BULL. AUSTRAL. MATH. SOC. 93E15
VOL. 10 (1974), 271-276.

Submartingales and stochastic stability

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Sufficient conditions, involving the existence of a Lyapunov function which is a submartingale of special type, are given for the instability of stochastic discrete time systems.

1. Introduction

The stability of the origin for a random nonlinear system on R^n ,

(1) $$x_{n+1} = f\left(x_n, n, r_n\right) ,$$

where $x_1 = x$ is a given random vector variable and r_n markovian, has been studied by Bucy [1] and Kushner [5], using Lyapunov functions which are supermartingales along the solutions $\phi(n, x)$:

$$E\left(V\left(\phi(n, x)\right) \mid \phi(n-1, x)\right) \leq V\left(\phi(n-1, x)\right) .$$

On the other hand, instability theorems have been discussed by Hahn [2] for deterministic systems

$$x_{n+1} = g\left(x_n, n\right) ,$$

requiring the existence of a Lyapunov function which is, roughly speaking, increasing on the trajectories. By analogy with these ideas, it is reasonable to conjecture that the system (1) is unstable at the origin if there exists a Lyapunov function which is a submartingale (expectation increasing) along the solutions. However, further conditions on the submartingale are needed to prove instability. An example is given below to show that these conditions cannot be greatly weakened.

Received 22 November 1973.

2. Definitions and notation

Solutions of the system (1), with initial vector $x_1 = x$, are denoted by $\phi(n, x)$. The probability of an event A, in the probability measure P, is $P(A)$, its expectation $E(A)$, whilst $\|x\|$ is the usual norm of $x \in R^n$. The following are more or less standard:

DEFINITION 1. A sequence of random variables v_n, sigma fields F_n and probability measure P, form a submartingale iff F_n is the minimal sigma field over $(v_1, \ldots, v_n)$ and $E(v_n \mid F_{n-1}) \geq v_{n-1}$ almost everywhere P.

DEFINITION 2. The origin is stable with probability 1 if, for any $\delta > 0$, $\varepsilon > 0$ there is a $\rho(\delta, \varepsilon) > 0$ such that if $\|x\| \leq \rho(\delta, \varepsilon)$ then $P(\|\phi(n, x)\| \geq \varepsilon) \leq \delta$. If the system is not stable it is termed unstable.

DEFINITION 3. The integrals of a sequence of random variables v_n are uniformly continuous if $\int_A v_n dP \to 0$ uniformly in n as $P(A) \to 0$.

DEFINITION 4. A real valued, positive definite continuous function v on R^n such that $v(0) = 0$ and $v(x) \to \infty$ as $\|x\| \to \infty$ is called a Lyapunov function. Let $v_n = v(\phi(n, x))$.

3. Results

THEOREM 1. *Suppose there exists a Lyapunov function v such that v_n is a submartingale, the integrals of v_n are uniformly continuous and $E(v_n \mid x)$ is unbounded as $n \to \infty$. Then the system* (1) *is unstable.*

Proof. Let $A = \left\{\sup_n v_n \geq \varepsilon\right\}$. Then

$$E(v_n \mid x) \leq \varepsilon P\left\{\sup_n v_n \leq \varepsilon\right\} + \int_A v_n dP .$$

If the origin were stable and $P(A) < \delta$ then, for small δ, $\int_A v_n dP$ is small by uniform continuity. So $E(v_n \mid x)$ is bounded, a contradiction.

COROLLARY. *Let* v_n *be a submartingale, the integrals of* v_n *uniformly continuous and*

$$E\left(v_n \mid F_{n-1}\right) - v_{n-1} \geq q\left(v_{n-1}\right) ,$$

where q *is a non-decreasing positive convex function on the positive reals. Then the origin is unstable.*

Proof. Taking expectations conditional on x, and assuming that $E\left(v_n\right) \nrightarrow \infty$ for finite n, it follows that

$$E\left(v_n\right) - E\left(v_{n-1}\right) \geq E\left(q\left(v_{n-1}\right)\right) . \tag{2}$$

But since q is convex, $E\left(q\left(v_k\right)\right) \geq q\left(E\left(v_k\right)\right)$, [6, p. 159]. Summing the inequality (2) gives

$$\begin{aligned} E\left(v_n\right) &\geq E\left(v_1\right) + \sum_{1}^{n-1} q\left(E\left(v_k\right)\right) \\ &\geq E\left(v_1\right) + (n-1)q\left(E\left(v_1\right)\right) , \end{aligned}$$

and $E\left(v_n\right)$ is unbounded as $n \to \infty$. The conditions of the previous theorem are satisfied.

THEOREM 2. *Suppose there exists a Lyapunov function* v *such that* v_n *is a submartingale,* $E\left(v_n \mid x\right)$ *is unbounded as* $n \to \infty$, *and*

$$\operatorname{var}\left(v_n \mid x\right) = o\left(E\left(v_n \mid x\right)\right) , \quad n \to \infty .$$

Then the system (1) *is unstable.*

Proof. From the Chebyshev inequality [6, p. 11],

$$P\left(\left|v_n - E\left(v_n \mid x\right)\right| \leq B\right) \geq 1 - \left(\operatorname{var}\left(v_n \mid x\right)\right)/B^2 .$$

For arbitrary $0 < \delta < 1$, set $B = \left(\left(\operatorname{var}\left(v_n \mid x\right)\right)/\delta\right)^{\frac{1}{2}}$. Then

$$P\left(E\left(v_n \mid x\right) - B \leq v_n \leq E\left(v_n \mid x\right) + B\right) \geq 1 - \delta .$$

Since $B = o\left(E\left(v_n \mid x\right)\right)$, the probability that v_n is greater than ε, for large n, is close to 1. But it is known [3] that there exist real continuous positive non-decreasing functions α and β such that

$$\alpha(0) = \beta(0) = 0 ,$$

$$\alpha(\|x\|),\ \beta(\|x\|) \to \infty \quad \text{as} \quad \|x\| \to \infty ,$$

$$\alpha(\|x\|) \le v(x) \le \beta(\|x\|) .$$

It follows that, with probability close to 1, $\alpha(\|\phi(n, x)\|)$, and equally $\|\phi(n, x)\|$, is greater than ε for large n. The origin is thus unstable.

4. An example

If the condition on the variance of v_n is dropped in Theorem 2, the remaining conditions are not sufficient for instability. To illustrate this, consider the scalar equation

$$x_{n+1} = \lambda_n x_n , \quad x_1 = x ,$$

where $\{\lambda_n\}$ is a sequence of independent random variables satisfying

$$P\left(\lambda_n = 1/a^3\right) = (a-1)/a = q ,$$

$$P\left(\lambda_n = a^2\right) = 1/a = p , \quad a > 1 ,$$

for each n. Let $v(x) = |x|$. Clearly

$$E\left(v_{n+1} \mid x\right) - v_n = |x_n|\left(a-1+1/a^3-1/a^4\right) ,$$

and $E\left(v_{n+1} \mid x\right) > a^n|x|$. For any given $x_1 = x$,

$$P\left(x_{n+1} = a^{2(n-r)-3r}x\right) = \binom{n}{r}p^{n-r}q^r .$$

It follows that $|x_{n+1}| > a^{-t}|x|$ if $2n - 5r > -t$. Put $s = [(2n+t)/5] + 1$. Then

$$P\left(|x_{n+1}| > a^{-t}|x|\right) < p^n + \binom{n}{1}p^{n-1}q + \ldots + \binom{n}{s}p^{n-s}q^s . \tag{3}$$

But it is known [4, 5.7] that the sum on the right of the inequality may be expressed as

$$B_p(s,\ n-s+1)\Gamma(n+1)/\left(\Gamma(s)\Gamma(n-s+1)\right) ,$$

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where B_p is the incomplete Beta function:

$$\int_0^{p/(1-p)} v^{s-1}(1+v)^{-n-1}dv .$$

Since $1/(1+v) < 1$ in the interval of integration, the integral is greater than

$$(4) \qquad (p/(1-p))^s/s = O(p/(1-p))^{2n/5} .$$

Stirling's Formula gives

$$\Gamma(n+1)/\Gamma(s)\Gamma(n-s+1) = O\left(n^{n+\frac{1}{2}}s^{-s+\frac{1}{2}}(n-s+1)^{-n+s-\frac{1}{2}}\right) .$$

Since $s = [(2n+t)/5] + 1$, the expression above is majorised by

$$5^n n^{n+\frac{1}{2}}/\left(n^n 2^a(1+t/2n)^a 3^b(1-t/3n)^b\right) ,$$

where $a = (2n+t)/5-1/2$, $b = (3n-t)/5 + 1/2$. The majorant is just $O\left(n^{\frac{1}{2}}\left(5.3^{-3/5}.2^{-2/5}\right)^n\right)$ and so, from (3) and (4),

$$P\left(|x_{n+1}| > a^{-t}|x|\right) = O\left(n^{\frac{1}{2}}c^n\right) ,$$

where $c = (p/(1-p))^{2/5}.5.3^{-3/5}.2^{-2/5}$. It follows that the origin is asymptotically stable with probability 1 (stable with probability 1 and $x_n \to 0$ with probability 1 in a neighbourhood of the origin) if $c < 1$. But this is certainly the case if $a > 1 + (5^5/108)^{\frac{1}{2}}$.

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https://doi.org/10.1017/S0004972700040880 Published online by Cambridge University Press