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A REMARK ON THE STABLE REAL FORMS OF COMPLEX VECTOR BUNDLES OVER MANIFOLDS

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Abstract

Let M be an n-dimensional closed oriented smooth manifold with $n \equiv 4 \bmod 8$, and η be a complex vector bundle over M. We determine the final obstruction for η to admit a stable real form in terms of the characteristic classes of M and η. As an application, we obtain the criteria to determine which complex vector bundles over a simply connected four-dimensional manifold admit a stable real form.

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1. Introduction

First we introduce some notation. For a topological space X, let $\mathrm{Vect}_{\mathbb{C}}(X)$ (respectively, $\mathrm{Vect}_{\mathbb{R}}(X)$) be the set of isomorphism classes of complex (respectively, real) vector bundles on X, and let $\widetilde{K}(X)$ (respectively, $\widetilde{KO}(X)$) be the reduced KU-group (respectively, reduced KO-group) of X, which is the set of stable equivalence classes of complex (respectively, real) vector bundles over X. For a map $f : X \to Y$ between topological spaces X and Y, denote by

$$f_u^* : \widetilde{K}(Y) \to \widetilde{K}(X) \quad \text{and} \quad f_o^* : \widetilde{KO}(Y) \to \widetilde{KO}(X)$$

the induced homomorphisms. For $\xi \in \mathrm{Vect}_{\mathbb{R}}(X)$ (respectively, $\eta \in \mathrm{Vect}_{\mathbb{C}}(X)$), we will denote by $\tilde{\xi} \in \widetilde{KO}(X)$ (respectively, $\tilde{\eta} \in \widetilde{K}(X)$) the stable class of ξ (respectively, η) (see Hilton [5, page 62]), $w_i(\xi)$ the ith Stiefel–Whitney class of ξ, $c_i(\eta)$ the ith Chern class of η and $\mathrm{ch}(\tilde{\eta})$ the Chern character of $\tilde{\eta}$. In particular, if X is a smooth manifold, then $w_i(X) = w_i(TX)$ is the ith Stiefel–Whitney class of X, where TX is the tangent bundle of X.

It is known that there is a complexification homomorphism

$$\tilde{c}_X : \widetilde{KO}(X) \to \widetilde{K}(X),$$

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which is induced from the complexification map

$$c_X : \mathrm{Vect}_{\mathbb{R}}(X) \to \mathrm{Vect}_{\mathbb{C}}(X)$$

defined by $c_X(\xi) = \xi \otimes \mathbb{C}$.

Let $\eta \in \mathrm{Vect}_{\mathbb{C}}(X)$ be a complex vector bundle over X. We say that η admits a *real form* (respectively, *stable real form*) over X if there exists a real vector bundle ξ over X such that $c_X(\xi) = \eta$ (respectively, $\tilde{c}_X(\tilde{\xi}) = \tilde{\eta}$). Clearly, if η admits a real form over X, then it admits a stable real form over X. It is known that if η admits a stable real form over X, then we must have

$$2c_{2i+1}(\eta) = 0$$

for any $i \in \mathbb{Z}$ (see Milnor [8, page 174]).

On the one hand, we know that the tangent bundle of a complex manifold admits a real form if the complex manifold admits a real form (see Kulkarni [6] or Totaro [11]). On the other hand, index theory tells us that the complex vector bundles which admit a stable real form may have beautiful properties (see, for example, Atiyah and Hirzebruch [1, Corollary 2(ii)], [7, page 286, Theorem 2.6]). This leads us to investigate which complex vector bundles η over X admit a stable real form.

Let $U = \lim_{n\to\infty} U(n)$ (respectively, $O = \lim_{n\to\infty} O(n)$) be the stable unitary (respectively, orthogonal) group. Denote $\Gamma = U/O$. Let X^q be the q-skeleton of X and denote by $i : X^q \to X$ the inclusion map. Suppose that $\eta \in \mathrm{Vect}_{\mathbb{C}}(X)$ admits a stable real form over X^q, that is, there exists a real vector bundle ξ over X^q such that

$$i_u^*(\tilde{\eta}) = \tilde{c}_{X^q}(\tilde{\xi}).$$

Then the obstruction to extending ξ over the $(q+1)$-skeleton of X is denoted by

$$\mathfrak{o}_{q+1}(\xi) \in H^{q+1}(X, \pi_q(\Gamma))$$

where (see Bott [2])

$$\pi_q(\Gamma) = \begin{cases} \mathbb{Z}, & q \equiv 1 \bmod 4, \\ \mathbb{Z}/2, & q \equiv 2, 3 \bmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the obstructions $\mathfrak{o}_{q+1}(\xi)$ may depend on the selection of ξ.

In order to determine whether or not η admits a stable real form, the obstructions $\mathfrak{o}_{q+1}(\xi)$ must be investigated. One approach is to study the Postonikov decomposition of the canonical map $BO \to BU$, where BU (respectively, BO) is the classifying space of U (respectively, O). For example, this gives the first nontrivial obstruction

$$\mathfrak{o}_2(\xi) = 2c_1(\eta), \tag{1.1}$$

which does not depend on the selection of ξ. However, we will not develop this point here. Instead, by combining the Atiyah–Hirzebruch spectral sequence with the Riemann–Roch theorem for differentiable manifolds (similar to the approach in [13]), we will determine the final obstruction.

Throughout this paper, M will denote an n-dimensional closed oriented smooth manifold with $n \equiv 4 \bmod 8$. We will denote by M^q the q-skeleton of M, $i : M^q \hookrightarrow M$ the inclusion map of the q-skeleton of M, $[M]$ the fundamental class of M and $\langle \cdot, \cdot \rangle$

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the Kronecker product. As in [1],

$$\hat{\mathfrak{A}}(M) = \prod_i \frac{x_i/2}{\sinh x_i/2}$$

denotes the $\mathfrak{A}$-class of M, where the Pontryagin classes of M are the elementary symmetric functions of x_i^2. Our main result can be stated as follows.

THEOREM 1.1. *Let M be an n-dimensional closed oriented smooth manifold with $n \equiv 4 \bmod 8$ and let η be a complex vector bundle over M. Suppose that η admits a stable real form over M^{n-1}. Then the necessary and sufficient conditions for η to admit a stable real form over M are:*

(1) *M is not spin; or*
(2) *M is spin and $\langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \equiv 0 \bmod 2$.*

REMARK 1.2. If M is *spin*, the rational number $\langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle$ is an integer (see Atiyah and Hirzebruch [1, Corollary 2(i)]), so it make sense to take congruence classes modulo 2.

Theorem 1.1 tell us that the final obstruction to the existence of the stable real form of η is

$$\mathfrak{o}_n(\xi) = \begin{cases} 0, & M \text{ is not } spin, \\ \langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \bmod 2, & M \text{ is } spin. \end{cases}$$

Note that this does not depend on the selection of ξ.

Suppose that M is *spin*. The Riemann–Roch theorem for differentiable manifolds [1, Corollary 2(i)] tells us that the modulo 2 congruence class

$$\langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \bmod 2$$

is equal to zero if η admits a stable real form over M, whereas Theorem 1.1 tells us that it is just the final obstruction to the existence of the stable real form of η.

REMARK 1.3. Denote by $\mathrm{Sq}^2 : H^i(M; \mathbb{Z}/2) \to H^{i+2}(M; \mathbb{Z}/2)$ the Steenrod square. It is known that the following three assertions are equivalent:

(1) M is *spin*;
(2) $w_2(M) = 0$;
(3) $\mathrm{Sq}^2 H^{n-2}(M; \mathbb{Z}/2) = 0$.

As an application, combining Theorem 1.1 with (1.1) gives the following result.

COROLLARY 1.4. *Let M a simply connected four-dimensional smooth manifold and let η be a complex vector bundle over M. Then η admits a stable real form if and only if one of the following conditions is satisfied:*

(1) *M is not spin and $c_1(\eta) = 0$; or*
(2) *M is spin, $c_1(\eta) = 0$ and $c_2(\eta) \equiv 0 \bmod 2$.*

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According to the definition of a stable real form, in order to prove Theorem 1.1, it is necessary for us to investigate the image of the complexification homomorphism $\tilde{c}_M$. So this paper is arranged as follows. After some preliminaries in Section 2, a relation between the complexification homomorphism $\tilde{c}_M$ and the second Stiefel–Whitney class $w_2(M)$ is given in Section 3, and then Theorem 1.1 is proved in Section 4.

2. Preliminaries

Since U/O is homotopy equivalent to $\Omega^{-1}BO$ (see Bott [2]), the canonical fibring

$$U/O \hookrightarrow BO \to BU$$

gives rise to a long exact sequence of K-groups (which we call the Bott exact sequence)

$$\cdots \to \widetilde{KO}^{q+1}(M) \to \widetilde{KO}^{q}(M) \xrightarrow{\tilde{c}_M} \widetilde{K}^q(M) \xrightarrow{\gamma} \widetilde{KO}^{q+2}(M) \to \cdots \tag{2.1}$$

which is the exact sequence given by Bott in [3, page 75].

According to Switzer [10, pages 336–341], the Atiyah–Hirzebruch spectral sequence of $KO^*(M)$ is the spectral sequence $\{E_r^{p,q}, d_r\}$ with

$$E_2^{p,q} \cong H^p(M; KO^q), \quad E_\infty^{p,q} \cong F^{p,q}/F^{p+1,q-1}, \tag{2.2}$$

where

$$F^{p,q} = \mathrm{Ker}\,[i_o^* : KO^{p+q}(M) \to KO^{p+q}(M^{p-1})], \tag{2.3}$$

and the coefficient ring of KO-theory is (see Bott [3, page 73])

$$KO^* = \mathbb{Z}[\alpha, x, \gamma, \gamma^{-1}]/(2\alpha, \alpha^3, \alpha x, x^2 - 4\gamma)$$

with degrees $|\alpha| = -1$, $|x| = -4$ and $|\gamma| = -8$.

It is well known that the differentials d_2 of the Atiyah–Hirzebruch spectral sequence of $KO^*(M)$ are as follows (see, for example, Fujii [4, formula (1.3)]):

$$d_2^{*,q} = \begin{cases} \mathrm{Sq}^2\rho_2, & q \equiv 0 \bmod 8, \\ \mathrm{Sq}^2, & q \equiv -1 \bmod 8, \\ 0 & \text{otherwise}. \end{cases} \tag{2.4}$$

Since $n \equiv 4 \bmod 8$, the following proposition follows from Atiyah and Hirzebruch [1, Corollary 2].

Proposition 2.1 (Riemann–Roch theorem for differentiable manifolds). *Suppose that M is spin and let $\tilde{\eta} \in \widetilde{K}(M)$ (respectively, $\tilde{\xi} \in \widetilde{KO}(M)$) be a stable complex (respectively, real) vector bundle over M. Then the rational number*

$$\langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle$$

is an integer. Moreover the integer

$$\langle \mathrm{ch}(\tilde{c}_M(\tilde{\xi})) \cdot \hat{\mathfrak{A}}(M), [M] \rangle$$

is even.

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3. A relation between complexification and $w_2(M)$

In this section we give a relation between the complexification homomorphism

$$\tilde{c}_M : \widetilde{KO}(M) \to \widetilde{K}(M)$$

and the second Stiefel–Whitney class of M.

According to Wall [12, Theorem 2.4], M is homotopy equivalent to a CW-complex

$$M^{n-1} \cup_f \mathbb{D}^n,$$

where $f \in \pi_{n-1}(M^{n-1})$ is the attaching map of the n-disc $\mathbb{D}^n$. Denote by

$$p : M \to S^n$$

the map collapsing the $(n-1)$-skeleton of M^{n-1} to the base point. Then, by the naturality of the Puppe sequence, we have the exact ladder

$$\begin{array}{ccccc}
\widetilde{KO}(S^n) & \xrightarrow{p_o^*} & \widetilde{KO}(M) & \xrightarrow{i_o^*} & \widetilde{KO}(M^{n-1}) \\
\downarrow{\scriptstyle \tilde{c}_{S^n}} & & \downarrow{\scriptstyle \tilde{c}_M} & & \downarrow{\scriptstyle \tilde{c}_{M^{n-1}}} \\
\widetilde{K}(S^n) & \xrightarrow{p_u^*} & \widetilde{K}(M) & \xrightarrow{i_u^*} & \widetilde{K}(M^{n-1})
\end{array} \tag{3.1}$$

Let $\mathbb{Z}\zeta$ be the infinite cyclic group generated by ζ. Recall that when $n \equiv 4 \bmod 8$,

$$\widetilde{K}(S^n) \cong \mathbb{Z}\tilde{\omega}^n_{\mathbb{C}}, \quad \widetilde{KO}(S^n) \cong \mathbb{Z}\tilde{\omega}^n_{\mathbb{R}}$$

(see Mimura and Toda [9, Theorem 5.12, page 209]). Here, $\tilde{\omega}^n_{\mathbb{C}}$ and $\tilde{\omega}^n_{\mathbb{R}}$ are the generators and they can be so chosen such that

$$\tilde{c}_{S^n}(\tilde{\omega}^n_{\mathbb{R}}) = 2\tilde{\omega}^n_{\mathbb{C}}.$$

According to the exact ladder (3.1), in order to investigate the image of the complexification homomorphism

$$\tilde{c}_M : \widetilde{KO}(M) \to \widetilde{K}(M)$$

in the case $n \equiv 4 \bmod 8$, it is helpful to find the necessary and sufficient conditions (in terms of the cohomology of M) for $\operatorname{Im} p_u^* \subseteq \operatorname{Im} \tilde{c}_M$.

THEOREM 3.1. $\operatorname{Im} p_u^* \subseteq \operatorname{Im} \tilde{c}_M$ *if and only if* $w_2(M) \neq 0$.

PROOF. Since M is homotopy equivalent to $M^{n-1} \bigcup_f \mathbb{D}^n$, by the naturality of the Puppe sequence and the Bott exact sequence (2.1) we have the commutative diagram

$$\begin{array}{ccccccc}
\widetilde{KO}(S^n) & \xrightarrow{p_o^*} & \widetilde{KO}(M) & \xrightarrow{i_o^*} & \widetilde{KO}(M^{n-1}) & \xrightarrow{f_o^*} & \widetilde{KO}(S^{n-1}) \\
\downarrow{\scriptstyle \tilde{c}_{S^n}} & & \downarrow{\scriptstyle \tilde{c}_M} & & \downarrow{\scriptstyle \tilde{c}_{M^{n-1}}} & & \downarrow{\scriptstyle \tilde{c}_{S^{n-1}}} \\
\widetilde{K}(S^n) & \xrightarrow{p_u^*} & \widetilde{K}(M) & \xrightarrow{i_u^*} & \widetilde{K}(M^{n-1}) & \xrightarrow{f_u^*} & \widetilde{K}(S^{n-1}) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
KO^2(S^n) & \xrightarrow{p_o^*} & KO^2(M) & \xrightarrow{i_o^*} & KO^2(M^{n-1}) & \xrightarrow{f_o^*} & KO^2(S^{n-1})
\end{array} \tag{3.2}$$

where the vertical and horizontal sequences are all exact.

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Diagram (3.2) establishes a relationship between the complexification homomorphism $\tilde{c}_M$ and the Atiyah–Hirzebruch spectral sequence of $KO^*(M)$ as follows. Since $\tilde{c}_{S^n}$ is a multiplication by 2, the homomorphism

$$\gamma : \widetilde{K}(S^n) \to KO^2(S^n)$$

is an epimorphism. Then diagram (3.2), together with (2.2) and (2.3), yields

$$\text{Im } p_u^* \subseteq \text{Im } \tilde{c}_M \quad \text{if and only if Im } [p_o^* : KO^2(S^n) \to KO^2(M)] = 0.$$

That is

$$\text{Im } p_u^* \subseteq \text{Im } \tilde{c}_M \quad \text{if and only if } F^{n,-n+2} = 0.$$

Then it follows from $E_\infty^{n,-n+2} = F^{n,-n+2}$ that

$$\text{Im } p_u^* \subseteq \text{Im } \tilde{c}_M \quad \text{if and only if } E_\infty^{n,-n+2} = 0. \tag{3.3}$$

Suppose that $w_2(M) \neq 0$, so that $\mathrm{Sq}^2 H^{n-2}(M; \mathbb{Z}/2) \neq 0$ by Remark 1.3. The differentials evaluated in (2.4) now imply that

$$E_\infty^{n,-n+2} = E_3^{n,-n+2} = 0.$$

Therefore $\text{Im } p_u^* \subseteq \text{Im } \tilde{c}_M$ by the equivalence (3.3).

Conversely, suppose that $\text{Im } p_u^* \subseteq \text{Im } \tilde{c}_M$, that is, $p_u^*(\tilde{\omega}_{\mathbb{C}}^n) \in \text{Im } \tilde{c}_M$. Let $\tilde{\xi} \in \widetilde{KO}(M)$ be the element such that $\tilde{c}_M(\tilde{\xi}) = p_u^*(\tilde{\omega}_{\mathbb{C}}^n)$. Since

$$\mathrm{ch}(\tilde{\omega}_{\mathbb{C}}^n) \in H^n(S^n; \mathbb{Z})$$

is a generator (see Bott [3, page 28, Theorem 6.1]) and the degree of the map $p : M \to S^n$ is one, it follows that

$$\begin{aligned}
\langle \mathrm{ch}(\tilde{c}_M(\tilde{\xi})) \cdot \hat{\mathfrak{A}}(M), [M] \rangle &= \langle \mathrm{ch}(p_u^*(\tilde{\omega}_{\mathbb{C}}^n)) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \\
&= \langle p_u^*(\mathrm{ch}(\tilde{\omega}_{\mathbb{C}}^n)) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \\
&= \langle p_u^*(\mathrm{ch}(\tilde{\omega}_{\mathbb{C}}^n)), [M] \rangle \\
&= \langle \mathrm{ch}(\tilde{\omega}_{\mathbb{C}}^n), [S^n] \rangle \\
&= \pm 1.
\end{aligned}$$

Now suppose that $w_2(M) = 0$, which means that the manifold M is *spin*. From Proposition 2.1,

$$\langle \mathrm{ch}(\tilde{c}_M(\tilde{\xi})) \cdot \hat{\mathfrak{A}}(M), [M] \rangle$$

is an even integer, which is a contradiction. Hence we must have $w_2(M) \neq 0$, and the proof is complete. □

REMARK 3.2. Let N be an k-dimensional closed oriented smooth manifold. Denote by $p : N \to S^k$ the map collapsing the $(k-1)$-skeleton of N^{k-1} to the base point. Since the complexification homomorphism $\tilde{c}_{S^k}$ is epimorphic in the cases $k \not\equiv 2, 4, 6 \bmod 8$ (see [9, Corollary 5.7, Theorem 5.12, pages 201–209]), $\text{Im } p_u^* \subseteq \text{Im } \tilde{c}_N$ is always true in these cases. In the cases $k \equiv 2, 6 \bmod 8$, we can only get that

$$w_2(N) \neq 0 \quad \text{implies Im } p_u^* \subseteq \text{Im } \tilde{c}_N.$$

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In fact, from the proof of Theorem 3.1, we can obtain more information about the differentials of the Atiyah–Hirzebruch spectral sequence of $KO^*(M)$ as follows.

Corollary 3.3. *Let M be an n-dimensional closed oriented smooth manifold with $n \equiv 4 \bmod 8$ and let $\{E_r^{p,q}, d_r\}$ be the Atiyah–Hirzebruch spectral sequence of $KO^*(M)$. Then the following three assertions are equivalent:*

(1) $w_2(M) \neq 0$;
(2) $E_3^{n,-n+2} = 0$;
(3) $E_\infty^{n,-n+2} = 0$.

Hence the differentials $d_r : E_r^{n-r,-n+r+1} \to E_r^{n,-n+2}$ with $r \geq 3$ are all zero. That is, $E_\infty^{n,-n+2} = E_3^{n,-n+2}$.

Proof. Note that

$$w_2(M) \neq 0 \quad \text{if and only if } E_3^{n,-n+2} = 0 \tag{3.4}$$

which follows from Remark 1.3 and Equation (2.4). The corollary can be deduced easily from the equivalences (3.3), (3.4) and the assertion of Theorem 3.1. □

4. Proof of the main result

Proof of Theorem 1.1. By the definition of a stable real form, if η admits a stable real form over M^{n-1} then there exists a real vector bundle ξ over M^{n-1} such that

$$\tilde{c}_{M^{n-1}}(\tilde{\xi}) = i_u^*(\tilde{\eta}).$$

Note that $\widetilde{KO}(S^{n-1}) = 0$ and the horizontal sequence of the diagram (3.2) is exact. Then there must exists an element $\tilde{\zeta} \in \widetilde{KO}(M)$ such that $i_o^*(\tilde{\zeta}) = \tilde{\xi}$ and

$$\tilde{\eta} - \tilde{c}_M(\tilde{\zeta}) = k p_u^*(\tilde{\omega}_{\mathbb{C}}^n) \tag{4.1}$$

for some $k \in \mathbb{Z}$.

If M is not *spin*, then $w_2(M) \neq 0$ and we have $\operatorname{Im} p_u^* \subseteq \operatorname{Im} \tilde{c}_M$ by Theorem 3.1. Therefore, from (4.1), η always admits a stable real form in this case.

If M is *spin*, then $w_2(M) = 0$. On the one hand,

$$p_u^*(\tilde{\omega}_{\mathbb{C}}^n) \notin \operatorname{Im} \tilde{c}_M$$

by Theorem 3.1, and

$$2p_u^*(\tilde{\omega}_{\mathbb{C}}^n) = \tilde{c}_M(p_o^*(\tilde{\omega}_{\mathbb{R}}^n)) \in \operatorname{Im} \tilde{c}_M,$$

and (4.1) implies that

$$\tilde{\eta} \in \operatorname{Im} \tilde{c}_M \quad \text{if and only if } k \equiv 0 \bmod 2. \tag{4.2}$$

On the other hand, since M is *spin*, Proposition 2.1 shows that the rational number

$$\langle \operatorname{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle$$

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is an integer, and the integer

$$\langle \mathrm{ch}(\tilde{c}_M(\tilde{\zeta})) \cdot \hat{\mathfrak{A}}(M), [M] \rangle$$

is even. Moreover,

$$\begin{aligned}\langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle &= \langle \mathrm{ch}(kp_u^*(\tilde{\omega}_{\mathbb{C}}^n) + \tilde{c}_M(\tilde{\zeta})) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \\ &= k + \langle \mathrm{ch}(\tilde{c}_M(\tilde{\zeta})) \cdot \hat{\mathfrak{A}}(M), [M] \rangle.\end{aligned}$$

Therefore

$$k \equiv 0 \bmod 2 \quad \text{if and only if } \langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \equiv 0 \bmod 2.$$

Hence, by the equivalence (4.2),

$$\tilde{\eta} \in \mathrm{Im}\, \tilde{c}_M \quad \text{if and only if } \langle \mathrm{ch}(\tilde{\eta}) \cdot \hat{\mathfrak{A}}(M), [M] \rangle \equiv 0 \bmod 2.$$

This completes the proof. □

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