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LOGARITHMIC CONVEXITY OF AREA INTEGRAL MEANS FOR ANALYTIC FUNCTIONS II

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Abstract

For $0 < p < \infty$ and $-2 \le \alpha \le 0$ we show that the L^p integral mean on $r\mathbb{D}$ of an analytic function in the unit disk $\mathbb{D}$ with respect to the weighted area measure $(1 - |z|^2)^\alpha \, dA(z)$ is a logarithmically convex function of r on $(0, 1)$.

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1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. For any $f \in H(\mathbb{D})$ and $0 < p < \infty$, the classical integral means of f are defined by

$$M_p(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta, \quad 0 \le r < 1.$$

The well-known Hardy convexity theorem asserts that $M_p(f, r)$, as a function of r on $[0, 1)$, is nondecreasing and logarithmically convex. Recall that the logarithmic convexity of $g(r)$ simply means that $\log g(r)$ is a convex function $\log r$. The case $p = \infty$ corresponds to the Hadamard three-circles theorem. See [1, Theorem 1.5] for example.

In this paper we will consider integral means of analytic functions in the unit disk with respect to weighted area measures. Thus, for any real number α, we consider the measure

$$dA_\alpha(z) = (1 - |z|^2)^\alpha \, dA(z),$$

where dA is area measure on $\mathbb{D}$. For any $f \in H(\mathbb{D})$ and $0 < p < \infty$, we define

$$M_{p,\alpha}(f, r) = \frac{\int_{r\mathbb{D}} |f(z)|^p \, dA_\alpha(z)}{\int_{r\mathbb{D}} dA_\alpha(z)}, \quad 0 < r < 1,$$

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and call them area integral means of f.

The study of area integral means of analytic functions began in [6], where it was shown that for $\alpha \leq -1$, the function $M_{p,\alpha}(f, r)$ is bounded in r if and only if f belongs to the Hardy space H^p and, for $\alpha > -1$, $M_{p,\alpha}(f, r)$ is bounded in r if and only if f belongs to the weighted Bergman space A^p_α. See [1, 2] for the theories of Hardy and Bergman spaces, respectively.

It was also shown in [6] that each function $r \mapsto M_{p,\alpha}(f, r)$ is strictly increasing unless f is constant. Furthermore, for $p \geq 1$ and $\alpha \in \{-1, 0\}$, the function $\log M_{p,\alpha}(f, r)$ is convex in $\log r$. However, an example in [6] shows that $\log M_{2,1}(z, r)$ is concave in $\log r$. Consequently, the following conjecture was made in [6]: the function $\log M_{p,\alpha}(f, r)$ is convex in $\log r$ when $\alpha \leq 0$ and it is concave in $\log r$ when $\alpha > 0$.

It turned out that the logarithmic convexity of $M_{p,\alpha}(f, r)$ is much more complicated than was conjectured in [6]. Somewhat surprisingly, the problem is highly nontrivial even in the Hilbert-space case $p = 2$. More specifically, it was proved in [4] that for $p = 2$ and any $f \in H(\mathbb{D})$ the function $M_{2,\alpha}(f, r)$ is logarithmically convex when $-3 \leq \alpha \leq 0$, and this range for α is best possible. It was also proved in [4] that when $p \neq 2$ and f is a monomial, the function $M_{p,\alpha}(z^k, r)$ is logarithmically convex for $-2 \leq \alpha \leq 0$.

Area integral means of analytic functions were also studied in [3, 5].

The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that $0 < p < \infty$, $-2 \leq \alpha \leq 0$, and f is analytic in $\mathbb{D}$. Then the function $M_{p,\alpha}(f, r)$ is logarithmically convex.*

We have been unable to determine whether or not the range $\alpha \in [-2, 0]$ is best possible. In other words, we do not know if there exists a set Ω properly containing $[-2, 0]$ such that $M_{p,\alpha}(f, r)$ is logarithmically convex on $(0, 1)$ for all $p \in (0, \infty)$, all $\alpha \in \Omega$, and all $f \in H(\mathbb{D})$. It is certainly reasonable to expect that the logarithmic convexity of $M_{p,\alpha}(f, r)$ for all f will depend on both p and α. The ultimate problem is to find the precise dependence.

2. Preliminaries

The proof of Theorem 1.1 is 'elementary' but very laborious. It requires several preliminary results that we collect in this section. Throughout the paper we use the symbol $\equiv$ whenever a new notation is being introduced.

The next lemma was stated in [4] without proof. We provide a proof here for the sake of completeness.

Lemma 2.1. *Suppose that f is positive and twice differentiable on $(0, 1)$. Then:*

(i) *$f(x)$ is convex in $\log x$ if and only if*

$$f'(x) + x f''(x) \geq 0$$

for all $x \in (0, 1)$;

(ii) $f(x)$ *is convex in* $\log x$ *if and only if* $f(x^2)$ *is convex in* $\log x$*;*

(iii) $\log f(x)$ *is convex in* $\log x$ *if and only if*

$$D(f(x)) \equiv \frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} - x\left(\frac{f'(x)}{f(x)}\right)^2 \geq 0$$

for all $x \in (0, 1)$.

Proof. Let $t = \log x$. Then $y = f(x) = f(e^t)$. The convexity of y in t is equivalent to $d^2y/dt^2 \geq 0$. Since

$$\frac{dy}{dt} = f'(e^t)e^t$$

and

$$\frac{d^2y}{dt^2} = f''(e^t)e^{2t} + f'(e^t)e^t = x(xf''(x) + f'(x)),$$

we obtain the conclusion in part (i).

If $g(x) = f(x^2)$, then it is easy to check that

$$g'(x) + xg''(x) = 4x[f'(x^2) + x^2 f''(x^2)].$$

So, part (ii) follows from part (i).

Similarly, part (iii) follows if we apply part (i) to the function $h(x) = \log f(x)$. □

Recall that $M_{p,\alpha}(f, r)$ is a quotient of two positive functions. It is thus natural that we will need the following result.

Lemma 2.2. *Suppose that* $f = f_1/f_2$ *is a quotient of two positive and twice-differentiable functions on* $(0, 1)$. *Then*

$$D(f(x)) = D(f_1(x)) - D(f_2(x)) \tag{2.1}$$

for $x \in (0, 1)$. *Consequently,* $\log f(x)$ *is convex in* $\log x$ *if and only if*

$$D(f_1(x)) - D(f_2(x)) \geq 0 \tag{2.2}$$

on $(0, 1)$.

Proof. Observe that

$$D(f(x)) = \left(\frac{xf'(x)}{f(x)}\right)' = (x(\log f(x))')'.$$

Since $\log f = \log f_1 - \log f_2$, we obtain the identity in (2.1). By part (iii) of Lemma 2.1, $\log f(x)$ is convex in $\log x$ if and only if inequality (2.2) holds. □

To simplify notation, we are going to write

$$x = r^2, \quad M(r) = M_p(f, \sqrt{r}).$$

Without loss of generality, we assume throughout the paper that f is not a constant, so that M and M' are always positive.

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We also write

$$h = h(x) = \int_0^r M_p(f,t)(1-t^2)^\alpha 2t\,dt = \int_0^x M(t)(1-t)^\alpha\,dt$$

and

$$\varphi = \varphi(x) = \int_0^r (1-t^2)^\alpha 2t\,dt = \int_0^x (1-t)^\alpha\,dt.$$

By part (ii) of Lemma 2.1, the logarithmic convexity of $M_{p,\alpha}(f,r)$ on $(0,1)$ is equivalent to the logarithmic convexity of $h(x)/\varphi(x)$ on $(0,1)$. According to Lemma 2.2, this will be accomplished if we can show that the difference

$$\Delta(x) \equiv D(h(x)) - D(\varphi(x)) \tag{2.3}$$

is nonnegative on $(0,1)$. This will be done in the next section.

We will need several preliminary estimates on the functions h and φ. The next lemma shows where and why we need the assumption $-2 \le \alpha \le 0$.

Lemma 2.3. *Suppose that $-2 \le \alpha \le 0$ and $x \in [0,1)$. Then:*

(i) $1 - (\alpha+1)\varphi(x) - (1-x)\varphi'(x) = 0$;
(ii) $\varphi(x) - x \ge 0$;
(iii) $g_1(x) \equiv x(1-x-\alpha x) - (1-x)\varphi(x) \ge 0$;
(iv) $g_2(x) \equiv (\alpha+2)\varphi^2(x) - 2(1+x+\alpha x)\varphi(x) + 2x \ge 0$;
(v) $g_3(x) \equiv \varphi^2(x) - (1+x+\alpha x)\varphi(x) + x \ge 0$.

Proof. If $\alpha \ne -1$, part (i) follows from the facts that

$$\varphi(x) = \frac{1-(1-x)^{\alpha+1}}{\alpha+1}, \quad \varphi'(x) = (1-x)^\alpha.$$

If $\alpha = -1$, part (i) follows from the fact that

$$\varphi'(x) = \frac{1}{1-x}.$$

Part (ii) follows from the fact that $(1-t)^\alpha \ge 1$ for $\alpha \le 0$ and $t \in [0,1)$.

A direct computation shows that

$$g_1'(x) = 1 - 2x - 2\alpha x + \varphi(x) - (1-x)\varphi'(x).$$

It follows from part (i) that

$$g_1'(x) = (\alpha+2)(\varphi(x) - x) - \alpha x.$$

By part (ii) and the assumption that $-2 \le \alpha \le 0$, we have $g_1'(x) \ge 0$ for $x \in [0,1)$. Thus, $g_1(x) \ge g_1(0) = 0$ for all $x \in [0,1)$. This proves (iii).

Another computation gives

$$\begin{aligned} g_2'(x) &= 2(\alpha+2)\varphi(x)\varphi'(x) - 2(\alpha+1)\varphi(x) - 2(1+x+\alpha x)\varphi'(x) + 2 \\ &= 2(\alpha+2)\varphi(x)\varphi'(x) - 2(1+x+\alpha x)\varphi'(x) + 2(1-x)^{\alpha+1} \\ &= 2(\alpha+2)(\varphi(x) - x)\varphi'(x). \end{aligned}$$

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Since

$$\alpha + 2 \geq 0, \quad \varphi(x) - x \geq 0, \quad \varphi'(x) = (1-x)^\alpha \geq 0,$$

we have $g_2'(x) \geq 0$ for all $x \in [0,1)$. Therefore, $g_2(x) \geq g_2(0) = 0$ for all $x \in [0,1)$. This proves (iv).

A similar computation produces

$$\begin{aligned} g_3'(x) &= 2\varphi(x)\varphi'(x) - (\alpha+1)\varphi - (1 + x + \alpha x)\varphi'(x) + 1 \\ &= 2\varphi(x)\varphi'(x) - (1 + x + \alpha x)\varphi'(x) + (1-x)^{\alpha+1} \\ &= (2\varphi(x) - (\alpha+2)x)\varphi'(x) \\ &\geq 2(\varphi(x) - x)\varphi'(x) \geq 0, \end{aligned}$$

which yields $g_3(x) \geq g_3(0) = 0$ for all $x \in [0,1)$. This proves (v) and completes the proof of the lemma. □

Let us write

$$\begin{aligned} A = A(x) &= \frac{\varphi(x) - x}{\varphi^2(x)}, \\ B = B(x) &= (1 - x - \alpha x) + x(1-x)\frac{M'(x)}{M(x)}, \\ C = C(x) &= x(1-x)^{\alpha+1}. \end{aligned}$$

By the proof of Lemma 2.3, $A(x)$ is positive on $(0,1)$. Also, $B(x)$ is positive on $(0,1)$ as $\alpha \leq 0$ and $M'/M > 0$. It is obvious that $C(x)$ is positive on $(0,1)$ as well.

Lemma 2.4. *We have* $B^2 - 4AC > 0$ *on* $(0,1)$.

Proof. We have

$$B^2 - 4AC = \left[(1 - x - \alpha x) + x(1-x)\frac{M'}{M}\right]^2 - 4x(1-x)^{\alpha+1}\frac{\varphi - x}{\varphi^2}.$$

It follows from part (i) of Lemma 2.3 and the identity $\varphi'(x) = (1-x)^\alpha$ that

$$(\alpha+1)x\varphi = x - x(1-x)^{\alpha+1}.$$

Rewrite this as

$$-(1 - x - \alpha x)\varphi + \varphi - x = -x(1-x)^{\alpha+1},$$

from which

$$-4x(1-x)^{\alpha+1}\frac{\varphi - x}{\varphi^2} = -\frac{4(\varphi - x)(1 - x - \alpha x)}{\varphi} + \frac{4(\varphi - x)^2}{\varphi^2}.$$

Combining this with the earlier expression for $B^2 - 4AC$, we see that $B^2 - 4AC$ is equal to the sum of

$$\left[(1 - x - \alpha x) - \frac{2(\varphi - x)}{\varphi}\right]^2$$

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and

$$x^2(1-x)^2\Big(\frac{M'}{M}\Big)^2 + 2x(1-x)(1-x-\alpha x)\frac{M'}{M}.$$

The first summand above is always nonnegative, while the second summand is always positive, because $\alpha \le 0$, $M' > 0$, and $M > 0$. This proves the desired result. □

3. Proof of main result

This section is devoted to the proof of Theorem 1.1. As was remarked in the previous section, we just need to show that the difference function $\Delta(x)$ defined in (2.3) is always nonnegative on $(0, 1)$. Continuing the convention in [4], we will also use the notation $A \sim B$ to mean that A and B have the same sign.

Since

$$\varphi' = (1-x)^{\alpha}, \quad \varphi'' = -\alpha(1-x)^{\alpha-1},$$

$$\begin{aligned} D(\varphi(x)) &= \frac{\varphi\varphi' + x\varphi\varphi'' - x(\varphi')^2}{\varphi^2} \\ &= \frac{(1-x)^{\alpha-1}}{\varphi^2}[\varphi - x[(\alpha+1)\varphi + (1-x)^{\alpha+1}]]. \end{aligned}$$

By part (i) of Lemma 2.3,

$$(\alpha+1)\varphi + (1-x)^{\alpha+1} = (\alpha+1)\varphi + (1-x)\varphi' = 1.$$

Therefore,

$$D(\varphi) = \frac{\varphi(x) - x}{\varphi^2(x)}(1-x)^{\alpha-1}.$$

On the other hand,

$$h' = h'(x) = M(x)(1-x)^{\alpha}$$

and

$$h'' = h''(x) = [(1-x)M'(x) - \alpha M(x)](1-x)^{\alpha-1}.$$

It follows from simple calculations that

$$D(h) = \frac{hh' + xhh'' - x(h')^2}{h^2} = \frac{(1-x)^{\alpha-1}M}{h^2}[hB - CM].$$

Therefore,

$$\begin{aligned} \Delta(x) &= \frac{(1-x)^{\alpha-1}M}{h^2}(hB - CM) - (1-x)^{\alpha-1}A \sim M(hB - CM) - Ah^2 \\ &= -Ah^2 + MBh - CM^2. \end{aligned}$$

The function $\Delta(x)$ is continuous on $[0, 1)$, so we just need to show that $\Delta(x) \ge 0$ for x in the open interval $(0, 1)$.

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For $x \in (0,1)$, we have $A > 0$ and $M > 0$, so

$$\begin{aligned}\Delta(x) \ge 0 &\iff Ah^2 + CM^2 \le hBM\\ &\iff \frac{h^2}{M^2} + \frac{C}{A} \le \frac{hB}{MA}\\ &\iff \frac{h^2}{M^2} - \frac{hB}{MA} + \frac{B^2}{4A^2} \le \frac{B^2}{4A^2} - \frac{C}{A}\\ &\iff \Big(\frac{h}{M} - \frac{B}{2A}\Big)^2 \le \frac{B^2 - 4AC}{4A^2}.\end{aligned}$$

Recall from Lemma 2.4 and the remark preceding it that $A > 0$ and $B^2 - 4AC \ge 0$. Thus, the proof of Theorem 1.1 will be completed if we can show that

$$-\frac{\sqrt{B^2 - 4AC}}{2A} \le \frac{h}{M} - \frac{B}{2A} \le \frac{\sqrt{B^2 - 4AC}}{2A}. \tag{3.1}$$

Since the function M is positive and increasing,

$$B(x) \ge 1 - x - \alpha x \ge 0, \quad h(x) \le \int_0^x M(x)(1-t)^\alpha \, dt = M(x)\varphi(x).$$

It follows from this, the proof of Lemma 2.4, part (ii) of Lemma 2.3, and the triangle inequality that

$$\begin{aligned}\frac{B + \sqrt{B^2 - 4AC}}{2A} &\ge \frac{(1 - x - \alpha x) + \big|1 - x - \alpha x - \frac{2(\varphi - x)}{\varphi}\big|}{2A}\\ &\ge \frac{\frac{2(\varphi - x)}{\varphi}}{2A} = \varphi \ge \frac{h}{M}.\end{aligned}$$

This proves the right half of (3.1).

To prove the left half of (3.1), we write

$$\delta = \delta(x) = h - M\frac{B - \sqrt{B^2 - 4AC}}{2A}$$

for $x \in (0,1)$ and proceed to show that $\delta(x)$ is always nonnegative. It follows from the elementary identity

$$\frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{2C}{B + \sqrt{B^2 - 4AC}}$$

that $\delta(x) \to 0$ as $x \to 0^+$. If we can show that $\delta'(x) \ge 0$ for all $x \in (0,1)$, then we will obtain

$$\delta(x) \ge \lim_{t \to 0^+} \delta(t) = 0, \quad x \in (0,1).$$

The rest of the proof is thus devoted to proving the inequality $\delta'(x) \ge 0$ for $x \in (0,1)$.

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By direct computation,

$$\delta'(x) = M\varphi' - \frac{M'A - MA'}{2A^2}[B - \sqrt{B^2 - 4AC}] \\ - \frac{M}{2A}\Big[B' - \frac{BB' - 2(A'C + AC')}{\sqrt{B^2 - 4AC}}\Big].$$

By part (ii) of Lemma 2.1 and Hardy's convexity theorem, M is logarithmically convex. According to part (iii) of Lemma 2.1, the logarithmic convexity of M is equivalent to

$$\Big(x\frac{M'}{M}\Big)' = D(M(x)) \geq 0.$$

It follows that

$$B' = -(\alpha + 1) - x\frac{M'}{M} + (1 - x)\Big(x\frac{M'}{M}\Big)' \\ \geq -(\alpha + 1) - x\frac{M'}{M} \equiv B_0.$$

Therefore,

$$\delta' \geq M\varphi' - \frac{M'A - MA'}{2A^2}[B - \sqrt{B^2 - 4AC}] - \frac{M}{2A}\Big[B_0 - \frac{BB_0 - 2(A'C + AC')}{\sqrt{B^2 - 4AC}}\Big] \\ \sim x(1 - x)\Big[2A^2\varphi' - A\Big(\frac{M'}{M}B + B_0\Big) + BA'\Big]\sqrt{B^2 - 4AC} \\ + x(1 - x)\Big[AB\Big(\frac{M'}{M}B + B_0\Big) - A'B^2 + 2AA'C\Big] - 2x(1 - x)A^2\Big(2C\frac{M'}{M} + C'\Big) \\ \equiv d.$$

Here $\sim$ follows from multiplying the expression on its left by the positive function

$$\frac{2x(1 - x)A^2}{M}.$$

We will show that $d \geq 0$ for all $x \in (0, 1)$. To this end, we are going to introduce seven auxiliary functions. More specifically, we let

$$y = x(1 - x)\frac{M'}{M}, \\ A_1 = x(1 - x)A'(x) \\ = \frac{x}{\varphi^3}[(\alpha + 1)\varphi^2 - (2 + x + 2\alpha x)\varphi + 2x], \\ B_1 = x(1 - x)\Big(\frac{M'}{M}B + B_0\Big) \\ = -(\alpha + 1)x(1 - x) + (1 - 2x - \alpha x)y + y^2, \\ C_1 = x(1 - x)\Big(2C\frac{M'}{M} + C'\Big) \\ = x(1 - (\alpha + 1)\varphi)(1 - 2x - \alpha x + 2y),$$

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$$E = 2A^2C - AB_1 + A_1B,$$
$$F = ABB_1 - A_1B^2 + 2AA_1C - 2A^2C_1,$$
$$S = \sqrt{B^2 - 4AC}.$$

Note that the computation for A_1 above uses part (i) of Lemma 2.3; the computation for B_1 uses the definitions of y, B, and B_0; and the computation for C_1 uses the identities

$$\begin{aligned} C &= x(1-x)\varphi', \\ C' &= (1-x)^{\alpha+1} - (\alpha+1)x(1-x)^{\alpha} \\ &= (1-x)\varphi' - (\alpha+1)x\varphi', \\ (1-x)\varphi' &= 1 - (\alpha+1)\varphi. \end{aligned}$$

In terms of these newly introduced functions, we can rewrite $d = ES + F$.

It is easy to see that we can write every function appearing in E, F, and S as a function of (x, y, φ). In fact,

$$\begin{aligned} E = {} & \frac{x^2}{\varphi^4}(1-(\alpha+1)\varphi)[(\alpha+2)\varphi^2 - 2(1+x+\alpha x)\varphi + 2x] \\ & + \frac{1}{\varphi^3}[(3x+2\alpha x-1)\varphi^2 - x(1+3x+3\alpha x)\varphi + 2x^2]y - \frac{\varphi - x}{\varphi^2}y^2 \end{aligned}$$

and

$$\begin{aligned} F = {} & \frac{x^2}{\varphi^5}(1-(\alpha+1)\varphi)[(\alpha+2)\varphi^2 - 2(1+x+\alpha x)\varphi + 2x][(1+x+\alpha x)\varphi - 2x] \\ & + \frac{1}{\varphi^4}[(1-2x+5x^2-\alpha x+8\alpha x^2+3\alpha^2x^2)\varphi^3 \\ & - x(1+6x+5x^2+5\alpha x+10\alpha x^2+5\alpha^2x^2)\varphi^2 \\ & + 4x^2(1+2x+2\alpha x)\varphi - 4x^3]y \\ & + \frac{1}{\varphi^3}[(2-4x-3\alpha x)\varphi^2 + 4(\alpha+1)x^2\varphi - 2x^2]y^2 + \frac{\varphi - x}{\varphi^2}y^3. \end{aligned}$$

Note that we have verified the formulas above for E and F with the help of Maple. Also, it follows from the proof of Lemma 2.4 that

$$S = \sqrt{y^2 + 2(1-x-\alpha x)y + \frac{1}{\varphi^2}((1+x+\alpha x)\varphi - 2x)^2}.$$

Another tedious calculation with the help of Maple shows that $F^2 - E^2S^2$ is equal to

$$\frac{4yx^2}{\varphi^8}(\varphi - x)^3(1-(\alpha+1)\varphi)[\varphi^2 - (1+x+\alpha x)\varphi + x](y - y_0),$$

where

$$y_0 = \frac{[x(1-x-\alpha x) - (1-x)\varphi][(\alpha+2)\varphi^2 - 2(1+x+\alpha x)\varphi + 2x]}{(\varphi - x)[\varphi^2 - (1+x+\alpha x)\varphi + x]}.$$

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This, together with Lemma 2.3, tells us that

$$F^2 - E^2S^2 \sim y - y_0. \tag{3.2}$$

By Lemma 2.3 again, we always have $y_0 \geq 0$.

Recall that E, F, and S are formally algebraic functions of (x, y, φ), where $x \in (0, 1)$, $y \geq 0$, and $\varphi > 0$. For the remainder of this proof, we fix x (hence φ as well) and think of $E = E(y)$, $F = F(y)$, and $S = S(y)$ as functions of a single variable y on $[0, \infty)$. Thus, E is a quadratic function of y, F is a cubic polynomial of y, and S is the square root of a quadratic function that is nonnegative for $y \in [0, \infty)$. There are two cases for us to consider: $0 \leq y \leq y_0$ and $y > y_0$.

Recall that

$$E(0) = \frac{x^2}{\varphi^4}(1 - (\alpha + 1)\varphi)[(\alpha + 2)\varphi^2 - 2(1 + x + \alpha x)\varphi + 2x].$$

It follows from Lemma 2.3 that $E(0) \geq 0$. Also, direct calculations along with Lemma 2.3 show that

$$\begin{aligned} E(y_0) &= \frac{(1 - (\alpha + 1)\varphi)(\varphi - x)^4[(\alpha + 2)\varphi^2 - 2(1 + x + \alpha x)\varphi + 2x]}{\varphi^4[\varphi^2 - (1 + x + \alpha x)\varphi + x]^2} \\ &\geq 0. \end{aligned}$$

Similarly, direct computations along with Lemma 2.3 give us

$$\begin{aligned} F(y_0) &= \frac{(1 - (\alpha + 1)\varphi)(\varphi - x)^3[(\alpha + 2)\varphi^2 - 2(1 + x + \alpha x)\varphi + 2x]}{\varphi^5[\varphi^2 - (1 + x + \alpha x)\varphi + x]^3} \\ &\quad \times [x[\varphi^2 - (1 + x + \alpha x)\varphi + x]^2 - (1 - (\alpha + 1)\varphi)(\varphi - x)^3] \\ &\sim x[\varphi^2 - (1 + x + \alpha x)\varphi + x]^2 - (1 - (\alpha + 1)\varphi)(\varphi - x)^3. \end{aligned}$$

For $x \in (0, 1)$ and $\alpha \in [-2, 0]$,

$$\begin{aligned} &[\varphi^2 - (1 + x + \alpha x)\varphi + x] - (1 - (\alpha + 1)\varphi)(\varphi - x) \\ &\quad = (\alpha + 2)\varphi^2 - 2(1 + x + \alpha x)\varphi + 2x > 0 \end{aligned}$$

and

$$\begin{aligned} &x[\varphi^2 - (1 + x + \alpha x)\varphi + x] - (\varphi - x)^2 \\ &\quad = \varphi[x(1 - x - \alpha x) - (1 - x)\varphi] > 0. \end{aligned}$$

It follows that $F(y_0) > 0$.

Since $E(y)$ is a quadratic function that is concave downward, it is nonnegative if and only if y belongs to a certain closed interval. This closed interval contains 0 and y_0, so it must contain $[0, y_0]$ as well. Therefore, $E(y) \geq 0$ for $0 \leq y \leq y_0$. It follows from this and (3.2) that

$$d \geq E(y)S(y) - |F(y)| \sim E^2(y)S^2(y) - F^2(y) \sim y_0 - y \geq 0$$

for $0 \leq y \leq y_0$.

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In the case when $y > y_0$,

$$F^2(y) - E^2(y)S^2(y) \sim y - y_0 > 0.$$

In particular, $F(y)$ is nonvanishing on (y_0, ∞). Since $F(y)$ is continuous on $[y_0, \infty)$ and $F(y_0) > 0$, we conclude that $F(y) > 0$ for all $y > y_0$. Combining this with (3.2),

$$d \geq F(y) - |E(y)|S(y) \sim F^2(y) - E^2(y)S^2(y) \sim y - y_0 > 0.$$

This shows that d is always nonnegative and completes the proof of Theorem 1.1.

4. Further results and remarks

The proof of Theorem 1.1 in the previous section actually gives the following more general result.

THEOREM 4.1. *Let* $0 < p < \infty$ *and* $-2 \leq \alpha \leq 0$. *If* $M(x)$ *is nondecreasing and* $\log M(x)$ *is convex in* $\log x$ *for* $x \in (0, 1)$, *then the function*

$$x \mapsto \log \frac{\int_0^x M(t)(1-t)^\alpha \, dt}{\int_0^x (1-t)^\alpha \, dt}$$

is also convex in $\log x$ *for* $x \in (0, 1)$.

The logarithmic convexity of $M_{p,\alpha}(f, r)$ is equivalent to following: if $0 < r_1 < r_2 < 1$, $0 < \theta < 1$, and $r = r_1^\theta r_2^{1-\theta}$, then

$$M_{p,\alpha}(f, r) \leq (M_{p,\alpha}(f, r_1))^\theta (M_{p,\alpha}(f, r_2))^{1-\theta}.$$

Furthermore, equality occurs if and only if $\log M_{p,\alpha}(f, r) = a \log r + b$ for some constants a and b, which appears to happen only in very special situations. For example, if $\alpha = 0$, then it appears that $M_{p,0}(f, r) = ce^{ar}$ (where c and a are constants) only when f is a monomial.

Finally, we mention that for $\alpha < -2$ and $y < y_0$,

$$\lim_{x \to 1} [(\alpha + 2)\varphi^2 - 2(1 + x + \alpha x)\varphi + 2x] = -\infty.$$

Thus, $E(y) < 0$ for x close enough to 1. This implies that $ES + F < 0$ for x close enough to 1, so d (and δ') is not necessarily positive for all $x \in [0, 1)$. Thus, the proof of Theorem 1.1 breaks down here in the case $\alpha < -2$. However, δ can still be positive. It is just that our approach does not work any more.

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