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LOCALIZATION, ALGEBRAIC LOOPS AND H-SPACES II

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In a previous work [**6**] it was shown that by imposing certain finiteness conditions on a nilpotent loop certain algebraic results yielded properties about $[X, Y]$ where X is finite CW and Y is an H-Space. In this sequel we further restrict the category of nilpotent loops to a full subcategory called H-loops which still contains all loops of the form $[X, Y]$. We prove that on this category there is a unique and universal P-localization if $P \neq \emptyset$ which corresponds to topological localization. We also show that if the H-loop is a group then the two concepts of localization agree.

The first section of this paper is devoted to the definition and basic properties of H-loops. In the second section we develop the localization construction and prove uniqueness. Finally, in the third section we consider the topological and group theoretic situations.

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Section 1. h-loops, pre-H-loops and H-loops. Recall [**6**] that a (centrally) nilpotent loop G is an h-loop if for any set of primes Q; $T_Q(G) = \{x \in G | x^n = e$, some association, some $n \in \langle Q \rangle\}$ is a finite normal subloop, called the Q-torsion of G. (By $\langle Q \rangle$ we mean the multiplicative set generated by Q. If Q is the set of all primes we will write $T(G)$ for $T_Q(G)$).

Definition 1.1. An h-loop G is a *Pre-H-loop* if it is residually h-finite, i.e., there exist a collection of epimorphisms $\{f_\alpha: G \to G_\alpha | \alpha \in I\}$ with each G_α a finite h-loop such that $i)$ $f = \prod f_\alpha: G \to \prod G_\alpha$ is one to one and $ii)$ if x is an element of G not in $T(G)$ then for any set of primes, Q, there is an $\alpha \in I$ such that $e \neq f_\alpha(x) \in T_Q(G_\alpha)$.

We will call $f: G \to \prod G_\alpha$ a *defining system* for G.

Note that any finite h-loop is trivially a pre-H-loop under the identity defining system.

LEMMA 1.2. *Let G be a pre-H-loop and let $\{G_\beta | \beta \in I\}$ be the collection of all finite h-loop quotients of G. Then $g: G \to \prod G_\beta$ is a defining system for G.*

Proof. Let $f: G \to \prod_{\alpha \in I} G_\alpha$ be a defining system for G. Then $f = (\prod p_\alpha)g$ where $\prod p_\alpha: \prod G_\beta \to \prod G_\alpha$ is the product of the projection maps. Since f is one to one so is g and property (i) of 1.1 holds. That condition (ii) of 1.1 holds is obvious.

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PROPOSITION 1.3. *Let P be a non-empty set of primes and let $f: G \to \prod G_\alpha$ be a defining system for the pre-H-loop G. Let $g_\alpha: G_\alpha \to G_\alpha/T_{P'}(G_\alpha)$ be the quotient map. Then* Ker $(qf) = T_{P'}(G)$ *where $q = \prod g_\alpha$. (P' is the compliment of P.)*

Proof. By ([**6**], 3.5) $T_{P'}(G_\alpha)$ is a normal subloop of G_α so that $G_\alpha/T_{P'}(G_\alpha)$ is a loop. If P', the compliment of P, is empty q is the identity and the proposition is trivially true.

If $x \in T_{P'}(G)$ then for each α in I, $f_\alpha(x)$ is in $T_{P'}(G_\alpha)$ and hence Ker $(qf) \supseteq T_{P'}(G)$.

Conversely, if $x \in$ Ker (qf) then for all α in I, $f_\alpha(x) \in T_{P'}(G_\alpha)$. Thus by (*ii*) of 1.1 $x \in T(G)$ since P is not empty.

Assume x is not in $T_{P'}(G)$. Then by ([**6**], 2.9) $x = x'y$ with $e \neq y \in T_P(G)$ for some $p \in P$ and $T_P(G)$ is a loop of p-power order.

Let $\langle y \rangle$ denote the loop generated by y. By ([**2**], V 2.2) $\langle y \rangle$ has p-power order and the same holds for all the loops $f_\alpha(\langle y \rangle)$, $\alpha \in I$. But by ([**6**], 2.3) the order of $T_{P'}(G_\alpha)$ is prime to p. Since the order of $f_\alpha(\langle y \rangle)$ must divide the order of $T_{P'}(G_\alpha)$ we get that $f_\alpha(y) = e$ for all α in I so that $f(y) = e$. But this contradicts the fact that f is one to one.

Definition 1.4. Let G be a loop, N a subloop and Q a set of primes. Define the *Q-isolator of N in G*, $S_Q(N, G)$ to be the set $\{x \in G \mid x^m \in N$ for some association, some $m \in Q\}$.

While if G is a nilpotent group the Q-isolator is a subgroup ([**7**], 3.25) the same is not true for nilpotent loops.

Definition 1.5. Let G be a pre-H-loop. Then G is an H-loop if there exists a defining system f: $G \to \prod G_\alpha$ such that (*i*) if g_β: $G \to G_\beta$ is epic with G_β a finite h-loop then there exists $\tilde{g}$: $\prod G_\alpha \to G_\beta$ such that $g_\beta = \tilde{g}f$ and (*ii*) for any set of primes Q, $S_Q(qf(G), \prod G_\alpha/T_Q(G_\alpha))$ is a loop.

Note that any finite h-loop is an H-loop and that while for any pre-H-loop there is always a defining system such that (*i*) holds, it is not clear if (*ii*) holds. We will call a defining system satisfying (*i*) and (*ii*) of 1.5 an *H-defining system*.

Let P be a set of primes. Recall that a loop G is *P-local* if the mapping defined by $x \to x^n$ is a bijection for any association, any $n \in \langle P' \rangle$ and a homomorphism f: $M \to N$ is a *P-equivalence* if ker $f \subseteq T_{P'}(M)$ (f is *P-monic*) and for any $x \in N$ there is an $r \in \langle P' \rangle$ and an association such that $x^r \in$ im f (f is *P-epic*).

LEMMA 1.6. *Let G be a pre-H-loop with defining system f: $G \to \prod G_\alpha$. Let P be a set of primes and let S be the subloop of $\prod G_\alpha/T_{P'}(G_\alpha)$ generated by $S_{P'}(qf(G), \prod G_\alpha/T_{P'}(G_\alpha))$. Then S is the P-local subloop of $\prod G_\alpha/T_{P'}(G_\alpha)$ generated by $qf(G)$.*

Proof. By 3.4 of [**6**] $\prod G_\alpha/T_{P'}(G_\alpha)$ is P-local and hence the mapping $x \to x^n$ for $n \in \langle P' \rangle$ is one to one on S. Further if $x \in S$ and $y^n = x$, $n \in \langle P' \rangle$ then by definition $y \in S$ so that $x \to x^n$ is onto on S. Thus S is P-local.

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On the other hand any P-local subloop of $\Pi\, G_\alpha/T_{P'}(G_\alpha)$ which contains $gf(G)$ must also contain $S_{P'}(qf(G), \Pi\, G_\alpha/T_{P'}(G_\alpha))$. The result follows.

PROPOSITION 1.7. *Let G be an H-loop. Then the defining system $g: G \to \prod_{\beta\in J} G_\beta$ of all finite h-loop quotients is an H-defining system.*

Proof. Trivially 1.5 (i) holds so that we need only demonstrate 1.5 (ii). Let $f: G \to \prod_{\alpha\in I} G_\alpha$ be an H-defining system and note that $I \subseteq J$. For any $\beta \in J - I$ let $\tilde{f}: \Pi\, G_\alpha \to G_\beta$ satisfy (i) of 1.5 and define $i: \prod_{\alpha\in I} G_\alpha \to \prod_{\beta\in J} G_\alpha$ by the product of the projections P_β if $\beta \in I$ and by $\tilde{f}_\beta$ if $\beta \in J - I$. Then the following diagram commutes:

$$\begin{array}{ccccc} G & \xrightarrow{f} & \prod\limits_{\alpha\in I} G_\alpha & \xrightarrow{q'} & \prod\limits_{\alpha\in I} G_\alpha/T_{P'}(G_\alpha) \\ & \searrow^{g} & \downarrow i & & \downarrow j \\ & & \prod\limits_{\beta\in J} G_\beta & \xrightarrow{q} & \prod\limits_{\beta\in J} G_\beta/T_{P'}(G_\beta) \end{array}$$

where j is induced by i.

It is easily seen that i and j are both monic.

Thus by 1.6 $j(S_{P'}(q'f\ (G), \Pi\, G_\alpha/T_{P'}(G_\alpha))$ is a P-local subloop of $\Pi\, G_\beta/T_{P'}(G_\beta)$ which contains $qg(G)$. But it is clear that

$$\prod\nolimits_{\alpha\in I} p_\alpha(S_{P'}(qg(G), \prod G_\beta/T_{P'}(G_\beta))) \subseteq S_{P'}(q'f(G), \prod G_\alpha/T_{P'}(G_\alpha))$$

and that $(\prod_{\alpha\in I} p_\alpha)j$ is the identity on $\Pi\, G_\beta/T_{P'}(G_\beta)$. Thus

$$j(S_{P'}(q'f(G)), \prod G_\alpha/T_{P'}(G_\alpha)) \subseteq S_{P'}(qg(G), \prod G_\beta/T_{p'}(G_\beta)).$$

Once again applying 1.6 yields the required result.

THEOREM 1.8. *Let G be an H-loop and let $f: G \to \Pi\, G_\alpha$ be any defining system. Then for any set of primes $Q_P S_Q(qf\ (G), \Pi\, G_\alpha/T_Q(G_\alpha))$ is a loop.*

Proof. By 1.7 the defining system of all finite h-loop quotients, $g: G \to \Pi\, G_\beta$ is an H-defining system and we may factor $f = (\Pi\, p_\alpha)g$ where p_α is the product of the projections. The same technique as in 1.7 now yields the result.

Section 2. Localization of H-loops. Let G be an H-loop and let g: G $\to \Pi\, G_\beta$ be the defining system of all finite H-loop quotients. Let P be a set of primes and g: $\Pi\, G_\beta \to G_\beta/T_{P'}(G_\beta)$ be the product of the quotient maps.

Definition 2.1. If $P \neq \emptyset$ the *P-localization* of an H-loop G, $L: G \to G_P$ is $S_{P'}(qg(G), \Pi\, G_\beta/T_{P'}(G_\beta))$.

Note that by 1.3 $\ker L = T_{p'}(G)$, by 1.6 G_p is the P-local subloop of $\Pi G_\beta/T_{p'}(G_\beta)$ generated by $qg(G)$ and by definition $x \in G_p$ implies that $x^n \in L(G)$ for some association, some $n \in \langle P' \rangle$. Thus we get:

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THEOREM 2.2. *If P is a non-empty set of primes and G is an H-loop then P-localization $L: G \to G_P$ is a P-equivalence and hence is a P-localization in the sense of* ([**6**], 3.1).

Let G be an H-loop and let $f: G \to \prod G_\alpha$ be any (not necessarily H) defining system. Let $P \neq \emptyset$ and let

$$k: G \to G' = S_{P'}(qf(G), \prod G_\alpha/T_{P'}(G_\alpha)).$$

PROPOSITION 2.3. *There is a unique isomorphism $k: G_P \to G'$ such that $k = \tilde{k}L$. Thus up to canonical equivalence $k: G \to G'$ is a P-localization.*

Proof. The product of the projections $\prod p_\alpha: \prod G_\beta \to \prod G_\alpha$ as defined in 1.8 induces a map $\tilde{k}: G_P \to G'$ such that $k = \tilde{k}L$ with the uniqueness of $\tilde{k}$ a triviality of the construction.

By 1.6 and 1.8 both L and K are P-equivalences. A trivial modification of ([**4**], *I* 1.4) to loops shows that $\tilde{k}$ is a P-equivalence. But both G_P and G' are P-local loops and by ([**4**], *I* 1.5) a P-equivalence between P-local loops is an isomorphism.

THEOREM 2.4. *Let $f: G \to M$ be a homomorphism of H-loops and let $P \neq \emptyset$ be a set of primes. Then there is a unique $f_P: G_P \to M_P$ such that $f_P L_G = L_M f$.*

Proof. If K is a normal subloop of M of finite index then $f^{-1}(K)$ is normal in G of finite index. Thus given a defining system $g: M \to \prod_{\alpha \in I} M_\alpha$ extend $G \to \prod_{\alpha \in I} (G/f^{-1}(\ker g_\alpha))$ to a defining system $\tilde{g}: G \to \prod_{\beta \in J} G_\beta$ and define $\tilde{f}: \prod G_\beta \to \prod M_\alpha$ to be trivial if $\beta \in J - I$ and the obvious map if $\beta \in I$.

In this manner we get the following commutative diagram:

$$\begin{array}{ccc}
G & \xrightarrow{\;f\;} & M \\
\big\downarrow{\scriptstyle \tilde{g}} & & \big\downarrow{\scriptstyle g} \\
\prod G_\beta & \xrightarrow{\;\tilde{f}\;} & \prod M_\alpha \\
\big\downarrow{\scriptstyle q_G} & & \big\downarrow{\scriptstyle q_M} \\
\prod G_\beta/T_{P'}(G_\beta) & \xrightarrow{\;f'\;} & \prod M_\alpha/T_{P'}(M_\alpha)
\end{array}$$

Let $x \in S_{P'}(q_G\tilde{g}(G), \prod G_\beta/T_{P'}(G_\beta)) = G_P$ then $x^n = L_G(y)$ for some $y \in G$, $n \in \langle P' \rangle$. Thus

$$(f'(x))^n = f'(x^n) = q_M g f(y)$$

and this implies that $f'(x)$ lies in

$$S_{P'}(q_M g(M), \prod M_\alpha/T_{P'}(M_\alpha)) = M_P.$$

Let f_P be the restriction of f' to G_P. Since f_P is unique on the image of G it is unique on its localization G_P.

COROLLARY 2.5. (Universality) *Let $f: G \to H$ be a homomorphism of H-loops with M P-local ($P \neq \emptyset$). Then there is a unique $\tilde{f}: G_P \to M$ such that $\tilde{f} L_G = f$.*

COROLLARY 2.6. *Let $K \overset{f}{\rightarrowtail} G \overset{g}{\twoheadrightarrow} N$ be a short exact sequence of H-loops. Then the sequence $K_P \overset{f_P}{\rightarrowtail} G_P \overset{g_P}{\twoheadrightarrow} N_P$ is short exact. Further if K is central then so is K_P.*

Proof. By 2.4 we have the following commutative diagram

$$\begin{array}{ccccc} K & \overset{f}{\rightarrowtail} & G & \overset{g}{\twoheadrightarrow} & N \\ \downarrow{\scriptstyle L_K} & & \downarrow{\scriptstyle L_G} & & \downarrow{\scriptstyle L_N} \\ K_P & \xrightarrow{f_P} & G_P & \xrightarrow{g_P} & N_P \end{array}$$

with the top row short exact. By 2.2 the vertical maps are all P isomorphisms. The proof now follows by the same arguments as ([**4**]. *I* 1.10).

COROLLARY 2.7. *On the category of H-loops the P-localization ($P \neq \emptyset$) of G is characterized by any P-equivalence $f: G \to G'$ where G' is a P-local H-loop.*

Proof. This is just a restatement of 2.2–2.5 combined with noting that G_P is an H-loop.

By combining the results of this section we get the following:

THEOREM 2.8. *On the category of H-loops there exists a P-localization functor ($P \neq \emptyset$) which is universal and exact. Furthermore the localization of an H-loop of nilpotency class $\leqq n$ is also of class $\leqq n$.*

Section 3. Topological and group theoretic considerations.

THEOREM 3.1. *Let G be a finitely generated nilpotent group. Then G is an H-loop and loop localization is equivalent to loop localization.*

Proof. Trivially G is an h-loop. By combining the results of [**5**] and [**3**] we get that G is a pre-H-loop. By ([**7**], 3.25) G is an H-loop and the localization construction of ([**7**], 8.5) is easily seen to be equivalent to the construction in this paper.

THEOREM 3.2. *Let X be a finite* CW *complex and Y be an H-Space with finitely generated homotopy groups in each dimension. Then $[X, Y]$ is an H-loop and for every set of primes, $P \neq \emptyset$, $[X, Y_P] \cong [X, Y]_P$.*

Proof. That $[X, Y]$ is an h-loop was shown in ([**6**], 4.1). By ([**1**], *VI* 8.1) the map $[X, Y] \to \prod_P [X, (Z/p)_\infty Y)$ is an injection where $(Z/p)_\infty Y$ is the Z/p completion of Y. Furthermore ([**1**], 4.3) $(Z/p)_\infty Y = \varinjlim (Z/p)_s Y$ and all the

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homotopy groups of $(Z/p)_s\, Y$ are finite when s is finite. Thus $[X, Y]$ includes into $\prod_P \prod_s [X, (Z/p)_s\, Y]$ with all the sets $[X, (Z/p)_s\, Y]$ finite.

But a trivial modification of ([**1**], *I* 7.3) shows that all the $(Z/p)_s\, Y$ are H-spaces with compatible structures so that $[X, Y]$ is residually finite. That $[X, Y]$ is a pre-H-loop follows from ([**1**], *VI* 8.1).

Property (*ii*) of 1.5 follows from ([**4**], 6.2). To prove that $[X, Y]$ satisfies (*i*) of 1.5 let K be normal in $[X, Y]$ of finite index. Since $[X, Y] \to \varinjlim (X, \prod_P (Z/p)_s Y]$ is an inclusion there must exist an s such that $\ker [X, Y] \to [X, \prod (Z/p)_s Y]$ is contained in K which is equivalent to 1.5 (*i*). That $[X, Y_P] \cong [X, Y]_P$ follows from the fact that $[X, Y_P]$ is P-local and $[X, Y] \to [X, Y_P]$ is a P isomorphism.

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