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THE CHARACTERIZATION OF A LATTICE HOMOMORPHISM

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1. Introduction. We shall give a simple characterization of a lattice homomorphism from a linear lattice E to a linear lattice F. This paper is motivated by the following two theorems in Kaplan [**2**]:

(1) If ϕ is a lattice homomorphism, then $\phi^t(F^b)$ is an ideal in E^b.

(2) If ϕ is a lattice homomorphism, then ϕ^{tt} is a lattice homomorphism from F^{bb} into E^{bb}.

The main theorem is stated and proved in section 3. In section 1, we shall give notations and in section 2, we shall prove a main lemma. For details, we refer to Vulikh [**3**].

The author wishes to thank the referee for revising the statements of the main theorem.

2. Notations and definitions. Throughout this paper E and F will be linear lattices. We shall denote by $[x, y]$ an interval $\{z \in E | x \leqq z \leqq y\}$. The complete linear lattice of all the order-bounded linear functionals on E will be denoted by E^b. We shall denote by E^c the band in E^b of all the order-continuous linear functionals on E. For any subset S of E, we define the disjoint complement S' of S by the set

$$S' = \{x \in E | \, |x| \wedge |y| = 0 \text{ for any } y \in S\}.$$

When A is an ideal in E^c or in E^b, $A^\perp$ will denote the null space of A in E. We shall use the following definitions.

Definition. A subcone A of E^+ is called a positive ideal if $x \in A$ and $0 \leqq y \leqq x$ implies $y \in A$.

When E and F are complete linear lattices and ϕ is a bounded linear mapping from E into F, we denote by ϕ^t the transpose of ϕ from F^b into E^b or from F^c into E^c when $\phi^t(F^c) \subset E^c$.

When E is a direct sum of two ideals I and J, we set $E = I \oplus J$.

For any $x \in E$, its components in I and J will be denoted by x_I and x_J.

3. Lemmas. We shall use the following lemma contained in Kaplan [**1**] without proof.

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LEMMA 1. *Let E be a complete linear lattice such that E^c separates the points on E. If $E^c = I \oplus J$ for two ideals I and J, then $E = I^\perp \oplus J^\perp$.*

We prove the following lemma.

LEMMA 2. *Let E and F be complete linear lattices such that F^c separates the points on F. Let ϕ be a continuous linear mapping from E into F such that for every positive ideal A of E, $\phi(A)$ is a positive ideal in F. Then $\phi^t : F^c \to E^c$ is a lattice homomorphism.*

Proof. Since ϕ is positive, ϕ^t is positive. Since ϕ is continuous, $\phi^t(F^c) \subset E^c$. Let $f, g \in F^c$, $f, g \geqq 0$ and $f \wedge g = 0$. We want to show that $\phi^t f \wedge \phi^t g = 0$. Let I be the closed principle ideal generated by f and let $J = I'$. Then $F^c = I \oplus J$ and $F = I^\perp + J^\perp$. We note that $g \in J$.

Now for any $x \in E^+$,

$$(\phi^t f \wedge \phi^t g)(x) = \inf_{\substack{x_1+x_2=x \\ x_1, x_2 \geqq 0}} \{\phi^t f(x_1) + \phi^t g(x_2)\}.$$

Therefore, if we can prove that

$$E^+ = \phi^{-1}(I^\perp) \cap E^+ + \phi^{-1}(J^\perp) \cap E^+,$$

then it follows that $\phi^t f \wedge \phi^t g = 0$.

Let $N = \{x \in \phi^{-1}(0) | x \geqq 0\}$. Then $E = N'' + N'$ such that $\phi(x) = 0$ for $x \in N''$ and

$$\phi(x) > 0 \quad \text{for } x > 0 \text{ and } x \in N'.$$

Since $(\phi^t f \wedge \phi^t g)(x) = 0$ for $x \in N''$, it is enough to show that $(\phi^t f \wedge \phi^t g)(x) = 0$ for $x \in (N')^+$, that is, we may take $E = N'$, without loss of generality, or, equivalently, we may assume that

(1) $x > 0$ implies $\phi(x) > 0$.

Let us set $A = \phi^{-1}(I^\perp) \cap E^+$ and $B = \phi^{-1}(J^\perp) \cap E^+$. Then A and B are closed positive ideals. And it follows easily that $A + B$ is a closed positive ideal.

We shall prove that $(A + B)'' = (A + B) - (A + B)$. In fact, $(A + B)''$ is the smallest closed ideal containing $A + B$. We noted that $(A + B) - (A + B)$ is an ideal and it can be easily shown that $(A + B) - (A + B)$ is closed when $A + B$ is closed. Hence we obtain our equality. It follows that $(A + B)'' \cap E^+ = A + B$.

Now let us show that $E^+ = A + B$. If we can show that $(A + B)' \cap E^+ = \{0\}$, then for any $x \in E^+$ $x = x_{(A+B)''} \in A + B$. Therefore it is enough to show that $(A + B)' \cap E^+ = \{0\}$. Consider $x \in (A + B)' \cap E^+$; then $\phi(x) = (\phi(x))_{I^\perp} + (\phi(x))_{J^\perp}$; hence there exist positive elements y and z in I_x, the principal ideal generated by x such that $\phi(y) = (\phi(x))_{I^\perp}$ and $\phi(z) = (\phi(x))_{J^\perp}$. But $y \in A \subset A + B$ and $y \in I_x \in (A + B)'$. Hence $y = 0$. Similarly $z = 0$. This shows that $\phi(x) = 0$. Therefore by (1) $x = 0$. This completes the proof.

3. Main theorem.

THEOREM. *Let E and F be linear lattices. Assume that E^b (respectively F^b) is separating on E (respectively F). If ϕ is a linear mapping from E into F, then the following are equivalent:*

(1) *ϕ is a lattice homomorphism;*

(2) *if $x \wedge y = 0$, then $\phi(x) \wedge \phi(y) = 0$;*

(3) *for any $f \in (F^b)^+$, $\phi^t[0,f] = [0, \phi^t f]$;*

(4) *for any positive ideal I in F^b, $\phi^t(I)$ is a positive ideal in E^b; and*

(5) *$\phi(E)$ is a linear sublattice of F, $\phi(E^+) = (\phi(E))^+$, and for any ideal I in F^b, $\phi^t(I)$ is an ideal in E^b.*

Proof. (1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (1). This is well-known.

(1) $\Rightarrow$ (3). Let $f \in (F^b)^+$ and $g \in [0, \phi^t f]$. We want to show that there exists $h \in [0,f]$ such that $g = \phi^t(h)$. $\phi^{-1}(0)$ is an ideal; denote it by I. Then $\phi^t(F^b) \subset I^\perp$ and $I^\perp$ is isomorphic with $(E/I)^b$, (E/I) is isomorphic with $\phi(E)$. Therefore $I^\perp$ is isomorphic with $\phi(E)^b$. Moreover, ϕ^t can be identified with the mapping $\pi : F^b \rightarrow (\phi(E))^b$ defined by $\pi f = f|\phi(E)$. Therefore it is enough to show that if $g \in [0, \pi f]$, then there exists $h \in [0,f]$ such that $g = \pi h$. But if ϕ is a lattice homomorphism, then $\phi(E)$ is a linear sublattice of F and hence g can be extended to a linear functional h on F such that $0 \leqq h \leqq f$. Then $\pi h = g$. This completes the proof that (1) implies (3).

(3) $\Rightarrow$ (1). We shall prove that the bitranspose $\phi^{tt} : E^{bc} \rightarrow F^{bc}$ is a lattice homomorphism. Once this is done, then since E (respectively F) can be regarded as a linear sublattice of E^{bc} (respectively F^{bc}), it follows that $\phi = \phi^{tt}|E$ is a lattice homomorphism.

To prove that ϕ^{tt} is a lattice homomorphism, it is enough to show that for every $x \in E^{bc}$, $\phi^{tt}x^+ = (\phi^{tt}x)^+$. Since ϕ^{tt} preserves order, $\phi^{tt}x^+ \geqq \phi^{tt}x$, hence $\phi^{tt}x^+ \geqq (\phi^{tt}x)^+$. To prove the opposite inequality, we shall show that $(\phi^{tt}x^+)(f) \leqq (\phi^{tt}x)^+(f)$ for all $f \in F^b, f \geqq 0$. Now

$$(\phi^{tt}x^+)(f) = (\phi^t f)(x^+) = \sup_{0 \leqq g \leqq \phi^t f} g(x),$$

while

$$(\phi^{tt}x)^+(f) = \sup_{0 \leqq h \leqq f} h(\phi^{tt}x) = \sup_{0 \leqq h \leqq f} (\phi^t h)(x).$$

By the assumption (4), if $0 \leqq g \leqq \phi^t(f)$, then $g = \phi^t(h)$ for some h satisfying $0 \leqq h \leqq f$. Therefore

$$\sup_{0 \leqq g \leqq \phi^t f} g(x) \leqq \sup_{0 \leqq h \leqq f} (\phi^t h)(x).$$

Hence (3) implies (1).

(3) $\Rightarrow$ (4). This is clear.

(4) $\Rightarrow$ (1). If ϕ^t maps positive ideal to a positive ideal, then ϕ^t is continuous.

Since E^{bc} is separating on E, ϕ^{tt} is a lattice homomorphism by the Lemma 2. Therefore ϕ is a lattice homomorphism.

(1) $\Rightarrow$ (5). If ϕ is a lattice homomorphism, then $\phi(E)$ is a linear sublattice and $\phi(E^+) = (\phi(E))^+$. Let I be any ideal in F^b. Then $I^+ = \{x \in I | x \geqq 0\}$ is a positive ideal and $I = I^+ - I^+$. Hence $\phi(I) = \phi(I^+) - \phi(I^+)$. By (4) $\phi(I^+)$ is a positive ideal. Hence $\phi(I)$ is an ideal in E^b.

(5) $\Rightarrow$ (3). Let $f \in (F^b)^+$ and $g \in [0, \phi^t f]$. Let I_f be the ideal generated by f in F^b. Then $g \in \phi^t I_f$, since $\phi^t I_f$ is an ideal. Therefore there exists $h \in I_f$ such that $g = \phi^t h$. Let us set $k = h|\phi(E)$. Then $0 \leqq k \leqq h|\phi(E)$ on $(\phi(E))$. In fact, for any $y \in (\phi(E))^+$, let $y = \phi(x)$ for some $x \in E^+$. Then

$$k(y) = k(\phi(x)) = h(\phi(x)) = \phi^t h(x) = g(x) \geqq 0.$$

Moreover, $k = h|\phi(E) \leqq f$ on $\phi(E)$. In fact, for any $g \in (\phi(E))^+$, let $y = \phi(x)$ for some $x \in E^+$. Then

$$k(y) = k(\phi(x)) = h(\phi(x)) = \phi^t h(x) = g(x) \leqq \phi^t f(x) = f(\phi(x)) = f(y).$$

Hence $h|\phi(E) = k$ can be extended to a linear functional, say k again, defined on F such that $0 \leqq k \leqq f$.

We have $\phi^t k = \phi^t h = g$. Therefore (5) implies (3). This completes our proof.

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