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Weakly collisional steady state linear and nonlinear plasma waves

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Landau's collisionless result for a weakly damped plasma wave is precisely recovered in a weakly collisional, steady state plasma by treating the physics of the narrow collisional boundary layer associated with the resonant electrons. To recover Landau's results, the collision frequency must be large enough that islands are unable to form and/or the wave amplitude must be small enough to allow linearization. However, the Landau treatment fails once the collision frequency becomes too weak and/or the wave amplitude too large. Remarkably, Landau's weakly damped plasma wave results require collisions and are shown to be inappropriate in the collisionless limit for a nonlinear, finite amplitude, steady state wave!

Key words: plasma nonlinear phenomena, plasma waves

Landau (1946) first realized that the singular behavior of the collisionless linearized kinetic equation describing weakly damped plasma waves could be resolved by Laplace transforming in time to solve a causal initial value problem. Subsequent collisionless work by Dawson (1961), O'Neil (1965) and Villani (2014) provided additional insights into the linear and nonlinear temporal evolution of the resonant electrons. Here, the focus is on an applied steady state plasma wave where weak collisions must be retained to restore the regularity needed to avoid the singular behavior dealt with in the preceding formulations by focusing on temporal evolution. Then, the need to solve an initial value problem or invoke causality by Laplace transforming in time is removed. In this weak collision limit a narrow boundary layer resolves the singular behavior, and exactly recovers the collisionless results as long as the wave amplitude is small and the collision frequency finite. However, the deceptive collisionless behavior is removed once the collision frequency becomes small enough and/or the wave amplitude becomes sufficiently large that a nonlinear treatment is required. The nonlinear solution found herein for a monochromatic wave retains the island structure, but Landau's results (1946) are no longer valid when the collision frequency goes to zero for a non-vanishing plasma wave amplitude. The collisional treatment remains valid as long as there are many electrons in a Debye sphere.

Consider a weakly dissipative plasma wave with an applied electric field

$$\vec{E} = \vec{z}\Im\mathrm{m}[\tilde{E}e^{i(k_{||}z-\omega t)}] \approx \vec{z}E_{||}\sin(k_{||}z - \omega t), \tag{1}$$

where the wave frequency ω is very much larger than the collision frequency, $\vec{z}$ is a unit vector, $\tilde{E}$ is a complex amplitude, $\Im\mathrm{m}$ denotes imaginary part and ω, $k_{||}$ and $E_{||}$ are positive with $k_{||}$ the z directed wave vector component. The steady state electron kinetic equation with the collision operator, $C\{f\}$, retained is

$$\partial f/\partial t + \vec{v}\cdot\nabla f - (e/m)\vec{E}\cdot\nabla_v f = C\{f\}, \tag{2}$$

where f, e and m are the electron distribution function, charge magnitude, and mass. If the plasma is homogeneous with stationary ions and the electrons are a stationary Maxwellian $f_0(v)$ to lowest order, then $C\{f_0\} = 0$ and the kinetic equation becomes

$$\partial f_1/\partial t + v_{||}\partial f_1/\partial z - (e/m)\vec{E}\cdot\vec{z}\partial(f_0 + f_1)/\partial v_{||} = C\{f_1\}, \tag{3}$$

where $f = f_0 + f_1$, only spatial variation in z enters, and $\vec{z}\cdot\vec{v} = v_{||}$ with $v^2 = v_{||}^2 + v_\perp^2$. The ordering allows $\partial f_1/\partial v_{||} \sim \partial f_0/\partial v_{||}$ to retain nonlinear behavior. The solution of this equation can be viewed as the weakly collisional, steady state counterpart of the collisionless, early temporal evolution model considered by O'Neil (1965).

The effects of resonant electrons, $k_{||}v_{||} \approx \omega \approx \omega_p$, on the plasma or Langmuir wave are retained by this ordering, with the electron plasma frequency defined by $\omega_p^2 = 4\pi e^2 n_e/m$. For the weakly dissipative plasma waves of interest $\omega^2/k_{||}^2 v_e^2 \gg 1$, with $v_e = (2T_e/m)^{1/2}$ the electron thermal speed. This inequality implies $\omega^2/k_{||}^2 \approx v_{||}^2 \sim v^2 \gg v_e^2$ are of interest in $C\{f_1\}$. As a result, the high speed expansion ($v^2 \gg v_e^2$) of the collision operator

$$\begin{aligned} C\{f_1\} &= \nabla_v\cdot\left\{\frac{\nu_e}{2x^3}\left[(v^2\overleftrightarrow{I} - \vec{v}\vec{v})\cdot\nabla_v f_1 + \frac{2Tf_0}{(Z+1)m}\nabla_v\left(\frac{f_1}{f_0}\right)\right]\right\} \\ &\approx \frac{\nu_e}{2x^3}\left[v_\perp^2 + \frac{v_e^2}{(Z+1)}\right]\frac{\partial^2 f_1}{\partial v_{||}^2}, \end{aligned} \tag{4}$$

is adequate, where $f_0 = n_e\pi^{-3/2}v_e^{-3}e^{-v^2/v_e^2}$, $x = v/v_e$, $\nu_e = 3\sqrt{\pi}(Z+1)\nu_{ee}/4$ and $\nu_{ee} = 4\sqrt{2\pi}e^4 n_e \ell n\Lambda/3m^{1/2}T^{3/2} = \nu_{ei}/Z$, for a quasineutral plasma, with Z the ion charge number, and a large Coulomb logarithm, $\ell n\Lambda \gg 1$, assuring there are many electrons in a Debye sphere. The final form is adequate for a resonance where rapid $v_{||}$ variation occurs in the thin collisional boundary layer. Collisions are negligible elsewhere.

Neglecting the nonlinear term leads to an inhomogeneous Airy equation with solution

$$f_1|_{\mathrm{res}} = -\frac{e\omega f_0}{T(k_{||}^5 v_{\perp z}^2\nu)^{1/3}}\Im\mathrm{m}\left[\tilde{E}e^{i(k_{||}z-\omega t)}\int_0^\infty \mathrm{d}\tau\, e^{-is\tau-\tau^3/3}\right], \tag{5}$$

with $s = (k_{||}/v_{\perp z}^2\nu)^{1/3}(v_{||} - \omega/k_{||})$, $\nu = \nu_e/2x^3$ and $v_{\perp z}^2 = v_\perp^2 + v_e^2/(Z+1)$. This solution can be checked by direct substitution and integration by parts, and is valid near $v_{||} \approx \omega/k_{||}$. It retains collisions (via s) and slightly generalizes the earlier weak collisions solutions of Su & Oberman (1968), Johnston (1971) and Auerbach (1977) by keeping $v_\perp^2$ dependence, and of Catto (2020) and Catto & Tolman (2021) by keeping $v_e^2/(Z+1)$ in $v_{\perp z}^2$.

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The collisional boundary-layer width of the resonance from $s=1$ is $(v_{||}-\omega/k_{||})_\nu = (\Delta v_{||})_\nu = |\nu v_{\perp z}^2/k_{||}|^{1/3}$. This corresponds to a normalized width $(\Delta v_{||})_\nu/v_e \sim (\nu_e k_{||}^2 v_e^2/\omega^3)^{1/3}$ and an effective resonant collision frequency $\nu_{\mathrm{eff}} \sim \nu_e[v_e/(\Delta v_{||})_\nu]^2 \gg \nu_e$ for $v_{\perp z}^2 \sim v_e^2$.

For $|s| \gg 1$, $\int_0^\infty \mathrm{d}\tau e^{-is\tau-\tau^3/3} \to 1/is$. As a result, away from resonance $f_1|_{\mathrm{res}}$ matches the usual collisionless non-resonant solution

$$f_1|_{\mathrm{non}} = eE_{||}v_{||}f_0\cos\phi/T(k_{||}v_{||}-\omega), \tag{6}$$

where $\phi \equiv k_{||}z-\omega t$. The perturbed non-resonant density $n_{\mathrm{non}} = \int d^3v f_1\big|_{\mathrm{non}}$, is then calculated from the principal value integral by expanding for $\omega/k_{||}v_{||} \gg 1$ to obtain

$$n_{\mathrm{non}} = \frac{en_e E_{||}\cos\phi}{\pi^{1/2}v_e T k_{||}} P\int_{-\infty}^{\infty} \mathrm{d}v_{||}\frac{v_{||}e^{-v_{||}^2/v_e^2}}{v_{||}-\omega/k_{||}} \approx -\frac{en_e k_{||}E_{||}\cos\phi}{m\omega^2}\left(1+\frac{3k_{||}^2v_e^2}{2\omega^2}+\ldots\right). \tag{7}$$

The resonant contribution to the perturbed density, $n_{\mathrm{res}} = \int d^3v f_1\big|_{\mathrm{res}}$, is

$$n_{\mathrm{res}} \approx -\frac{e\omega E_{||}}{Tk_{||}^2}\Im\mathrm{m}\left[e^{i\phi}\int d^3v \frac{k_{||}^{1/3}f_0}{(v_{\perp z}^2\nu)^{1/3}}\int_0^\infty \mathrm{d}\tau e^{-is\tau-\tau^3/3}\right]. \tag{8}$$

The localization of the resonance makes it convenient to perform the $v_{||}$ integral first by letting $\omega/k_{||} \gg w \gg (\Delta v_{||})_\nu = |\nu v_{\perp z}^2/k_{||}|^{1/3}$ to find

$$\int_{-w}^{w}\mathrm{d}u\frac{\mathrm{d}s}{\mathrm{d}u}f_0\int_0^\infty \mathrm{d}\tau e^{-is\tau-\tau^3/3} \approx 2f_0(u=0)\int_0^\infty \mathrm{d}\tau e^{-\tau^3/3}\frac{\sin(W\tau)}{\tau} \approx \pi f_0(u=0), \tag{9}$$

for $u = v_{||}-\omega/k_{||}$ and $W \equiv (k_{||}/v_{\perp z}^2\nu)^{1/3}w \gg 1$. As a result, collisions cancel to yield

$$4\pi e n_{\mathrm{res}} \approx -\frac{2\pi^{1/2}\omega_p^2\omega E_{||}}{k_{||}^2 v_e^3}e^{-\omega^2/k_{||}^2v_e^2}\sin\phi, \tag{10}$$

due to the delta function character of $\pi^{-1}\int_0^\infty \mathrm{d}\tau e^{-\tau^3/3}\cos(s\tau)$ when integrated over s.

The perturbed Poisson equation, $\nabla\cdot\vec{E} = -4\pi e(n_{\mathrm{non}}+n_{\mathrm{res}})$, then gives

$$\left[\omega^2-\omega_p^2\left(1+\frac{3k_{||}^2v_e^2}{2\omega^2}+\ldots\right)\right]E_{||}\cos\phi = \omega^2\left[\Im\mathrm{m}(\tilde{E})+\frac{2\pi^{1/2}\omega_p^2\omega E_{||}}{k_{||}^3v_e^3}e^{-\omega^2/k_{||}^2v_e^2}\right]\sin\phi, \tag{11}$$

where a small imaginary part of $\tilde{E}$ is required to balance the dissipation – as in the initial value Landau (1946) and collisional Auerbach (1977) solutions. Here, the temporal decay of Landau prescription is replaced by the need to provide power to maintain the steady state. To verify the preceding dispersion relation in a weakly collision kinetic simulation requires resolving parallel velocity scales of the order of $(\Delta v_{||})_\nu = |\nu v_{\perp z}^2/k_{||}|^{1/3} \sim v_e(\nu_e k_{||}^2 v_e^2/\omega^3)^{1/3}$.

The collisional power absorbed is evaluated using $\langle\ldots\rangle_\phi = \oint \mathrm{d}\phi(\ldots)/2\pi$ to find

$$P = -eE_{||}\left\langle\sin\phi\int d^3v v_{||}f_1\right\rangle_\phi \approx -(eE_{||}\omega/k_{||})\left\langle\sin\phi\int d^3v\ f_1|_{\mathrm{res}}\right\rangle_\phi, \tag{12}$$

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since $\langle \sin\phi \ f_1|_{non}\rangle_\phi = 0$ and only the $\sin\phi$ term in $f_1|_{res}$ contributes. As a result

$$P_0 = \frac{e^2 E_{||}^2 \omega^2}{m v_e^2 k_{||}^{8/3}} \int d^3 v \frac{f_0(u=0)}{(v_{\perp z}^2 \nu)^{1/3}} \int_0^\infty \mathrm{d}\tau e^{-is\tau - \tau^3/3}. \tag{13}$$

Then $d^3 v \rightarrow 2\pi v_\perp \mathrm{d}v_\perp \mathrm{d}s (v_{\perp z}^2 \nu / k_{||})^{1/3}$ leads to the usual 'collisionless' O'Neil (1965) result

$$P_0 = \frac{2\pi^2 e^2 E_{||}^2 \omega^2}{m v_e^2 k_{||}^3} \int_0^\infty \mathrm{d}v_\perp v_\perp f_0(u=0) = \left(\frac{E_{||}^2}{4\pi}\right) \frac{\pi^{1/2} \omega_p^2 \omega^2}{k_{||}^3 v_e^3} e^{-\omega^2/k_{||}^2 v_e^2}, \tag{14}$$

even though collisions are crucial! But the absence is misleading as confirmed next.

The full nonlinear problem is considered for a monochromatic wave by allowing $\partial f_1/\partial v_{||} \sim \partial f_0/\partial v_{||}$. Employing the phase $\phi = k_{||} z - \omega t$, and noting $v_\perp$ enters as a parameter, the solution to the lowest order kinetic equation must be of the form $f_1 = f_1(\phi, u)$ and satisfy

$$k_{||} u \partial f_1/\partial \phi - (eE_{||}/m) \sin\phi (\partial f_0/\partial v_{||} + \partial f_1/\partial u) = \nu v_{\perp z}^2 \partial^2 f_1/\partial u^2. \tag{15}$$

Noticing $\partial f_0/\partial v_{||} \approx -(m\omega/T k_{||}) f_0(u=0)$ is a slowly varying function of $v_{||}$, inserting

$$f_1 = g(u, \phi) - (u - \sigma\alpha) \partial f_0/\partial v_{||}, \tag{16}$$

where $\sigma = u/|u| = \pm 1$ or 0, and α is a constant speed to be determined, leads to

$$k_{||} u \partial g/\partial \phi - (eE_{||}/m) \sin\phi \partial g/\partial u = \nu v_{\perp z}^2 \partial^2 g/\partial u^2. \tag{17}$$

Letting

$$j = |m/eE_{||} k_{||}|^{1/2} k_{||} u = |m/eE_{||} k_{||}|^{1/2} (k_{||} v_{||} - \omega), \tag{18}$$

and

$$\Delta = \nu k_{||}^2 v_{\perp z}^2 |m/eE_{||} k_{||}|^{3/2} \propto \nu_{ee}/E_{||}^{3/2}, \tag{19}$$

gives the equation to be the same as the one solved numerically by Hamilton *et al.* (2023),

$$j \frac{\partial g}{\partial \phi} - \sin\phi \frac{\partial g}{\partial j} = \Delta \frac{\partial^2 g}{\partial j^2}. \tag{20}$$

where $\Delta > 0$ since $v_\perp^2 \approx v^2 - \omega^2/k_{||}^2 > 0$ in $v_{\perp z}^2$. Their steady state solution is found by temporally evolving the full equation for g with a slow time derivative inserted.

The nonlinear equation treats the island structure as well as collisions. Taking $j \sim 1$ gives the width of the velocity space island $(\Delta v_{||})_{is} = (v_{||} - \omega/k_{||})_{is}$ to be

$$(\Delta v_{||})_{is} = |eE_{||}/m k_{||}|^{1/2}. \tag{21}$$

A linearized treatment is appropriate when the collisional boundary layer is wider than any island structure, requiring $\Delta = [(\Delta v_{||})_\nu/(\Delta v_{||})_{is}]^3 \gg 1$. However, for a larger amplitude plasma wave in an extremely weak collisionality plasma, the $\Delta \ll 1$ limit is of interest. This limit is considered next and will not lead to the Landau (1946) or Auerbach (1977) results.

The weak collision limit is different than the collisionless limit because the island structure must be retained. In this limit

$$1 \gg |eE_{||}/m k_{||} v_e^2|^{3/2} \gg \nu/k_{||} v_e \sim \nu_e k_{||}^2 v_e^2/\omega^3, \tag{22}$$

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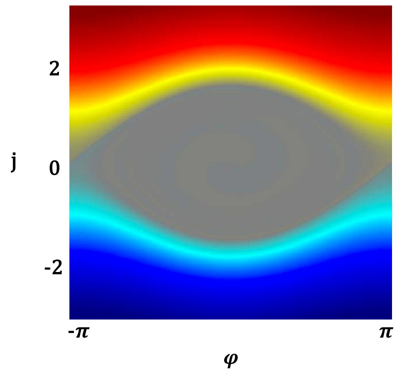

FIGURE 1. Contours of constant $g(j, \varphi)$, with the flattened bound region inside the separatrix at $h = 1$ and the two unbound regions above (in red and yellow) and below (in dark and light blue). Very narrow collisional boundary layers surround the separatrix at $h = 1$ for this $\varDelta = 0.001$ case. (Reprinted with permission from Hamilton *et al.* 2023.)

for $v_{\perp z}^2 \sim v_e^2$. The collisional boundary layers are now much narrower than the islands. They surround the separatrix between the bound (or librating) and unbound (or circulating) electron motion. All the dissipation occurs in these narrow boundary layers.

Hamilton *et al.* (2023) solved the nonlinear equation for g with slow temporal evolution for an astrophysical application. The steady skew symmetric solution satisfies

$$g(j, \phi) = -g(-j, -\phi). \tag{23}$$

An analytic treatment is possible for $\varDelta \ll 1$ by introducing the reduced constant of the motion

$$h(j, \phi) = j^2/2 - \cos\phi, \tag{24}$$

with $h = 1$ the location of the separatrix enclosed by a narrow collisional boundary layer separating the bound $(-1 < h < 1)$ and the unbound $(h > 1)$ electrons. The $\varDelta = 0.001$ case shown in their figure 2(*a*) is reproduced here with their kind permission as figure 1. The reduced Hamiltonian allows the kinetic equation to be rewritten in terms of the new variables h, ϕ as

$$\left.\frac{\partial g}{\partial \phi}\right|_h = \varDelta \left.\frac{\partial}{\partial h}\right|_\phi \left(j \left.\frac{\partial g}{\partial h}\right|_\phi \right). \tag{25}$$

The $\varDelta \ll 1$ limit allows a solution of the form $g = g_1(h) + g_2(h, \phi) + \cdots$ to be found. To lowest order $\partial g_1/\partial \phi|_h = 0$, while to next order

$$\left.\frac{\partial g_2}{\partial \phi}\right|_h = \varDelta \left.\frac{\partial}{\partial h}\right|_\phi \left(j \left.\frac{\partial g_1}{\partial h}\right|_\phi \right). \tag{26}$$

The solution to this equation, g_1, must satisfy the collisional solubility constraint

$$\left.\frac{\partial}{\partial h}\right|_\phi \left[\left(\oint_h \mathrm{d}\phi\, j \right) \left.\frac{\partial g_1}{\partial h}\right|_\phi \right] = 0. \tag{27}$$

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It is independent of collisions, but its form is collisionally constrained. The partial solution of Hamilton *et al*. (2023) for $\Delta \ll 1$ was completed by Catto (2025*a*) for a stellarator application. It vanishes for bound orbits. For the unbound, $\int_{-\pi}^{\pi} \mathrm{d}\varphi j = \sigma 8k^{-1}E(k)$, with $k=\sqrt{2/(h+1)}$ and $E(k)$ a complete elliptic integral. To match f_1 to the non-resonant solution $f_1|_{non}$ as $k \to 0$

$$\left.\frac{\partial g_1}{\partial h}\right|_{\phi} = \left|\frac{eE_{||}}{mk_{||}}\right|^{1/2} \frac{\sigma \pi k}{4E(k)} \frac{\partial f_0}{\partial v_{||}}. \tag{28}$$

Integrating from the separatrix at $k=1$ using $\mathrm{d}h = -4\mathrm{d}k/k^3$ leads to

$$g_1 = \sigma\pi \left|\frac{eE_{||}}{mk_{||}}\right|^{1/2} \frac{\partial f_0}{\partial v_{||}} \int_k^1 \frac{\mathrm{d}t}{t^2 E\left(t\right)} \underset{k\to 0}{\rightarrow} \sigma \left|\frac{eE_{||}}{mk_{||}}\right|^{1/2} \frac{\partial f_0}{\partial v_{||}} \left[\frac{2}{k} - 1.379 - \frac{k}{2} + O\left(k^3\right)\right]$$

$$\approx \left|\frac{eE_{||}}{mk_{||}}\right|^{1/2} \frac{\partial f_0}{\partial v_{||}} \left[j - 1.379\sigma - \frac{\cos\phi}{j}\right] = \left(u - \sigma\alpha - \frac{eE_{||}\cos\phi}{mk_{||}u}\right) \frac{\partial f_0}{\partial v_{||}}, \tag{29}$$

with $j = \pm\sqrt{2(h+\cos\phi)} = \pm(2/k)\sqrt{1-k^2\sin^2(\phi/2)}$, and $\alpha = 1.379|eE_{||}/mk_{||}|^{1/2}$ now determined. The unbound solution satisfies $g_1 \to 0$ at the separatrix ($h=1$), but $\partial g_1/\partial h|_{\varphi}$ and f_1 step crossing it. The narrow, unresolved collisional boundary layer about the separatrix provides the smooth matching. Far from the resonance layer $f_1 = g_1 - (u-\sigma\alpha)\partial f_0/\partial v_{||} \to (eE_{||} f_0/k_{||}u)\cos\phi = f_1|_{non}$ as desired.

The power absorbed is evaluated using $P = -eE_{||}\langle \sin\phi \int d^3v v_{||} f_1\rangle_{\phi}$. The details differ slightly from a recent lower hybrid current drive calculation (Catto 2025*b*) as the plasma wave evaluation is for an unmagnetized plasma. Using skew symmetry gives $\langle \int_{-\infty}^{\infty} \mathrm{d}u u \sin\phi f_1(u,\phi)\rangle_{\phi} = 0$. Integrating by parts using $2\sin\phi = -\mathrm{d}j^2/\mathrm{d}\phi|_h$ at fixed h, and inserting the kinetic equation leads to

$$P = \frac{\omega eE_{||}}{2k_{||}} \left\langle \int d^3v f_1 \left.\frac{\mathrm{d}j^2}{\mathrm{d}\phi}\right|_h \right\rangle_{\phi} = -\frac{\omega eE_{||}}{2k_{||}} \left\langle \int d^3v j^2 \left.\frac{\partial g_2}{\partial\phi}\right|_h \right\rangle_{\phi}$$

$$= -\frac{\omega eE_{||}}{2k_{||}} \left\langle \int d^3v \Delta j^2 \left.\frac{\partial}{\partial h}\right|_{\phi} \left(j \left.\frac{\partial g_1}{\partial h}\right|_{\phi}\right)\right\rangle_{\phi}. \tag{30}$$

In addition

$$\left\langle \int_{-\infty}^{\infty} \mathrm{d}j j^2 \left.\frac{\partial}{\partial h}\right|_{\phi} j \left.\frac{\partial g_1}{\partial h}\right|_{\phi}\right\rangle_{\phi} = 2\left\langle \int_1^{\infty} \mathrm{d}h j \left.\frac{\partial}{\partial h}\right|_{\phi} j \left.\frac{\partial g_1}{\partial h}\right|_{\phi}\right\rangle_{\phi}$$

$$= 2\int_1^{\infty} \mathrm{d}h \left[\left.\frac{\partial}{\partial h}\right|_{\phi} \langle j^2\rangle_{\phi} \left.\frac{\partial g_1}{\partial h}\right|_{\phi} - \left.\frac{\partial g_1}{\partial h}\right|_{\phi}\right]$$

$$= 4h \left.\frac{\partial g_1}{\partial h}\right|_{\phi,h=1}^{h\to\infty} - 2\left. g_1\right|_{h=1}^{h\to\infty} = 0.384 \left|\frac{eE_{||}}{mk_{||}}\right|^{1/2} \frac{\partial f_0}{\partial v_{||}}. \tag{31}$$

Using $d^3v \to 2\pi v_{\perp}\mathrm{d}v_{\perp}\mathrm{d}j/(\mathrm{d}j/\mathrm{d}u)$ and $\partial f_0/\partial v_{||} \approx -(m\omega/Tk_{||})f_0(u=0)$, yields

$$P = 0.384\frac{mn\nu_e\omega^2}{\sqrt{\pi}k_{||}^2 v_e^5} \left|\frac{eE_{||}}{mk_{||}}\right|^{1/2} e^{-\omega^2/k_{||}^2 v_e^2} \int_{\omega/k_{||}}^{\infty} \mathrm{d}v_{\perp} v_{\perp} \frac{v_{\perp z}^2}{x^3} e^{-v_{\perp}^2/v_e^2}. \tag{32}$$

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The power absorbed for $\omega^2/k_{||}^2 v_e^2 \gg 1$ is explicitly collisional and given by

$$P \approx 0.144(Z+2)mnv_e^2\nu_{ee}|eE_{||}k_{||}/m\omega^2|^{1/2}e^{-\omega^2/k_{||}^2v_e^2}. \tag{33}$$

Normalizing P to the result P_0 yields

$$\frac{P}{P_0} \approx 0.081(Z+2)\frac{\nu_{ee}k_{||}^5v_e^5}{\omega^3}\left|\frac{m}{eE_{||}k_{||}}\right|^{3/2} \sim \varDelta. \tag{34}$$

Consequently, Landau's (1946) collisionless results do not hold as $\nu_{ee} \to 0$ for a finite $E_{||}$.

Remarkably, the limits considered here demonstrate weak collisions play the key role in obtaining what is viewed as 'collisionless Landau damping' of a resonant plasma wave. The behavior is a characteristic of any kinetic equation involving a velocity space dependent resonance that must be resolved by diffusive collisions in a boundary layer. It is only when the collision frequency is very small and/or the amplitude becomes sufficiently large that departures from the deceptively looking 'collisionless' Landau (1946) limit arise. For a larger amplitude monochromatic plasma wave, nonlinearity occurs because of the appearance of island structure whose separatrix is enclosed by a very narrow collisional boundary layer that must be present – even as the collision frequency becomes very small. Consequently, this velocity space fine structure prevents Landau's limit from being recovered in the limit of vanishing collision frequency for a plasma wave of non-vanishing amplitude. Moreover, energy must cascade to these ever finer velocity scales to be dissipated. The key role of collisions implies the seemingly 'collisionless' resonant Landau (1946) limit is actually a collisional plateau (or resonant plateau) regime, with $\varDelta > 1$, located between the small $\varDelta < 1$ nonlinear regime treated here and a fully collisional plasma limit with $\nu_{ee} \gtrsim \omega$. Thus, the Landau (1946) limit is valid when $1 \gg \nu_{ee}/\omega \gg (\omega/k_{||}v_e)^2|eE_{||}/k_{||}T_e|^{3/2}$.

Note added in proofs. François Waelbroeck has kindly brought to my attention a classic paper by Zakharov and Karpman (1963) in which they solved the temporally evolving plasma wave problem of Landau but with collisions. The treatment in this paper solves the steady state driven plasma wave problem with collisions. Some details differ (like some details of the collision operator and the coefficient of the power absorbed in the weakly collisional, large plasma wave amplitude limit), but many other details are broadly the same. Their pioneering treatment should be consulted for full details.

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Declaration of interests

The author reports no conflict of interest.

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