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ON THE NUMBER OF BINOMIAL COEFFICIENTS WHICH ARE DIVISIBLE BY THEIR ROW NUMBER

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ABSTRACT. If n is a natural number, let $A(n)$ be the number of integers, k, such that $0<k<n$ and n divides $\binom{n}{k}$. Then $\phi(n) \leq A(n) \leq n-1-2\omega(n)+\varepsilon$, where $\omega(n)$ denotes the number of distinct prime factors of n, and $\varepsilon=0$ unless n is twice a prime, in which case $\varepsilon=1$.

Introduction

DEFINITION 1. If n is a natural number, let $A(n)$ be the number of integers, k, such that $0<k<n$ and n divides $\binom{n}{k}$.

Although an extensive literature exists concerning the divisibility properties of binomial coefficients, just one recent article [2] dealt directly with $A(n)$, only to obtain an asymptotic result. In this article, we develop some properties of $A(n)$; p always designates a prime.

Preliminaries

DEFINITION 2. $0_n(m)=k$ if $n^m \mid k$ but $n^{m+1} \nmid k$.

DEFINITION 3. $t_p(n)=\sum_{i=0}^r a_i$ if $n=\sum_{i=0}^r a_i p^i$.

(1) $0_p(ab)=0_p(a)+0_p(b)$

(2) $[a+b]-1 \leq [a]+[b] \leq [a+b]$

(3) $\binom{n}{k}=\binom{n}{n-k}$

(4) $t_p(ap^j)=t_p(a)$ if $p \nmid a$

(5) $t_p(a)=1$ if and only if $a=p^j$ for some $j \geq 0$

(6) $t_2(n-k)=t_2(n)-t_2(k)$ if $n=2^m-1$ and $0 \leq k \leq n$

(7) $0_p(n!)=\sum_{j=1}^{\infty}[n/p^j]$

(8) $0_p\left(\binom{n}{k}\right)=\{t_p(k)+t_p(n-k)-t_p(n)\}/(p-1)$

(9) $n\binom{n-1}{k-1}=k\binom{n}{k}$

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THEOREM 1. *If* $0<k<n$ *and* $(k,n)=1$, *then* $n/\binom{n}{k}$.

Proof. Follows directly from hypothesis and (9).

COROLLARY 1. $A(n)\geq\phi(n)$ *for all* n.

Proof. Follows from Definition 1 and Theorem 1.

REMARK 1. It is possible for n^2 to divide $\binom{n}{k}$, for example if $n=30$ and $k=7$ or 11.

THEOREM 2. *If* $0<k<p^e$ *and* $p\nmid k$, *then* $0_p\left(\binom{p^e}{k}\right)=e$.

Proof.

$$(1),(7)\to 0_p\left(\binom{p^e}{k}\right)=\sum_{j=1}^{\infty}\{[p^e/p^j]-[k/p^j]-[(p^e-k)/p^j]\}$$
$$=\sum_{j=1}^{e}\{p^{e-j}-[k/p^j]-[(p^e-k)/p^j]\}$$

Hypothesis $\to p\nmid k(p^e-k)\to[k/p^j]<k/p^j$, $[(p^e-k)/p^j]<(p^e-k)/p^j$, so that $[k/p^j]+[(p^e-k)/p^j]<p^{e-j}$. Since

$$[p^{e-j}]=p^{e-j}\quad\text{for}\quad 1\leq j\leq e,\qquad (2)\to p^{e-j}-[k/p^j]-[(p^e-k)/p^j]=1,$$

so that $0_p\left(\binom{p^e}{k}\right)=\sum_{j=1}^{e}1=e$.

THEOREM 3. *If* $p\nmid ab$, $j\leq k$, *and* $0<a<bp^{k-j}$, *then* $0_p\left(\binom{bp^k}{ap^j}\right)=0_p\left(\binom{bp^{k-j}}{a}\right)$.

Proof.

$$(8),(4)\to 0_p\left(\binom{bp^k}{ap^j}\right)=\{t_p(ap^j)+t_p(bp^k-ap^j)-t_p(bp^k)\}/(p-1)$$
$$=\{t_p(a)+t_p(bp^{k-j}-a)-t_p(bp^{k-j})\}/(p-1)=0_p\left(\binom{bp^{k-j}}{a}\right).$$

THEOREM 4. *If* $p\nmid a$, $j\leq e$, *and* $0<ap^{e-j}$, *then* $0_p\left(\binom{p^e}{ap^j}\right)=e-j$.

Proof. Apply Theorem 3 with $b=1$, then apply Theorem 2.

THEOREM 5. $A(p^e)=\phi(p^e)=p^{e-1}(p-1)$.

Proof. If $0<k<p^e$ and $p\mid k$, let $k=ap^j$, where $p\nmid a$ and $1\leq j<e$.

Theorem $4\to 0_p\left(\binom{p^e}{k}\right)=e-j<e\to p^e\nmid\binom{p^e}{k}$. The conclusion now follows from Corollary 1.

REMARK 2. Theorem 5 also follows from [1, 4.12].

THEOREM 6. *If* $p \mid n$, *then* $n \nmid \binom{n}{p}$.

Proof. Let $n = bp^e$, where $b \geq 1$ and $p \nmid b$.

$$0_p\left(\binom{n}{p}\right) = 0_p\left(\binom{bp^e}{p}\right) = 0_p\left(\binom{bp^{e-1}}{1}\right) = 0_p(bp^{e-1}) = e-1 \to p^e \nmid \binom{n}{p} \to n \nmid \binom{n}{p}.$$

COROLLARY 2. $A(n) \leq n - 1 - 2\omega(n) + \epsilon$, *where* $\omega(n)$ *denotes the number of distinct prime factors of* n, *and* $\epsilon = \begin{cases} 1 & \text{if } n = 2p \\ 0 & \text{if } n \neq 2p \end{cases}$.

Proof. Follows from Theorem 6 and (3).

THEOREM 7. *If* $n = 2p$, *where* $p = 2^k - 1$, *then* $A(n) = \phi(n)$.

Proof. By Theorems 1 and 6, and by (3), it suffices to show that $2 < 2m < p$ implies $2 \nmid \binom{2p}{2m}$. Using (8), (4), and (6), we have

$$0_2\left(\binom{2p}{2m}\right) = t_2(2m) + t_2(2p-2m) - t_2(2p) = t_2(m) + t_2(p-m) - t_2(p) = 0.$$

REMARK 3. There exist integers n such that $A(n) = \phi(n)$, yet n is neither a prime power nor twice a Mersenne prime, for example $n = 15$ or 51.

THEOREM 8. If $n = 2p$, where $p = 2^k + 1$, then $A(n) = n - 2\omega(n) = n - 4$.

Proof. By Theorems 1 and 6 and by (3), it suffices to show that $2 < 2m < p$ implies $2p \mid \binom{2p}{2m}$. Clearly, $p \mid \binom{2p}{2m}$. As in the proof of Theorem 7, we have $0_2\left(\binom{2p}{2m}\right) = t_2(m) + t_2(p-m) - t_2(p)$. Now hypothesis implies $t_2(p) = 2$; p odd implies $p - m \not\equiv m \pmod 2$. Thus (5) implies $t_2(m) + t_2(p-m) \geq 3$, hence $0_2\left(\binom{2p}{2m}\right) \geq 1$, and $2 \mid \binom{2p}{2m}$.

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