an atom of B in K.*
(vi) *F is a direct summand of B.*

PROOF. (i) implies (ii) is trivial.

(ii) implies (iii). As noted above, K is always a relatively complemented lattice and so by (ii), K is a generalized Boolean lattice. Thus by Corollary 2.12, K is isomorphic to F.

(iii) implies (iv). If K is ismorphic to F, K has a least element c_0. Now F is generated by $\{c + d \mid c, d \in K$ and $c \leqq d\}$. For $c, d \in K$ with $d \geqq c$, $c \geqq c_0$ and so $0 \leqq c_0 \wedge (c + d) \leqq c \wedge (c + d) = 0$. It follows that $c_0 F = \{0\}$.

(iv) implies (v). Let $c \in K$ such that $cF = \{0\}$. Let $x \in B$ with $0 \leqq x \leqq c$. If $x \in F$, then $0 = cx = x$. If $x \in K$, then $c + x \in F$ and $0 = (c + x)c = c + x$. Therefore $x = c$ and hence c is an atom of B.

(v) implies (vi). Let c be an atom of B in K. Then $(c]_B \cap F = \{0\}$. Since F is a maximal ideal of B, $F + (c]_B = B$ and so F is a direct summand of B.

(vi) implies (i). Suppose that B is the direct sum of F and M, where M is an ideal of B. Since the index of F in B is two, it follows that $M = \{0, c\}$ for some $c \in K$. If $x \in K$, then $x \wedge c = c$ or $x \wedge c = 0$. Since F is prime, $x \wedge c \neq 0$. Therefore c is the smallest element of K and hence K is a generalized Boolean lattice.

In concluding this section, we note that Theorem 2.15 exemplifies a reason for requiring in the definition of L R-generates B that if L has a smallest element, it must be the zero of B.

3. Extending congruences

Throughout this section let L denote a lattice without a smallest element, L_0 denote the lattice L with a smallest element 0 adjoined, and let $\mathscr{C}_0(L) = \{\theta \mid \theta \in \mathscr{C}(L)$ and L/θ has a least element$\}$. J. Hashimoto (1973; Theorem 5.1) proved that $\mathscr{C}_0(L)$ is a dual ideal of $\mathscr{C}(L)$. The hypothesis that L is distributive is not needed for the proofs of Lemma 3.1, Corollary 3.2, and Theorems 3.3 and 3.4.

LEMMA 3.1. *If θ is a congruence relation of L, then θ can be extended to a congruence relation of L_0. Moreover,*

(i) *if $\theta \notin \mathscr{C}_0(L)$, then θ has a unique extension to L_0;*

(ii) *if $\theta \in \mathscr{C}_0(L)$, then θ has exactly two extensions to L_0.*

PROOF. If $\theta_0 = \theta \cup \{(0,0)\}$, then θ_0 is a congruence relation of L_0 that extends θ. If $\theta \notin \mathscr{C}_0(L)$, then it is easily verified that θ_0 is the only extension of θ. If $\theta \in \mathscr{C}_0(L)$ and T is the smallest element of L/θ, then let $\theta_1 = \theta \cup (T \times \{0\}) \cup (\{0\} \times T) \cup \{(0,0)\}$. Then θ_1 is a congruence relation of L_0 that extends θ and $\theta_1 > \theta_0$. Again it is easily verified that these are the only extensions of θ in $\mathscr{C}(L_0)$.

COROLLARY 3.2. *A congruence relation θ of L has a unique extension to a congruence relation of L_0 if and only if $\theta \notin \mathscr{C}(L_0)$.*

If T is an ideal of L, we observed in Section 2 that $\theta_T \in \mathscr{C}(L)$ and trivially T is the least element of L/θ_T. Thus by the theorem, θ_T has two extensions to L_0. It is evident that if B is a generalized Boolean lattice R-generated by L_0, then B is also R-generated by L. We note that Lemma 5 (Grätzer (1971; page 104)) is valid if the lattice has a smallest element and hence each congruence of L_0 has a unique extension to B, but Lemma 3.1 shows that each element of $\mathscr{C}_0(L)$ has two extensions to B. Since $\mathscr{C}(L)$ is isomorphic to $\mathscr{C}(E)$ and $\omega(L_0)$ is isomorphic to $\mathscr{C}(B)$, this raises the following question: which congruences of E have two extensions to B? (By Grätzer (1971; Theorem 6, page 90) each congruence of E has at least one extension to B.) In this section we give an answer to this question.

Let f_0 and f_1 be the mappings of $\mathscr{C}(L)$ into $\mathscr{C}(L_0)$ given by

$$\theta f_0 = \theta_0$$

and

$$\theta f_1 = \begin{cases} \theta_1 & \text{if } \theta \in \mathscr{C}_0(L) \\ \theta f_0 & \text{otherwise,} \end{cases}$$

where $\theta \in \mathscr{C}(L)$ and θ_0 and θ_1 are given above.

THEOREM 3.3. (i) f_0 *is an isomorphism of* $\mathscr{C}(L)$ *into* $\mathscr{C}(L_0)$.

(ii) f_1 *is a one-to-one inclusion preserving mapping of* $\mathscr{C}(L)$ *into* $\mathscr{C}(L_0)$ *and* $f_1 \mid \mathscr{C}_0(L)$ *is an isomorphism.*

PROOF. The verification of (i) and that f_1 is a one-to-one inclusion preserving mapping is straightforward and will be omitted. Let θ, $\Psi \in \mathscr{C}_0(L)$ and let S and T denote the smallest elements of θ and Ψ respectively. Then $S \cap T$ is the smallest element of $\theta \wedge \Psi$ (Hashimoto (1952; Lemma 5.1)). Then

$$\begin{aligned} \theta_1 \wedge \Psi_1 &= (\theta \cup (S \times \{0\}) \cup (\{0\} \times S) \cup \{(0,0)\} \cap (\Psi \cup (T \times \{0\}) \cup (\{0\} \times T) \\ &\qquad \cup \{(0,0)\}) \\ &= (\theta \cap \Psi) \cup ((S \cap T) \times \{0\}) \cup (\{0\} \times (S \cap T)) \cup \{(0,0)\} \\ &= (\theta \wedge \Psi)_1 . \end{aligned}$$

Since f_1 is inclusion preserving $\theta_1 \vee \Psi_1 \subseteq (\theta \vee \Psi)_1$. Let $(a,b) \in (\theta \vee \Psi)_1$ and U be the least element of $L/(\theta \vee \Psi)$. If $(a,b) \in \theta \vee \Psi$, then clearly $(a,b) \in \theta_1 \vee \Psi_1$. If $(a,b) \in (U \times \{0\}) \cup (\{0\} \times U)$, then, since $(\theta \vee \Psi)_1$ is symmetric, we assume that $a \in U$ and $b = 0$. Now $a \in U$ implies that there exists $b_0, \cdots, b_m \in L$ such that $a = b_0$, $b_m \in T$, and $(b_i, b_{i+1}) \in \theta \cup \Psi$. Hence $(a, b_m) \in \theta_1 \vee \Psi_1$. Since $T \subseteq U$ (Hashimoto (1952; Lemma 5.1)). $(b_m, 0) \in \Psi_1$. Therefore $(a, 0) \in \theta_1 \vee \Psi_1$. Trivially, if $a = b = 0$, $(a,b) \in \theta_1 \vee \Psi_1$.

We now describe the lattice $\mathscr{C}(L_0)$ in terms of $\mathscr{C}(L)$.

THEOREM 3.4. $\mathscr{C}(L_0)$ *is the disjoint union of* $(\mathscr{C}(L))f_0$ *and* $(\mathscr{C}_0(L))f_1$. *Moreover,* $(\mathscr{C}_0(L))f_1$ *is a prime dual ideal of* $\mathscr{C}(L_0)$ *and for* $\theta \in \mathscr{C}(_L)$ *and* $\Psi \in \mathscr{C}_0(L)$,

$$\theta_0 \wedge \Psi_1 = (\theta \wedge \Psi)_0$$

and

$$\theta_0 \vee \Psi_1 = (\theta \vee \Psi)_1 .$$

PROOF. Using Lemma 3.1, Theorem 3.3, and the fact that L is a sublattice of L_0, it is easily verified that $\mathscr{C}(L_0)$ is the disjoint union of $(\mathscr{C}(L)) f_0$ and $(\mathscr{C}_0(L))f_1$ and that $(\mathscr{C}_0(L))f_1$ is a prime dual ideal of $\mathscr{C}(L_0)$.

Let $\theta \in \mathscr{C}(L)$, $\Psi \in \mathscr{C}_0(L)$, and T be the least element of L/Ψ. Then

$$\begin{aligned} \theta_0 \wedge \Psi_1 &= (\theta \cup \{(0,0)\}) \cap (\Psi \cup (T \times \{0\}) \cup (\{0\} \times T) \cup \{(0,0)\}) \\ &= (\theta \cap \Psi) \cup \{(0,0)\} \\ &= (\theta \wedge \Psi)_0 . \end{aligned}$$

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Since f_1 is inclusion preserving $\theta_0 \subseteq (\theta \vee \Psi)_1$ and $\Psi_1 \subseteq (\theta \vee \Psi)_1$. Therefore $\theta_0 \vee \Psi_1 \subseteq (\theta \vee \Psi)_1$. Let $(a,b) \in (\theta \vee \Psi)_1$ and let U be the least element of $L/(\theta \vee \Psi)$. If $(a,b) \in \theta \vee \Psi$, then clearly $(a,b) \in \theta_0 \vee \Psi_1$. Suppose that $(a,b) \in U \times \{0\}$. Then there exists $b_0, \cdots, b_m \in L$ such that $a = b_0$, $b_m \in T$, and $(b_i, b_{i+1}) \in \theta \cup \Psi$. As above, $(a, b_m) \in \theta_0 \vee \Psi_1$ and $(b_m, 0) \in \Psi_1$. Hence $(a,0) \in \theta_0 \vee \Psi_1$. It follows that $(\theta \vee \Psi)_1 \subseteq \theta_0 \vee \Psi_1$.

For the remainder of this section, let B be a generalized Boolean lattice R-generated by L and E be the ideal of B evenly generated by L. Then by Corollary 2.5, E is a maximal ideal of B. Obviously B is also R-generated by L_0 and by Theorem 2.4 the ideal of B evenly generated by L_0 is B. Let p be the isomorphism of $\mathscr{C}(L)$ onto $\mathscr{I}(E)$ given by $(\theta)p = I_\theta$ (p is the inverse of the isomorphism g of Theorem 2.8), let p_0 be the corresponding isomorphism of $\mathscr{C}(L_0)$ onto $\mathscr{I}(B)$, let ι be the inclusion mapping of $\mathscr{I}(E)$ into $\mathscr{I}(B)$, and let f_0 be as above.

LEMMA 3.5. *If* $\theta \in \mathscr{C}(L)$, *then* $(\theta)p\iota = (\theta)f_0 p_0$.

The proof of this lemma is routine and hence will be omitted.

LEMMA 3.6. *Let F be a maximal ideal of B and I be an ideal of F. Then there are at most two ideals of B whose intersections with F is I.*

PROOF. Let $M, N \in \mathscr{I}(B) \backslash \mathscr{I}(F)$ such that $M \cap F = N \cap F = I$. Since F is maximal, we have $M = M \wedge B = M \wedge (F \vee N) = (M \wedge F) \vee (M \wedge N) = I \vee (M \wedge N) \subseteq N$. Dually $N \subseteq M$. Trivially I is the only ideal of F whose intersection with F is I.

THEOREM 3.7. *Let I be an ideal of E. Then there exists $M \in \mathscr{I}(B) \backslash \mathscr{I}(E)$ such that $M \cap E = I$ if and only if $(I)p^{-1} \in \mathscr{C}_0(L)$.*

PROOF. Let $M \in \mathscr{I}(B) \backslash \mathscr{I}(E)$ such that $M \cap E = I$, $T = M \cap L$, and $\theta = Ip^{-1}$. Then $T \in \mathscr{I}(L)$ and to prove that $\theta \in \mathscr{C}_0(L)$, it suffices to show that $T \in L/\theta$. If $a, b \in T$, then $a + b \in M \cap E$ and hence by Lemma 2.7, $a\theta b$. Conversely let $a \in T$ and $b \in [a]\theta$. Then $a + b \in I \subseteq M$. Thus we have $b = a + a + b \in M$. Therefore $b \in M \cap L = T$ and so $T \in L/\theta$. Since $T \in \mathscr{I}(L)$, it is the smallest element of L/θ. Hence $\theta \in \mathscr{C}_0(L)$.

Next suppose that $\theta = Ip^{-1} \in \mathscr{C}_0(L)$. Then $\theta_0 = \theta f_1 \cap ((L \times L) \cup \{(0,0)\}) = \theta f_1 \cap (L \times L) f_0$. By Theorem 3.4, $\theta f_1 \cap (L \times L) f_0 = \theta f_0$ and by Lemma 3.5, $I = \theta p\iota = \theta f_0 g_0 = \theta f_1 g_0 \cap (L \times L) f_0 g_0 = \theta f_1 g_0 \cap E$. Now $\theta f_1 g_0 \in \mathscr{I}(B)$ and if T is the smallest element of L/θ, then $T \subseteq \theta f_1 g_0$. Therefore $\theta f_1 g_0 \notin \mathscr{I}(E)$.

It follows from this theorem that a congruence of E which has exactly two extensions to B is induced (see the discussion after Corollary 2.12) by an element from $(\mathscr{C}_0(L))p$.

4. Extension Property

Let K be a distributive lattice. We say that K has the *extension property* (EP) over L if L is a sublattice of K and each congruence of L can be uniquely extended to a congruence of K. If L is a sublattice of K, then K has (EP) over L if and only if the mapping of $\mathscr{C}(K)$ into $\mathscr{C}(L)$, which sends χ onto $\chi \cap (L \times L)$, is one-to-one. (It is well known that this mapping is onto (Grätzer (1971; Theorem 6, page 90)). The next lemma is immediate from the definition.

LEMMA 4.1. *Let K be a distributive lattice and M and L be sublattices of K such that $L \subseteq M$. Then K has (EP) over L if and only if K has (EP) over M and M has (EP) over L.*

THEOREM 4.2. *Let B be a generalized Boolean lattice R-generated by L.*

(i) *If $0 \in L$ and K is a sublattice of B that contains L, then K has (EP) over L.*

(ii) *If $0 \notin L$ and K is a sublattice of $B \backslash E$ that contains L, then K has (EP) over L.*

PROOF. Let K be as either in (i) or (ii), $\theta \in \mathscr{C}(L)$ and $\chi \in \mathscr{C}(K)$ such that χ is an extension of θ. Trivially, K R-generates B and $E = \{c_1 + \cdots + c_{2n} \mid c_1, \cdots, c_{2n} \in K\}$, where E is the ideal of B evenly generated by L. Let I_θ and I_χ be the ideals of E generated by $\{a + b \mid a, b \in L \text{ and } a\theta b\}$ and $\{c + d \mid c, d \in K \text{ and } c\chi d\}$ respectively. Clearly $I_\theta \subseteq I_\chi$. Suppose (by way of contradiction) that $I_\theta \neq I_\chi$. Let $\Psi \in \mathscr{C}(L)$ such that $(\Psi)g^{-1} = I_\chi$, where g^{-1} is the isomorphism given in Theorem 2.8. Since $I_\chi \supset I_\theta$, we have by Lemma 2.6 that there exists $a, b \in L$ such that $a + b \in I_\chi \backslash I_\theta$ and $(a, b) \in \Psi \backslash \theta$. Now $a + b \in I_\chi$ implies $(a, b) \in \chi$ and hence $(a, b) \in \theta$, a contradiction. Therefore $I_\chi = I_\theta$. Since K generates B and E is the ideal of B evenly generated by K, we have by Theorem 2.8 that χ is unique.

COROLLARY 4.3. *Let B be a generalized Boolean lattice R-generated by L.*

(i) *If $0 \in L$, then B has (EP) over L.*

(ii) *If $0 \notin L$, then $B \backslash E$ has (EP) over L.*

Note that in the preceding corollary, if $0 \notin L$, then B/E is a relatively complemented lattice without a smallest element. Also, it gives a corrected version of Grätzer (1971; Lemma 5, page 104).

LEMMA 4.4. *Let K be a distributive lattice that has (EP) over L.*

(i) *If a_0 is the smallest element of L, then a_0 is the smallest element of K.*

(ii) *If c_0 is the smallest element of K, then $c_0 \in L$.*

PROOF. (i) Let $e \in K$ and $\chi = \{(c, d) \mid c, d \in K,\ c \wedge e \wedge a_0 = d \wedge e \wedge a_0,$ and $c \vee a_0 = d \vee a_0\}$. It is readily verified that $\chi \in \mathscr{C}(K)$ and $(e \wedge a_0, a_0) \in \chi$. If $a, b \in L$ and $a\chi b$, then $a = b$ as a_0 is zero the of L. Therefore $\chi \cap (L \times L) = \{(a, a) \mid a \in L\}$ and since K has (EP) over L, it follows that $\chi = \{(c, c) \mid c \in K\}$. Hence $e \wedge a_0 = a_0$.

(ii) If $c_0 \notin L$, then by (i), L does not have a smallest element. If D is the dual ideal of K generated by L, then $D = \{c \mid c \geqq a \text{ for some } a \in L\}$ and $D \cap \{c_0\} = \square$.

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By Grätzer (1971; Theorem 15, page 75) there exists a prime ideal P of K such that $c_0 \in P$ and $P \cap D = \square$. Let $\chi = \{(c, d \mid c, d \in P$ or $c, d \in K/P\}$. Then $\chi \in \mathscr{C}(K)$ and $K/\chi = \{P, K \backslash P\}$. Moreover, $\chi \cap (L \times L) = L \times L = (K \times K) \cap (L \times L)$. Thus K does not have (EP) over L.

We now prove a converse of Theorem 4.2.

THEOREM 4.5. *Let K be a distributive lattice that has (EP) over L. Then there exists a generalized Boolean lattice B that is R-generated by L and such that K is a sublattice of B. Moreover, if L does not have a smallest element, then $K \subseteq B \backslash E$, where E is the ideal of B evenly generated by L.*

PROOF. Let C be a generalized Boolean lattice R-generated by K and let $D = \{x \mid x \in C$ and there exists $a_1, \cdots, a_{2n} \in L$ such that $x = a_1 + \cdots + a_{2n}\}$. Since L is a sublattice of C, D is a subring of C. Let $c, d \in K$, $\chi_1 = \{(x, y) \mid x, y \in K$ and $x + y \leqq c + d\}$, and $\chi_2 = \{(x, y) \mid x, y \in K$ and $x + y \leqq e \leqq c + d$ for some $e \in D\}$. Clearly χ_1 is reflexive and symmetric. If (x, y), $(y, z) \in \chi_1$, then $x + y \leqq c + d$ and $y + z \leqq c + d$. Hence $(x + y) \vee (y + z) \leqq c + d$. Now $x + z = (x + y) + (y + z)$ is the relative complement of $(x + y) \wedge (y + z)$ in $[0, (x + y) \vee (y + z)]$ and hence $x + z \leqq c + d$. Therefore χ_1 is an equivalence relation on K. Let $(x, y) \in \chi_1$ and $z \in K$. Then $xz + yz = (x + y) \wedge z \leqq x + y \leqq c + d$ and so $(x \wedge z, y \wedge z) \in \chi_1$. Now $x \wedge z + y \vee z = x + z + xz + y + z + yz = (x + y) + (x + y)z \leqq (x + y) \vee ((x + y)z) \leqq x + y \leqq c + d$. Therefore, $(x \vee z, y \vee z) \in \chi_1$ and so $\chi_1 \in \mathscr{C}(K)$. A similar argument yields that $\chi_2 \in \mathscr{C}(K)$ and obviously $\chi_2 \subseteq \chi_1$. If $(a, b) \in \chi_1 \cap (L \times L)$, then $a + b \leqq c + d$ and $a + b \in D$. Therefore $(a, b) \in \chi_2$. Since K has (EP) over L, it follows that $\chi_1 = \chi_2$. Since $(c, d) \in \chi_1$, we have $c + d \leqq e \leqq c + d$ for some $e \in D$ so $c + d \in D$. It follows that D is the ideal of C evenly generated by K.

If K has a smallest element c_0, then by Lemma 4.4, $c_0 \in L$ and c_0 is the zero of C. Thus, since $0 \in K$, we have by Theorem 2.4 that $D = C$. Therefore C is a generalized Boolean lattice R-generated by L and D is the ideal evenly generated by L.

If K does not have a smallest element, then by Lemma 4.4, L does not have a smallest element. Since K generates C and $0 \notin K$, we have by Corollary 2.5 that D is a maximal ideal of index two in C and $K \cap D = \square$. Thus if $a \in L$, $D + a = C \backslash D$. Again we have that C is a generalized Boolean lattice R-generated by L and D is the ideal evenly generated by L.

Finally, if L does not have a smallest element, then K does not have a smallest element and so $K \subseteq C \backslash D$.

As a corollary to the proof of this theorem we have

COROLLARY 4.6. *Let K be a distributive lattice that has (EP) over L. If C is a generalized Boolean lattice R-generated by K, then C is R-generated by L.*

COROLLARY 4.7. *If K is a relatively complemented distributive lattice that has (EP) over L, then there exists a generalized Boolean lattice B that is R-generated by L and such that $K = B$ or $K = B\backslash E$.*

PROOF. By the theorem there is a generalized Boolean lattice R-generated by L such that K is a sublattice of B. If $0 \in L$, then $0 \in K$ and so K is a generalized Boolean lattice. Let $a, b \in L$ and let c be the relative complement of $a \wedge b$ in the interval $[0, a \vee b]_K$ of K. Then $c = a + b$ and it follows that $E \subseteq K$. By Theorem 2.5, $E = B$.

If $0 \notin L$, then by the theorem $K \subseteq B\backslash E$. Let $x \in B\backslash E$. Then $x = a_1 + \cdots + a_{2n-1}$ where $a_1, \cdots, a_{2n-1} \in L$ with $a_1 \leqq \cdots \leqq a_{2n-1}$. If $n = 1$, then $a_1 \in K$. Suppose that $n > 1$ and that $a_2 + \cdots + a_{2n-2} \in K$. Now $a_1 \leqq a_2 \leqq a_2 + \cdots + a_{2n-2} \leqq a_{2n-2} \leqq a_{2n-1}$. Let c be the relative complement of $a_2 + \cdots + a_{2n-2}$ in the interval $[a_1, a_{2n-1}]_K$ of K. Then $a_1 = c \wedge (a_2 + \cdots + a_{2n-2})$ and $a_{2n-1} = c \vee (a_2 + \cdots + a_{2n-2}) = c + a_2 + \cdots + a_{2n-2} + c(a_2 + \cdots + a_{2n-2})$. Hence, $a_1 + \cdots + a_{2n-1} = c \in K$.

COROLLARY 4.8. *Let K be a relatively complemented distributive lattice that has (EP) over L. Then no proper sublattice of K contains L and is relatively complemented.*

PROOF. Let M be a relatively complemented sublattice of K that contains L. By the preceding corollary, there is a generalized Boolean lattice B that is R-generated by L and such that $K = B$ or $K = B\backslash E$. Then the proof of Corollary 4.7 shows that $M = K$.

COROLLARY 4.9. *Let K_1 and K_2 be relatively complemented distributive lattices which have (EP) over L. Then there is an isomorphism of K_1 onto K_2 that is the identity on L.*

PROOF. Let B_1 and B_2 be generalized Boolean lattices that are R-generated by K_1 and K_2 respectively. By Corollary 4.6, B_1 and B_2 are R-generated by L. By Grätzer (1971; Theorem 6, page 104) there is an isomorphism q of B onto B_2 that is the identity on L.) (We note that Theorem 6 of Grätzer (1971) is valid even though Lemma 5 which is invalid, is used in the proof.) If $0 \in L$, then $B_1 = K_1$ and $B_2 = K_2$. If $0 \in L$, then $K_1 = B_1/E_1$ and $K_2 = B_2/E_2$, where E_i is the ideal of B_i evenly generated by L. Since $(K_1)q$ is a relatively complemented lattice that contains L, we ghave by Corollary 4.8 that $(K_1)q = K_2$.

5. An example

The motivation for many of the ideas in this note is the following example.

Let P denote the power set of the naturally ordered set of integers and F denote the collection of all finite subsets of the integers. For $n \in \mathbb{Z}$, $(n]$ will

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denote the ideal of $\mathbb{Z}$ generated by n. Let $B = F \cup \{(n] \cup S \mid n \in \mathbb{Z}$ and $S \in F\}$. Then B is a sublattice of the complete Boolean lattice P and it is readily verified that B is a generalized Boolean lattice.

Let r be the mapping of $\mathscr{C}(\mathbb{Z})$ into P given by $(\theta)r = \mathbb{Z}\backslash\{n \mid n = \text{l.u.b.}[n]\theta\}$. Then r is an isomorphism of $\mathscr{C}(\mathbb{Z})$ onto P and hence $\mathscr{C}(\mathbb{Z})$ is a complete Boolean lattice. If $L = \{(n] \mid n \in \mathbb{Z}\}$, then L is isomorphic to $\mathbb{Z}$, B is R-generated by L, and F is the ideal of B that is evenly generated by L. By Corollary 2.11, $\mathscr{C}(F)$ is isomorphic to $\mathscr{C}(L)$ and hence to $\mathscr{C}(\mathbb{Z})$. Since F satisfies the descending chain condition, there does not exist an isomorphism of L into F.

Suppose (by way of contradiction) that $\mathscr{C}(\mathbb{Z})$ is isomorphic to $\mathscr{C}(B)$. Since B is a generalized Boolean lattice, $\mathscr{I}(B)$ is isomorphic to $\mathscr{C}(B)$. Thus the ideals of B form a Boolean lattice. However, by the corollary to Theorem 4.3 (Hashimoto (1952; page 165)) this implies that B satisfies the descending chain condition. This is impossible as L is a sublattice of B.

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