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A SPECTRAL PROBLEM IN ORDERED BANACH ALGEBRAS

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We recall the definition and properties of an algebra cone C of a complex unital Banach algebra A. It can be shown that C induces on A an ordering which is compatible with the algebraic structure of A, and A is then called an ordered Banach algebra. The Banach algebra $\mathcal{L}(E)$ of all bounded linear operators on a complex Banach lattice E is an example of an ordered Banach algebra, and an interesting aspect of research in ordered Banach algebras is that of investigating in an ordered Banach algebra-context certain problems that originated in $\mathcal{L}(E)$. In this paper we investigate the problems of providing conditions under which (1) a positive element a with spectrum consisting of 1 only will necessarily be greater than or equal to 1, and (2) $f(a)$ will be positive if a is positive, where $f(a)$ is the element defined by the holomorphic functional calculus.

1. INTRODUCTION

An interesting problem in Banach algebra-theory is that of finding conditions under which an element a with $\mathrm{Sp}\,(a) = \{1\}$ will be the unit element; or, in an operator-context, provide conditions such that if T is a bounded linear operator on a Banach space with $\mathrm{Sp}\,(T) = \{1\}$, then T is necessarily the identity operator. Naturally, in certain cases the problem has an obvious answer. For instance, if a Banach algebra A is commutative and semisimple, then if $a \in A$ is any element with $\mathrm{Sp}\,(a) = \{1\}$, it follows from the Spectral Mapping Theorem that $a - 1 \in \mathrm{QN}\,(A) = \mathrm{Rad}\,(A) = \{0\}$, so that $a = 1$. Other interesting answers have been obtained in, for instance, [**4**] and [**3**].

Huijsmans and de Pagter (see [**12**]) asked the following more general question: under which conditions will it be true that if T is a positive bounded linear operator on a complex Banach lattice with $\mathrm{Sp}\,(T) = \{1\}$, then $T \geqslant I$? This question has been investigated by Zhang in his papers [**11**] and [**12**]. In this paper we introduce the problem in the context of an ordered Banach algebra. In [**8**] and [**7**], and later [**5**] and [**6**], some spectral theory of positive elements in ordered Banach algebras was developed. We recall some of this information in Section 3. In Section 4 we investigate the mentioned problem in an ordered Banach algebra-context, that is, find conditions under which a positive element a in an ordered Banach algebra with $\mathrm{Sp}\,(a) = \{1\}$ will be greater than or equal to the

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unit element. We extend the problem somewhat and provide some answers in the finite dimensional case, the case where the spectral radius of a is a pole of a certain order of the resolvent of a, and the case in which the algebra cone is inverse-closed.

We also consider the more general problem of obtaining conditions which imply that if $a \in C$, then $f(a) \in C$, where f is analytic in a neighbourhood of the spectrum of a.

Throughout we seek to obtain our results using only the intrinsic properties of Banach algebras, and therefore without using operator-theoretic arguments or relying on properties which are unique to Banach lattices.

2. Preliminaries

Throughout A (or B) will be a complex Banach algebra with unit 1. A *homomorphism* ϕ from a Banach algebra A into a Banach algebra B is a linear map $\phi : A \to B$ such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$ and $\phi(1) = 1$. The spectrum of an element a in A will be denoted by Sp (a), the spectral radius of a in A by $\rho(a)$ and the distance $d\big(0, \text{Sp }(a)\big)$ from 0 to the spectrum of a by $\delta(a)$ (or by Sp (a, A), $\rho(a, A)$ and $\delta(a, A)$ if necessary to avoid confusion). Recall that if a is invertible, then $\rho(a^{-1}) = 1/\big(\delta(a)\big)$ ([**1**, Theorem 3.3.5]). A map $\phi : A \to B$ is called *spectrum preserving* if Sp $(a, A) =$ Sp $\big(\phi(a), B\big)$ for all $a \in A$. It is easy to see that a bijective homomorphism is spectrum preserving. We denote the *peripheral spectrum* $\big\{\lambda \in \text{Sp }(a) : |\lambda| = \rho(a)\big\}$ of an element a in A by psp (a), the set of quasinilpotent elements in A by QN (A) and the radical of A by Rad (A). A Banach algebra is called *semisimple* if its radical consists of zero only.

3. Ordered Banach Algebras

In ([**8**, Section 3]) we defined an algebra cone C of a complex Banach algebra A and showed that C induced on A an ordering which was compatible with the algebraic structure of A. Such a Banach algebra is called an ordered Banach algebra. We recall those definitions now and also the additional properties that C may have.

Let A be a complex Banach algebra with unit 1. We call a nonempty subset C of A a *cone* of A if C satisfies the following:

1. $C + C \subseteq C$,
2. $\lambda C \subseteq C$ for all $\lambda \geqslant 0$.

If in addition C satisfies $C \cap -C = \{0\}$, then C is called a *proper* cone.

Any cone C of A induces an *ordering* "$\leqslant$" on A in the following way:

$$a \leqslant b \text{ if and only if } b - a \in C \tag{3.1}$$

$(a, b \in A)$. It can be shown that this ordering is a partial order on A, that is, for every $a, b, c \in A$

(a) $a \leqslant a$ ($\leqslant$ is *reflexive*),

(b) if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$ ($\leqslant$ is *transitive*).

Furthermore, C is proper if and only if this partial order has the additional property of being *antisimmetric*, that is, if $a \leqslant b$ and $b \leqslant a$, then $a = b$. Considering the partial order that C induces we find that $C = \{a \in A : a \geqslant 0\}$ and therefore we call the elements of C *positive*.

A cone C of a Banach algebra A is called an *algebra cone* of A if C satisfies the following conditions:

3. $C.C \subseteq C$,
4. $1 \in C$.

Motivated by this concept we call a complex Banach algebra with unit 1 an *ordered Banach algebra* if A is partially ordered by a relation "$\leqslant$" in such a manner that for every $a, b, c \in A$ and $\lambda \in \mathbb{C}$

1′. $a, b \geqslant 0 \Rightarrow a + b \geqslant 0$,

2′. $a \geqslant 0, \lambda \geqslant 0 \Rightarrow \lambda a \geqslant 0$,

3′. $a, b \geqslant 0 \Rightarrow ab \geqslant 0$,

4′. $1 \geqslant 0$.

Therefore if A is ordered by an algebra cone C, then A, or more specifically (A, C), is an ordered Banach algebra.

An algebra cone C of A is called *proper* if C is a proper cone of A and *closed* if it is a closed subset of A. Furthermore, C is said to be *normal* if there exists a constant $\alpha > 0$ such that it follows from $0 \leqslant a \leqslant b$ in A that $\|a\| \leqslant \alpha \|b\|$. It is well-known that if C is a normal algebra cone, then C is proper. If C has the property that if $a \in C$ and a is invertible, then $a^{-1} \in C$, then C is said to be *inverse-closed*.

The following theorem is well-known in an operator-context:

Theorem 3.2. ([8, Proposition 5.1]) *Let (A, C) be an ordered Banach algebra with C closed and normal. If $a \in C$, then $\rho(a) \in \operatorname{Sp}(a)$.*

It is interesting to note that also $\delta(a) \in \operatorname{Sp}(a)$, under the additional assumption that C is inverse-closed:

Theorem 3.3. *Let (A, C) be an ordered Banach algebra with C closed, normal and inverse-closed. If $a \in C$, then $\delta(a) \in \operatorname{Sp}(a)$.*

Proof: If a is not invertible, then $\delta(a) = 0 \in \operatorname{Sp}(a)$, so suppose that a is invertible. Since $a \in C$ and C is inverse-closed, it follows that $a^{-1} \in C$. The normality and closedness of C implies that $\rho(a^{-1}) \in \operatorname{Sp}(a^{-1})$, so that $\rho(a^{-1}) = 1/(\lambda_0)$, for some $\lambda_0 \in \operatorname{Sp}(a)$. Since $\rho(a^{-1}) = 1/(\delta(a))$, it follows that $\delta(a) = \lambda_0 \in \operatorname{Sp}(a)$. ∎

Note that the condition that C is inverse-closed in Theorem 3.3 is essential. Consider, for instance, the Banach algebra A of all 2×2 complex matrices. If C is the subset of A

of matrices with only non-negative entries, then C is a closed and normal algebra cone (see Example 3.5), but C is not inverse-closed and $\delta(a) \in \mathrm{Sp}\,(a)$ does not hold for all $a \in C$. This can be seen by considering the element $a = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in C$, which is invertible with $a^{-1} = -(1/3)\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \notin C$. Also, $\mathrm{Sp}\,(a) = \{-1, 3\}$, so that $\delta(a) = 1 \notin \mathrm{Sp}\,(a)$.

Let A and B be Banach algebras and $\phi : A \to B$ a homomorphism. If C is an algebra cone of A, then $\phi(C)$ is an algebra cone of B. If ϕ is injective, then if C is proper, so is $\phi(C)$. Furthermore, if ϕ is continuous and bijective, then if C is closed, so is $\phi(C)$.

We conclude this section with a number of examples, which serve to illustrate the concepts.

Let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X.

EXAMPLE 3.4. Let E be a complex Banach lattice and let $C := \{x \in E : x = |x|\}$. If $K := \{T \in \mathcal{L}(E) : TC \subset C\}$, then K is a closed, normal algebra cone of $\mathcal{L}(E)$. Therefore $(\mathcal{L}(E), K)$ is an ordered Banach algebra.

The nontrivial part of the above example follows from ([**9**, Lemma 3]).

Let $M_n(\mathbb{C})$ denote the (Banach) algebra of $n \times n$ complex matrices.

EXAMPLE 3.5. Let $n \in \mathbb{N}$, C the subset of $M_n(\mathbb{C})$ of matrices with only nonnegative entries and C' the subset of $M_n(\mathbb{C})$ of diagonal matrices with only non-negative entries. Then C and C' are closed, normal algebra cones of $M_n(\mathbb{C})$. Therefore $(M_n(\mathbb{C}), C)$ and $(M_n(\mathbb{C}), C')$ are ordered Banach algebra.

EXAMPLE 3.6. Let $n \in \mathbb{N}$ and A_i an ordered Banach algebra, with algebra cone C_i, for each $i = 1, \ldots, n$. Let $A := A_1 \oplus \cdots \oplus A_n$ and $C := \{(c_1, \ldots, c_n) \in A : c_i \in C_i \text{ for } i = 1, \ldots, n\}$. Then (A, C) is an ordered Banach algebra, and if C_i is closed (proper, normal) for all $i = 1, \ldots, n$ then C is closed (proper, normal).

The preceding two examples imply

EXAMPLE 3.7. Let $n \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathbb{N}$ and $A := M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$. Let $C := \{(c_1, \ldots, c_n) \in A : c_i \text{ is a } k_i \times k_i \text{ matrix with only non-negative entries, for all } i = 1, \ldots, n\}$ and $C' := \{(c_1, \ldots, c_n) \in A : c_i \text{ is a diagonal } k_i \times k_i \text{ matrix with only non-negative entries, for all } i = 1, \ldots, n\}$. Then both (A, C) and (A, C') are ordered Banach algebras and both C and C' are closed, normal algebra cones of A.

EXAMPLE 3.8. Let $A = l^\infty$ and $C = \{(c_1, c_2, \ldots) \in l^\infty : c_i \geqslant 0 \text{ for all } i \in \mathbb{N}\}$. Then (A, C) is an ordered Banach algebra, and C is a closed, normal and inverse-closed algebra cone of A.

A proof of part of the contents of this example was given in ([**5**, Example 4.14]). The closedness and inverse-closedness of C follow easily from the definition of C and the definition of the (sup-) norm in l^∞.

EXAMPLE 3.9. Let A be a commutative C^*-algebra, $C = \{x \in A : x = x^* \text{ and } \operatorname{Sp}(x) \subset [0,\infty)\}$. Then (A, C) is an ordered Banach algebra, and C is a closed, normal and inverse-closed algebra cone of A.

References giving the proof of part of the contents of this example was given in ([**6**, Example 3.3]). The inverse-closedness of C follows easily from the definition of C.

4. A spectral problem

Let A be an ordered Banach algebra with an algebra cone C. Under which conditions will it follow that if $a \in C$ with $\operatorname{Sp}(a) = \{1\}$, then $a - 1 \in C$? This problem is equivalent to the problem stated in the introduction, that is, the problem of providing conditions under which it will follow from a positive and $\operatorname{Sp}(a) = \{1\}$, that $a \geqslant 1$. Originally this problem has been investigated for bounded linear operators on a Banach lattice (see [**11**] and [**12**]).

Another way to look at this problem is by considering the analytic function $f(\lambda) = \lambda - 1$. Then $a - 1$ is $f(a)$, the element defined by the holomorphic functional calculus. So the problem becomes: provide conditions which imply that if $\operatorname{Sp}(a) = \{1\}$ and $a \in C$, then $f(a) \in C$. This problem will be investigated in a more general form.

Returning to the original problem, what can be said in the case that A is a finite dimensional Banach algebra? We begin by investigating the Banach algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices, in which case the following holds:

THEOREM 4.1. *Let $n \in \mathbb{N}$ and C the algebra cone of $M_n(\mathbb{C})$ consisting of all complex $n \times n$ matrices with only non-negative entries. If $a \in C$ and $\operatorname{Sp}(a) = \{1\}$, then $a - 1 \in C$.*

PROOF : Suppose $a = (\alpha_{ij})$. Then $\alpha_{ij} \geqslant 0$ for all $i, j \in \{1, \dots, n\}$. Let $b = a - 1$. In the matrix b^2 the i-th diagonal element is $\alpha_{i1}\alpha_{1i} + \alpha_{i2}\alpha_{2i} + \dots + \alpha_{i(i-1)}\alpha_{(i-1)i} + (\alpha_{ii} - 1)^2 + \alpha_{i(i+1)}\alpha_{(i+1)i} + \dots + \alpha_{in}\alpha_{ni}$, which is greater than or equal to zero. Since $\operatorname{Sp} b^2 = (\operatorname{Sp}(a-1))^2 = \{0\}$, the trace $\operatorname{Tr} b^2$ of b^2 is zero, and $\operatorname{Tr} b^2$ is the sum of all the diagonal elements of b^2. Hence each diagonal element of b^2 is zero. Also, each term in such an element is greater than or equal to zero, so that each term must be zero. In particular, $\alpha_{ii} = 1$ for all $i = 1, \dots, n$ (and $\alpha_{ij}\alpha_{ji} = 0$ for all $i \neq j$). Hence each entry of b is non-negative, so that $b \in C$. Therefore $a - 1 \in C$. ∎

The above proof is essentially the same as the one X.-D. Zhang used to prove a similar result for positive operators on finite dimensional Banach lattices (see [**12**, Theorem 4.1]).

THEOREM 4.2. *Let (A, C) denote the ordered Banach algebra $A_1 \oplus \dots \oplus A_n$ of Example 3.6, that is, each (A_i, C_i) is an ordered Banach algebra with an algebra cone C_i, and $C = \{(c_1, \dots, c_n) \in A : c_i \in C_i \text{ for } i = 1, \dots, n\}$. Suppose that for each $i = 1, \dots, n$ the following holds: if $c_i \in C_i$ with $\operatorname{Sp}(c_i) = \{1\}$, then $c_i - 1 \in C_i$. Then if $c \in C$ with $\operatorname{Sp} c = \{1\}$, then $c - 1 \in C$.*

PROOF: It follows easily by recalling that if $c = (c_1, \ldots, c_n)$, then Sp $c = \cup_{i=1}^{n}$Sp c_i. ∎

Using Theorems 4.1 and 4.2, we obtain

THEOREM 4.3. *Let $n, k_1, \ldots, k_n \in \mathbb{N}$ and let A denote the ordered Banach algebra $M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$, with algebra cone $C = \{(c_1, \ldots, c_n) \in A : c_i \in C_i$ for $i = 1, \ldots, n\}$, where C_i denotes the algebra cone of $M_{k_i}(\mathbb{C})$ consisting of all complex $k_i \times k_i$ matrices with only non-negative entries, for each $i = 1, \ldots, n$. If $c \in C$ with Sp $(c) = \{1\}$, then $c - 1 \in C$.*

An application of the Wedderburn-Artin Theorem ([**1**, Theorem 2.1.2]), together with Example 3.7 and Theorem 4.3, yield

THEOREM 4.4. *If B is a semisimple finite-dimensional Banach algebra, then B is isomorphic (as an algebra) to an ordered Banach algebra A (as in Theorem 4.3) with a closed and normal algebra cone C (as in Theorem 4.3) which has the property that if $c \in C$ and Sp $(c) = \{1\}$, then $c - 1 \in C$.*

Finally, we have

THEOREM 4.5. *Let B be an ordered Banach algebra with a proper algebra cone C_1 and with B isomorphic (as an algebra) to an ordered Banach algebra A, with a proper algebra cone C which has the property that if $c \in C$ and Sp $(c, A) = \{1\}$, then $c - 1 \in C$. If C is the only proper algebra cone of A, then if $c_1 \in C_1$ and Sp $(c_1, B) = \{1\}$, then $c_1 - 1 \in C_1$.*

PROOF: Suppose $\phi : B \to A$ is a bijective homomorphism. Then ϕ is spectrum-preserving. Let $c_1 \in C_1$ and Sp $(c_1, B) = \{1\}$. Then $\phi(c_1) \in \phi(C_1)$. The remarks preceding the examples in Section 3 show that $\phi(C_1)$ is a proper algebra cone of A. Hence, by the assumption, $\phi(C_1) = C$, so that $\phi(c_1) \in C$. Since Sp $(\phi(c_1), A) =$ Sp $(c_1, B) = \{1\}$, it follows by assumption that $\phi(c_1) - 1 \in C$, that is, $\phi(c_1 - 1) \in \phi(C_1)$. Since ϕ is injective, it follows that $c_1 - 1 \in C_1$. ∎

Unfortunately, it is not possible to say more than Theorem 4.4 about the semisimple finite-dimensional case (at least by using Theorem 4.5), since the algebra cone C in Theorem 4.4 is not the only proper algebra cone of A (see Example 3.7).

We now consider the case where the spectral radius of a is a pole of the resolvent $(\lambda 1 - a)^{-1}$ of a, and extend the problem to the case where Sp $(a) = \{\rho(a)\}$ with $\rho(a) \geqslant 1$ (see Corollaries 4.9 and 4.15). The following proposition is vital in solving this problem:

PROPOSITION 4.6. *Let (A, C) be an ordered Banach algebra with C closed, and let $a \in C$. If $\lambda > \rho(a)$, then $(\lambda 1 - a)^{-1} \geqslant 0$.*

PROOF: For $|\lambda| > \rho(a)$, the resolvent of a has a Neumann series representation $(\lambda 1 - a)^{-1} = \sum_{n=0}^{\infty}(a^n/\lambda^{n+1})$. Since $\lambda > \rho(a)$, all the terms of this series are positive, so that $(\lambda 1 - a)^{-1} \geqslant 0$, since C is closed. ∎

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Proposition 4.7. *Let A be a Banach algebra and $a \in A$ such that* $\mathrm{Sp}\,(a) = \{\lambda_0\}$*. If $\lambda \neq \lambda_0$, then*

$$(\lambda 1 - a)^{-1} = \sum_{n=1}^{\infty} b_{-n}(\lambda - \lambda_0)^{-n}$$

where $b_{-n} = (a - \lambda_0 1)^{n-1}$.

Proof: If $\lambda \neq \lambda_0$, then $|\lambda - \lambda_0| > 0 = \rho(a - \lambda_0 1)$, so that

$$(\lambda 1 - a)^{-1} = \big((\lambda - \lambda_0)1 - (a - \lambda_0 1)\big)^{-1} = \sum_{n=0}^{\infty} \frac{(a - \lambda_0 1)^n}{(\lambda - \lambda_0)^{n+1}} = \sum_{n=1}^{\infty} \frac{(a - \lambda_0 1)^{n-1}}{(\lambda - \lambda_0)^n}.$$

Hence the result follows. ∎

Since this series is clearly the Laurent series of the resolvent of a around λ_0, we have the following

Corollary 4.8. *Let A be a Banach algebra and $a \in A$ such that* $\mathrm{Sp}\,(a) = \{\lambda_0\}$*. If λ_0 is a pole of order k of the resolvent of a, then $(a - \lambda_0 1)^k = 0$ and* $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^k (\lambda 1 - a)^{-1} = (a - \lambda_0 1)^{k-1}$.

Proof: If λ_0 is a pole of order k of the resolvent of a, then by Proposition 4.7, the coefficient $b_{-(k+1)} = 0$. Hence $(a - \lambda_0 1)^k = 0$. Furthermore, since

$$(\lambda 1 - a)^{-1} = \frac{1}{\lambda - \lambda_0} + \frac{a - \lambda_0 1}{(\lambda - \lambda_0)^2} + \cdots + \frac{(a - \lambda_0 1)^{k-1}}{(\lambda - \lambda_0)^k},$$

the result follows. ∎

Using the preceding elementary result, we can state some conditions which imply that if $a \in C$ and $\mathrm{Sp}\,(a) = \{\rho(a)\}$ with $\rho(a) \geqslant 1$, then $a - 1 \in C$.

Corollary 4.9. *Let A be a Banach algebra and $a \in A$ such that* $\mathrm{Sp}\,(a) = \{\rho(a)\}$.

1. *If $\rho(a)$ is a pole of order k of the resolvent of a, then $\big(a - \rho(a)1\big)^k = 0$.*
2. *If $\rho(a)$ is a simple pole of the resolvent of a, then $a = \rho(a)1$. It follows that, if C is an algebra cone of A, then*

$$\rho(a) \geqslant 1 \quad \Rightarrow \quad a - 1 \in C.$$

Suppose, in addition, that (A, C) is an ordered Banach algebra with C closed, and $a \in C$.

3. *If $\rho(a)$ is a pole of order k of the resolvent of a, then $\big(a - \rho(a)1\big)^{k-1} \in C$.*
4. *If $\rho(a)$ is a pole of order 2 of the resolvent of a, then $a \geqslant \rho(a)1$. It follows that*

$$\rho(a) \geqslant 1 \quad \Rightarrow \quad a - 1 \in C.$$

Proof:

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1. Follows directly from Corollary 4.8.
2. Follows from 1.
3. It follows from Corollary 4.8 that $(a-\rho(a)1)^{k-1} = \lim_{\lambda\to\rho(a)} (\lambda-\rho(a))^k(\lambda 1 - a)^{-1}$. Restricting λ to an interval of the form $(\rho(a), \rho(a)+R)$, we obtain $(a-\rho(a)1)^{k-1} = \lim_{\lambda\to\rho(a)^+} (\lambda-\rho(a))^k(\lambda 1-a)^{-1}$. Since C is closed, it follows from Proposition 4.6 that $(a-\rho(a)1)^{k-1} \in C$.
4. Follows from 3. ∎

We note that Corollary 4.9 4 in one sense extends, and in another sense is included by, [**12**, Theorem 5.3], in the case $A = \mathcal{L}(E)$ (see Example 3.4).

Suppose that f is a complex valued function which is analytic on a neighbourhood Ω of the spectrum of a. Then an element $f(a) = (1/2\pi i)\int_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\,d\lambda$ in A is defined, where Γ is a contour in $\Omega\backslash\text{Sp}\,(a)$ surrounding Sp (a) ([**1**, p. 43]). An interesting question arises, namely: if $a \in C$, when does it follow that $f(a) \in C$? Naturally, for certain functions, answers can be obtained easily. We collect some of these in

PROPOSITION 4.10. *Let (A, C) be an ordered Banach algebra and $a \in C$.*

1. *If $p(\lambda) = \alpha_n\lambda^n + \cdots + \alpha_1\lambda + \alpha_0$ with $\alpha_n, \ldots, \alpha_0$ real and positive, then $p(a) \in C$.*
2. *Suppose, in addition, that C is closed. If $f(\lambda) = e^\lambda$, then $f(a) \in C$.*

PROOF:

1. By definition, C is closed under addition, multiplication and multiplication by positive scalars. Since $p(a) = \alpha_n a^n + \cdots + \alpha_1 a + \alpha_0$, it follows that $p(a) \in C$.
2. First note that $f(a) = e^a = \sum_{n=0}^{\infty}(1/n!)a^n$ ([**2**, p. 38]). Then $f(a) \in C$ follows from the defining properties of C, together with the fact that C is closed. ∎

We provide a more general result in Theorem 4.14, if a satisfies certain spectral properties. We begin with

THEOREM 4.11. *Let A be a Banach algebra and $a \in A$ such that $\rho(a)$ is a pole of order k of the resolvent of a. Suppose that f is a complex valued function, analytic at least on an open disk of the form $D(\rho(a), R)$. Let $g(\lambda) = f(\lambda)(\lambda 1 - a)^{-1}$ and let a_n denote the coefficient of $(\lambda-\rho(a))^n$ in the Laurent series of g around $\rho(a)$, for all $n \in \mathbb{Z}$.*

1. *If $f(\rho(a)) = 0$ and the order of f at $\rho(a)$ is k, then $a_{-1} = 0$.*

Suppose, in addition, that (A, C) is an ordered Banach algebra with C closed, $a \in C$ and $f(\lambda) > 0$ for all λ in the real interval $(\rho(a), \rho(a)+R)$.

2. *If $f(\rho(a)) > 0$, then $a_{-k} \in C$.*
3. *If $f(\rho(a)) = 0$ and the order of f at $\rho(a)$ is $k-1$, then $a_{-1} \in C$.*

PROOF:

1. If $f(\rho(a)) = 0$ and the order of f at $\rho(a)$ is k, then the order of g at $\rho(a)$ is zero, so that the residue of g at $\rho(a)$ is zero. Hence $a_{-1} = 0$.
2. If $f(\rho(a)) > 0$, then the order of g at $\rho(a)$ is $-k$, so that $a_{-k} = \lim_{\lambda \to \rho(a)} (\lambda - \rho(a))^k g(\lambda)$. Restricting λ to the interval $(\rho(a), \rho(a) + R)$, we obtain $a_{-k} = \lim_{\lambda \to \rho(a)^+} (\lambda - \rho(a))^k f(\lambda)(\lambda 1 - a)^{-1}$. Since C is closed, the assumption on f, together with Proposition 4.6, yield $a_{-k} \in C$.
3. If $f(\rho(a)) = 0$ and the order of f at $\rho(a)$ is $k - 1$, then the order of g at $\rho(a)$ is -1, so that $a_{-1} = \lim_{\lambda \to \rho(a)} (\lambda - \rho(a)) g(\lambda) = \lim_{\lambda \to \rho(a)^+} (\lambda - \rho(a)) f(\lambda)(\lambda 1 - a)^{-1}$. Once again the assumptions, together with Proposition 4.6, yield $a_{-1} \in C$. ∎

By taking $f(\lambda) = 1$ in Theorem 4.11 we rediscover a well-known ordered Banach algebra-result ([**7**, Theorem 3.2]):

COROLLARY 4.12. *Let (A, C) be an ordered Banach algebra with C closed, and $a \in C$ such that $\rho(a)$ is a pole of order k of the resolvent of a. Let $g(\lambda) = (\lambda 1 - a)^{-1}$ and let a_n denote the coefficient of $(\lambda - \rho(a))^n$ in the Laurent series of g around $\rho(a)$, for all $n \in \mathbb{Z}$. Then $a_{-k} \in C$.*

Recalling that $a_{-1} = p$, where p is the spectral idempotent associated with a and $\rho(a)$, we have

COROLLARY 4.13. *Let (A, C) be an ordered Banach algebra with C closed, and $a \in C$ such that $\rho(a)$ is a simple pole of the resolvent of a. If p is the spectral idempotent associated with a and $\rho(a)$, then $p \in C$.*

The following theorem gives some results of the form "if $a \in C$, then $f(a) \in C$".

THEOREM 4.14. *Let A be a Banach algebra and $a \in A$ such that $\mathrm{Sp}\,(a) = \{\lambda_1, \dots, \lambda_m\}$ $(m \geqslant 1)$ where $\lambda_1 = \rho(a)$ and λ_j is a pole of order k_j of the resolvent of a $(j = 1, \dots, m)$. Let f be any complex valued function, analytic at least on a neighbourhood of $\mathrm{Sp}\,(a)$, such that f has a zero of order k_j at λ_j $(j = 2, \dots, m)$.*

1. *If $f(\rho(a)) = 0$ and the order of f at $\rho(a)$ is k_1, then $f(a) = 0$.*

Suppose, in addition, that (A, C) is an ordered Banach algebra with C closed, $a \in C$ and $f(\lambda) > 0$ for all λ in a real interval of the form $(\rho(a), \rho(a) + R)$.

2. *If $f(\rho(a)) > 0$ and $k_1 = 1$, then $f(a) \in C$.*
3. *If $f(\rho(a)) = 0$ and the order of f at $\rho(a)$ is $k_1 - 1$, then $f(a) \in C$.*

PROOF: By the holomorphic functional calculus an element $f(a) = (1/2\pi i) \int_\Gamma g(\lambda)\, d\lambda \in A$ is defined, where $g(\lambda) = f(\lambda)(\lambda 1 - a)^{-1}$ and we may suppose that Γ is a union of small circles (say with radii $r_1, \dots, r_m$) with centres $\lambda_1, \dots, \lambda_m$. Therefore $f(a)$

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$= \sum_{j=1}^{m} (1/2\pi i) \int_{C(\lambda_j, r_j)} g(\lambda)\, d\lambda$. Since the order of g at λ_j is zero, it follows that $\int_{C(\lambda_j, r_j)} g(\lambda)$ $d\lambda = 0$, for $j = 2, \dots, m$, so that $f(a) = (1/2\pi i) \int_{C(\rho(a), r_1)} g(\lambda)\, d\lambda$. Since g is analytic in a deleted neighbourhood of $\rho(a)$ containing $C\big(\rho(a), r_1\big)$, the quantity $(1/2\pi i) \int_{C(\rho(a), r_1)} g(\lambda)$ $d\lambda$ is the residue of g at $\rho(a)$. Therefore, if a_n denotes the coefficient of $\big(\lambda - \rho(a)\big)^n$ in the Laurent series of g around $\rho(a)$, for all $n \in \mathbb{Z}$, then $f(a) = a_{-1}$. The results now follow from Theorem 4.11. ∎

Corollary 4.9 can now be obtained as a consequence of Theorem 4.14:

Corollary 4.15. *Let A be a Banach algebra and $a \in A$ such that* $\mathrm{Sp}\,(a) = \big\{\rho(a)\big\}$. *Let $k \in \mathbb{N}$.*

1. *If $\rho(a)$ is a pole of order k of the resolvent of a, then $\big(a - \rho(a)1\big)^k = 0$.*
2. *If $\rho(a)$ is a simple pole of the resolvent of a, then $a = \rho(a)1$. It follows that, if C is an algebra cone of A, then*

$$\rho(a) \geqslant 1 \quad \Rightarrow \quad a - 1 \in C.$$

Suppose, in addition, that (A, C) is an ordered Banach algebra with C closed, and $a \in C$.

3. *If $\rho(a)$ is a pole of order $k+1$ of the resolvent of a, then $\big(a - \rho(a)1\big)^k \in C$.*
4. *If $\rho(a)$ is a pole of order 2 of the resolvent of a, then $a \geqslant \rho(a)1$. It follows that*

$$\rho(a) \geqslant 1 \quad \Rightarrow \quad a - 1 \in C.$$

Proof: Let $f(\lambda) = \big(\lambda - \rho(a)\big)^k$. Then f is an entire function with a zero of order k at $\rho(a)$ and $f(\lambda) > 0$ for all real $\lambda > \rho(a)$. Furthermore, if $f(a) = (1/2\pi i) \int_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\, d\lambda$ (with Γ a small circle with centre $\rho(a)$), then $f(a) = \big(a - \rho(a)1\big)^k$.

1. If $\rho(a)$ is a pole of order k of the resolvent of a, then $f(a) = 0$, by Theorem 4.14 1. Hence $\big(a - \rho(a)1\big)^k = 0$.
2. Follows from 1.
3. If $\rho(a)$ is a pole of order $k+1$ of the resolvent of a, then $f(a) \in C$, by Theorem 4.14 3. Hence $\big(a - \rho(a)1\big)^k \in C$.
4. Follows from 3. ∎

We conclude this discussion by giving some more corollaries of Theorem 4.14, involving the sine and log functions.

Corollary 4.16. *Let A be a Banach algebra and $a \in A$ such that $\rho(a) = k\pi \in \mathrm{Sp}\,(a)$ with $k \in \mathbb{N}$ an even number, and*

$$\mathrm{Sp}\,(a) \backslash \big\{\rho(a)\big\} \subset \big\{n\pi : n \in \{0, \pm 1, \dots, \pm k\}\big\}.$$

1. *If each spectral value of a is a simple pole of the resolvent of a, then $\sin a = 0$.*

Suppose, in addition, that (A, C) is an ordered Banach algebra with C closed, and $a \in C$.

2. *If each element of* $\mathrm{Sp}\,(a)\backslash\{\rho(a)\}$ *is a simple pole and $\rho(a)$ is a pole of order 2 of the resolvent of a, then $\sin a \in C$.*

PROOF: Let $f(\lambda) = \sin\lambda$. Then f has simple zeroes at all spectral values of a and $f(\lambda) > 0$ for all λ in a real interval of the form $\big(\rho(a), \rho(a) + R\big)$. Since $f(a) = \sin a$,

1. Follows from Theorem 4.14 1.
2. Follows from Theorem 4.14 3. ∎

COROLLARY 4.17. *Let (A, C) be an ordered Banach algebra with C closed, and $a \in C$ such that $\rho(a) = \big(k + (1/2)\big)\pi \in \mathrm{Sp}\,(a)$ with $k \in \mathbb{N}$ an even number, and*

$$\mathrm{Sp}\,(a)\backslash\{\rho(a)\} \subset \big\{n\pi : n \in \{0, \pm 1, \dots, \pm k\}\big\}.$$

If each spectral value of a is a simple pole of the resolvent of a, then $\sin a \in C$.

PROOF: Let $f(\lambda) = \sin\lambda$. Then f has simple zeroes at all values in $\mathrm{Sp}\,(a)\backslash\{\rho(a)\}$. Furthermore, $f\big(\rho(a)\big) > 0$ and $f(\lambda) > 0$ for all λ in a real interval of the form $\big(\rho(a), \rho(a) + R\big)$. Since $f(a) = \sin a$, the result follows from Theorem 4.14 2. ∎

COROLLARY 4.18. *Let A be a Banach algebra and $a \in A$ such that $\mathrm{Sp}\,(a) = \{\rho(a)\}$ with $\rho(a) > 0$.*

1. *If $\rho(a) = 1$ is a simple pole of the resolvent of a, then $\log a = 0$.*

Suppose, in addition, that (A, C) is an ordered Banach algebra with C closed, and $a \in C$.

2. *If $\rho(a)$ is a simple pole of the resolvent of a and $\rho(a) > 1$, then $\log a \in C$.*
3. *If $\rho(a) = 1$ is a pole of order 2 of the resolvent of a, then $\log a \in C$.*

PROOF: Let $f(\lambda) = \log\lambda = \log|\lambda| + i\arg\lambda$. Then f is analytic on a neighbourhood of the spectrum of a, so that the element $\log a = (1/2\pi i)\int_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\,d\lambda \in A$ (where Γ is a small circle with centre $\rho(a)$ in the right half plane) is defined ([**2**, p. 40]). Furthermore, f has a simple zero at 1, and $f(\lambda) > 0$ for all real $\lambda > 1$. Hence the results follow from Theorem 4.14. ∎

The final corollary follows in a similar way from Theorem 4.14 2:

COROLLARY 4.19. *Let (A, C) be an ordered Banach algebra with C closed and $a \in C$ such that $\mathrm{Sp}\,(a) = \{1, \rho(a)\}$ (with $\rho(a) > 1$). If both 1 and $\rho(a)$ are simple poles of the resolvent of a, then $\log a \in C$.*

We now turn our attention to the case in which the algebra cone C of A is inverse-closed. (Some properties of inverse-closed algebra cones were investigated in the context of positive operators on Banach lattices in [**10**].)

Recalling the problem of providing conditions under which $f(a)$ will be positive if a is positive, we have the following result to complement Proposition 4.10 and Theorem 4.14:

Proposition 4.20. *Let (A, C) be an ordered Banach algebra with C inverse-closed, and $a \in C$. Let $p(\lambda) = \alpha_n\lambda^n + \cdots + \alpha_1\lambda + \alpha_0$ and $q(\lambda) = \beta_m\lambda^m + \cdots + \beta_1\lambda + \beta_0$ with $\alpha_n, \ldots, \alpha_0, \beta_m, \ldots, \beta_0$ real and positive. Suppose that $q(\lambda)$ has no zeroes in* Sp (a) *and let $r(\lambda) = \big(p(\lambda)/q(\lambda)\big)$. Then $r(a) \in C$.*

Proof: It follows from Proposition 4.10 1 that $p(a) \in C$ and $q(a) \in C$. By the Spectral Mapping Theorem $q(a)$ is invertible, and since C is inverse-closed, $\big(q(a)\big)^{-1} \in C$. Since $r(a) = p(a)\big(q(a)\big)^{-1}$ ([**1**, Lemma 3.3.1]), it follows that $r(a) \in C$. ∎

We now return to the problem of finding conditions such that if $a \in C$ and Sp $(a) = \{1\}$, then $a - 1 \in C$, under the assumption that C is inverse-closed. Here we extend the problem to the case $\delta(a) \geqslant 1$ (with no other restrictions on Sp (a)) (see Theorem 4.23).

The following lemma is obvious:

Lemma 4.21. *Let (A, C) be an ordered Banach algebra with a and b invertible elements of A such that $a \leqslant b$ and $a^{-1}, b^{-1} \geqslant 0$. Then $b^{-1} \leqslant a^{-1}$.*

Theorem 4.22. *Let (A, C) be an ordered Banach algebra with C closed and inverse-closed. If $a \in C$ and a is invertible, then*

1. *$a \geqslant \alpha 1$ for all $\alpha \geqslant 0$ with $\alpha < \delta(a)$, and*
2. *$a \leqslant \beta 1$ for all $\beta > \rho(a)$.*

Proof:

1. If $0 < \alpha < \delta(a)$, then $\big(1/\delta(a)\big) < (1/\alpha)$, so that $(1/\alpha) > \rho(a^{-1})$. It follows from Proposition 4.6 that $\big((1/\alpha)1 - a^{-1}\big)^{-1} \geqslant 0$. Therefore $(1/\alpha)1 - a^{-1} \geqslant 0$, so that $a^{-1} \leqslant (1/\alpha)1$, since C is inverse-closed. The result now follows by applying Lemma 4.21.
2. If $\beta > \rho(a)$, then $(\beta 1 - a)^{-1} \geqslant 0$, by Proposition 4.6. Since C is inverse-closed, it follows that $\beta 1 - a \geqslant 0$, and hence $a \leqslant \beta 1$. ∎

Using Theorem 4.22, we obtain results of the form "if $a \in C$ and $\delta(a) \geqslant 1$, then $a - 1 \in C$" and "if $a \in C$ and Sp $(a) = \{1\}$, then $a = 1$" (see Theorem 4.23). Let $C(0, 1)$ denote the circle with centre 0 and radius 1 in the complex plane.

Theorem 4.23. *Let (A, C) be an ordered Banach algebra with C closed and inverse-closed, and let $a \in C$. Then we have the following implications:*

1. *$\delta(a) > 1 \Rightarrow a > 1$ and $\delta(a) = 1 \Rightarrow a \geqslant 1$; hence $\delta(a) \geqslant 1 \Rightarrow a - 1 \in C$.*
2. *If a is invertible: $\rho(a) < 1 \Rightarrow a < 1$ and $\rho(a) = 1 \Rightarrow a \leqslant 1$; hence $\rho(a) \leqslant 1 \Rightarrow 1 - a \in C$.*

If, in addition, C is proper, then we also have:

3. Sp $(a) \subset C(0, 1) \Rightarrow a = 1$.
4. Sp $(a) = \{1\} \Rightarrow a = 1$.

PROOF:

1. Suppose $\delta(a) \geqslant 1$. Let (α_n) be a sequence of real numbers such that $0 \leqslant \alpha_n < \delta(a)$ and $\alpha_n \to 1$ as $n \to \infty$. Then $a \geqslant \alpha_n 1$, by Theorem 4.22 1. By taking limits as $n \to \infty$, it follows that $a \geqslant 1$, since C is closed. If $\delta(a) > 1$, the case $a = 1$ is not possible, so that then $a > 1$.
2. Suppose $\rho(a) \leqslant 1$. Let (β_n) be a sequence of real numbers such that $\rho(a) < \beta_n$ and $\beta_n \to 1$ as $n \to \infty$. Then $a \leqslant \beta_n 1$, by Theorem 4.22 2, so that $a \leqslant 1$, as in 1. If $\rho(a) < 1$, the case $a = 1$ is not possible, so that then $a < 1$.
3. If Sp $(a) \subset C(0,1)$, then $\delta(a) = 1 = \rho(a)$, so that both $a \geqslant 1$ and $a \leqslant 1$ hold. Since C is proper, it follows that $a = 1$.
4. Follows from 3. ∎

Finally we observe that in the case of a normal algebra cone C, the behaviour of the spectrum in 3 above is quite restricted.

If X is a set, let $\#X$ denote the number of elements in X.

LEMMA 4.24. *Let A be a Banach algebra and $a \in A$. If there exist a $k \in \mathbb{N}$ and a $0 \neq \lambda_0 \in \mathbb{C}$ such that* psp $(a^k) = \{\lambda_0\}$, *then* $\#$psp $(a) \leqslant k$.

PROOF: If $\lambda \in$ psp (a), then $\lambda^k \in$ psp (a^k). Equivalently, $\lambda^k = \lambda_0$ for all $\lambda \in$ psp (a). Hence psp (a) consists of some, or all, of the k-th complex roots of λ_0, so that the result follows. ∎

THEOREM 4.25. *Let (A, C) be an ordered Banach algebra with C closed and normal. If $a \in A$ and there exist a $k \in \mathbb{N}$ and an $\alpha > 0$ such that $a^k \geqslant \alpha 1$, then*

1. psp $(a^k) = \{\rho(a)^k\}$, *and*
2. $\#$psp $(a) \leqslant k$.

PROOF:

1. Since psp $(\beta a) = \beta$psp (a) for every $\beta \geqslant 0$, we may assume without loss of generality that $\rho(a) = 1$. Let $b = a^k - \alpha 1$. Then $b \geqslant 0$. Since $a^k = b + \alpha 1$, it follows that $1 = \rho(a^k) = \rho(b + \alpha 1)$, so that $1 = \sup\{|\lambda + \alpha| : \lambda \in \text{Sp}\,(b)\}$. Since $\rho(b) \in$ Sp (b), by Theorem 3.2, this supremum is exactly $\rho(b) + \alpha$. Hence $\rho(b) = 1 - \alpha$, so that Sp $(a^k) \subset \{\lambda + \alpha : |\lambda| \leqslant 1 - \alpha\}$.
   Now suppose $z \in$ psp (a^k). Then $z = \lambda + \alpha$ with $|\lambda| \leqslant 1 - \alpha$, so that $|z - \alpha| \leqslant 1 - \alpha$, and $|z| = 1$. Consequently $z \in \overline{D}(\alpha, 1 - \alpha) \cap C(0,1)$. Let $z = c + di$. Then $(c - \alpha)^2 + d^2 \leqslant (1 - \alpha)^2$ and $c^2 + d^2 = 1$, so that $2\alpha c \geqslant 2\alpha$, and hence $c \geqslant 1$, since $\alpha > 0$. Since $c^2 + d^2 = 1$, it follows that $c = 1$ and $d = 0$, so that $z = 1$. Hence the result follows.
2. Follows from Lemma 4.24. ∎

The proof of Theorem 4.25 1 follows the lines of the proof of [**12**, Theorem 2.10]. Theorems 4.22 1 and 4.25 1 now yield

THEOREM 4.26. *Let (A, C) be an ordered Banach algebra with C closed, normal and inverse-closed. If $a \in C$ is an invertible element, then* $\operatorname{psp}(a) = \{\rho(a)\}$.

The above theorem implies that if the algebra cone C in Theorem 4.23 is normal, then the only way in which the case $\operatorname{Sp}(a) \subset C(0,1)$ in 3 can occur, is if $\operatorname{Sp}(a) = \{1\}$, as in 4.

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