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A DECOMPOSITION FOR SETS HAVING A SEGMENT CONVEXITY PROPERTY

MARILYN BREEN

1. Introduction. Let S be a subset of Euclidean space. The set S is said to be *m-convex*, $m \geqq 2$, if and only if for every m distinct points of S, at least one of the line segments determined by these points lies in S. Clearly any union of $m - 1$ convex sets will be m-convex, yet the converse is false. However, several decomposition theorems have been proved which allow us to write any closed planar m-convex set as a finite union of convex sets, and actual bounds for the decomposition in terms of m have been obtained ([**6**], [**4**], [**3**]). Moreover, with the restriction that $(\text{int cl } S) \sim S$ contain no isolated points, an arbitrary planar m-convex set S may be decomposed into a finite union of convex sets ([**1**]).

Here we strengthen the m-convexity condition to define an analogous combinatorial property for segments. A set S in Euclidean space is said to have the *segment convexity property* $P(m)$, $m \geqq 2$, if and only if for every m segments s_i, $1 \leqq i \leqq m$, (possibly degenerate) in S, at least one of the corresponding convex hulls $\text{conv}\,(s_i \cup s_j)$, $1 \leqq i < j \leqq m$, lies in S. It is proved that if S is any planar set having property $P(m)$, then S is a union of $m - 1$ convex sets. The result is best possible for every m.

The following familiar terminology will be used. A point x in S is said to be a *point of local convexity of* S if and only if there is some neighborhood N of x such that $S \cap N$ is convex. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) of S. For points x and y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. Points $x_1, \ldots, x_n$ in S are *visually independent via* S if and only if for $1 \leqq i < j \leqq n$, x_i does not see x_j via S. Throughout the paper, conv S, cl S, bdry S, int S and ker S will be used to denote the convex hull, closure, boundary, interior, and kernel, respectively, of the set S.

2. The case for closed sets. We begin by restricting our attention to closed sets, and we have the following characterization theorem.

THEOREM 1. *Let* $S = \text{cl (int } S)$ *be a set in* R^d. *Then* S *is expressible as a union of* $m - 1$ *maximal convex sets* C_i *with* $C_i \cap C_j$ *at most a singleton set for* $1 \leqq i < j \leqq m - 1$ *if and only if for every* m *segments in* int S, *at least one of the corresponding pairs has its convex hull in* S.

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Proof. To prove necessity, assume that S is expressible as the required union of convex sets C_i, $1 \leqq i \leqq m - 1$, and let $s_1, \ldots, s_m$ denote m segments in $\operatorname{int} S$. We suppose that no corresponding pair has its convex hull in S to reach a contradiction. Then some segment, say s_i, has nonempty intersection with more than one of the convex sets C_i, and for an appropriate labeling, s_1 contains some point p in bdry $C_1 \cap$ bdry C_2.

It is easy to see that $p \in$ bdry S: Otherwise, if $C_1, \ldots, C_k$ are those C_i sets which contain p, we may select a convex neighborhood N of p with cl $N \subseteq C_1 \cup \ldots \cup C_k$. However, then cl N is a union of the k convex sets cl $N \cap C_i$, $1 \leqq i \leqq k$, clearly impossible since $C_i \cap C_j = \{p\}$ for $1 \leqq i < j \leqq k$. Thus $p \in$ bdry S.

Since $s_1 \subseteq \operatorname{int} S$ and $p \in s_1 \cap$ bdry S, we have a contradiction. Our supposition must be false, and one of the corresponding convex hulls conv $\{s_i \cup s_j\}$, $i \neq j$, lies in S, the desired result.

To prove sufficiency, assume that for every m segments in $\operatorname{int} S$, one of the corresponding pairs has its convex hull in S. Let Q denote the set of lnc points of S. We will show that $S \sim Q$ has at most $m - 1$ components, each with convex closure, and that these closures provide a suitable decomposition for S. To begin, let A be a component of $S \sim Q$ and let $K \equiv$ cl A. For $z \in A$, certainly $z \notin Q$ so for some neighborhood M of z, $S \cap M$ is convex (and hence disjoint from Q). Thus z sees each point of $S \cap M$ via $S \sim Q$, and $S \cap M = A \cap M$. Since $S =$ cl (int S), $z \in$ cl (int A), and it is easy to see that cl $A \equiv K =$ cl (int K). Using this observation, it is not hard to show that $S \sim Q$ has at most $m - 1$ components: Otherwise, we could select m segments in $\operatorname{int} S$, each from a distinct component of $S \sim Q$, and with no two segments collinear. Then none of the corresponding pairs could have its convex hull in S, contradicting our hypothesis. We conclude that $S \sim Q$ has at most $m - 1$ components.

It remains to show that each of these components has convex closure. Let K denote the closure of a component of $S \sim Q$, and assume that K is not convex to reach a contradiction. Standard arguments reveal that every lnc point of K is an lnc point of S. Then since $K \sim Q$ is connected, we may use [**2**, Theorems 2 and 3] to conclude that K has at least one essential lnc point q. That is, for every neighborhood U of q there is at least one component W of $K \cap U \sim \{q\}$ such that q is an lnc point of cl W. It is not hard to show that S is m-convex and hence locally starshaped [**5**, Lemma 2], so we may select a convex neighborhood N of q such that $S \cap N$ is starshaped at q. Using the fact that $K =$ cl(int K), we may prove that $K \cap N$ is starshaped at q. Moreover, since q is an essential lnc point of K, there is a component B' of $K \cap N \sim \{q\}$ such that q is an lnc point for $B \equiv$ cl B'. Again using the facts that S is locally starshaped and $K =$ cl (int K), it is easy to see that B' is locally starshaped. Then since B' is connected and locally starshaped, standard arguments may be applied to show that B' is polygonally connected. Furthermore, it is clear that B is locally starshaped, $q \in$ ker B, and $B =$ cl (int B).

Since q is an Inc point for B, select points x and y in $B \cap N$ such that $[x, y] \not\subseteq B$, and without loss of generality, assume $x, y \in \operatorname{int} B$. Hence there exist neighborhoods V and W of x and y, respectively, with $V \cup W \subseteq B$, and q sees every point of $V \cup W$ via B.

To finish the argument, we consider two cases:

Case 1. If some point of $[x, q)$ sees any point of $[y, q)$ via B, then by an easy geometric argument involving conv $(V \cup \{q\})$, x sees some point y' of (y, q) via B. Then $[x, y'] \cup [y', y] \subseteq B$, $[x, y] \not\subseteq B$, so by a lemma of [**7**, Corollary 2], conv $\{x, y', y\}$ contains an Inc point q_2 of B, and $q_2 \in N$. Then using our earlier argument, we may select points x_2 and y_2 near q_2 in B and neighborhoods V_2 and W_2 of x_2 and y_2, respectively, so that $[x_2, y_2] \not\subseteq B$, $V_2 \cup W_2 \subseteq B$, and q_2 sees every point of $V_2 \cup W_2$ via B. Hence

$$\operatorname{conv}(V_2 \cup \{q, q_2\}) \cup \operatorname{conv}(W_2 \cup \{q, q_2\}) \subseteq B.$$

Clearly for any two segments s_1 and s_2 containing q_2 and having maximal length in B, conv $\{s_1 \cup s_2\} \not\subseteq S$. (Otherwise, q_2 could not be an Inc point of B.) Hence for an appropriate choice of m segments in int B chosen sufficiently close to q_2, no pair of segments has its corresponding convex hull in S. We have a contradiction, our assumption is false, and Case 1 cannot occur.

Case 2. Suppose that no point of $[x, q)$ sees any point of $[y, q)$ via B. Recall that B' is polygonally connected. Clearly $x, y \in B'$, so there is a polygonal path λ in B' from x to y, and $q \notin \lambda$. Then since $q \in \ker B$, there is a simply connected subset D of B such that

$$D \cap \operatorname{int} \operatorname{conv} \{x, q, y\} = \emptyset \text{ and}$$

$$\{q\} \subseteq \operatorname{bdry} D \subseteq \lambda \cup [x, q] \cup [y, q].$$

Using the fact that $q \notin \lambda$ and repeating an argument in Case 1, we see that an appropriate selection of m segments in int B and sufficiently close to q gives the required contradiction. Therefore Case 2 cannot occur.

Our assumption that K is not convex must be false, and we conclude that every component of $S \sim Q$ has convex closure. Since there are at most $m - 1$ such components, this yields a decomposition of S into $m - 1$ closed convex sets $C_1, \ldots, C_{m-1}$. Furthermore, since $C_i = \operatorname{cl}(\operatorname{int} C_i)$, $1 \leqq i \leqq m - 1$, it is easy to show that each C_i set is maximal and that $C_i \cap C_j$ is at most a singleton set for $i \neq j$. This completes the proof of Theorem 1.

Remark. If we replace the requirement $S = \operatorname{cl} \operatorname{int} S$ with the weaker condition $\operatorname{cl} S = \operatorname{cl} \operatorname{int} S$, then the sufficiency in Theorem 1 fails and, in fact, S is not necessarily a finite union of convex sets. (Delete rational points from an edge of the unit square U to obtain an easy counterexample.)

Similarly, the necessity in Theorem 1 fails if the segments are required to lie in S instead of in int S. (Consider the union of the unit square U with $-U$.)

COROLLARY. *Let S be a close planar set having property $P(m)$. Then S is a union of $m - 1$ or fewer convex sets.*

Proof. If $S = \text{cl}\ (\text{int}\ S)$, the result is an immediate consequence of Theorem 1. Otherwise, techniques used in [**3**] may be used to write S as a union of k segments and a closed set S' having property $P(m - k)$ for some $1 \leqq k \leqq m - 2$. An obvious induction applied to S' completes the proof.

3. The general case.

THEOREM 2. *If S is a subset of $\mathbf{R}^2$ having property $P(m)$, then S is a union of $m - 1$ or fewer convex sets. The number $m - 1$ is best possible for every $m \geqq 2$.*

Proof. Without loss of generality, we may assume that $\text{cl}\ S = \text{cl}\ (\text{int}\ S)$, for otherwise S will be a union of k segments and a set S' having property $P(m - k)$ for some $1 \leqq k \leqq m - 2$, and an easy induction finishes the argument. Thus the set $\text{cl}\ S$, while not necessarily having property $P(m)$, will satisfy the segment condition in the hypothesis of Theorem 1. Hence $\text{cl}\ S$ will be a union of $m - 1$ or fewer maximal convex sets C_i, with $C_i \cap C_j$ at most a singleton set for $1 \leqq i < j \leqq m - 1$. Also, by the proof of Theorem 1, $C_i = \text{cl}\ (\text{int}\ C_i)$ for each i, and it is easy to see that $A_i = C_i \cap S$ has property $P(k_i)$ for some $k_i \leqq m$. Since $\text{cl}\ S = \text{cl}\ (\text{int}\ S)$, clearly $C_i = \text{cl}\ (\text{int}\ A_i) = \text{cl}\ A_i$.

We will examine the points of $C_i \sim A_i = \text{cl}\ A_i \sim A_i$, and first we consider points in $\text{int}\ (\text{cl}\ A_i) \sim A_i$. Let x be such a point. Since S is m-convex, by [**1**, Lemma 4] either x is an isolated point or x lies in a segment in $\text{int}\ (\text{cl}\ A_i) \sim A_i$. However, x cannot be isolated: Otherwise, for an appropriate collection of m segments in A_i having midpoints sufficiently close to x, none of the corresponding pairs would have its convex hull in S, contradicting our hypothesis. Hence x must lie in a segment in $\text{int}\ (\text{cl}\ A_i) \sim A_i$. Moreover, by [**1**, Corollary to Lemma 2], $\text{int}\ (\text{cl}\ A_i) \sim A_i$ contains at most $2^{m-2} - 1$ noncollinear segments, so clearly $\text{int}\ (\text{cl}\ A_i) \sim A_i$ is a finite union of segments.

Therefore, if x is a point in $\text{int}\ (\text{cl}\ A_i) \sim A_i$, then x lies in some polygonal path $\lambda \subseteq \text{int}\ (\text{cl}\ A_i) \sim A_i$, where λ is maximal. Now if z is an endpoint of λ, $z \notin \text{int}\ (\text{cl}\ A_i)$: Otherwise, since z lies in the closure of at most finitely many segments in $\text{int}\ (\text{cl}\ A_i) \sim A_i$, an earlier argument would yield m segments in S with no pair having its convex hull in S, which is impossible. Thus the points of $\text{int}\ (\text{cl}\ A_i) \sim A_i$ induce a partition of $\text{int}\ (\text{cl}\ A_i) \cap A_i$ into components T such that $T = \text{int}\ \text{cl}\ T$.

We assert that each set T is convex, and clearly it suffices to show that $\text{cl}\ T$ is convex: Otherwise, $\text{cl}\ T$ would have as an lnc point some vertex of a polygonal path in $\text{int}\ (\text{cl}\ A_i) \sim A_i$, and an appropriate choice of m segments in T would contradict the fact that S has property $P(m)$. Thus each component T of $\text{int}\ (\text{cl}\ A_i) \cap A_i$ is convex. Clearly if A_i has the property $P(k_i)$, there are at most $k_i - 1$ corresponding components T, and we let T_{ij}, $1 \leqq j \leqq k_i - 1$

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denote these sets. Finally, define $B_{ij} = \text{cl } T_{ij} \cap A_i$, so that $A_i = \bigcup\{B_{ij}: 1 \leqq j \leqq k_i - 1\}$, $1 \leqq i \leqq m - 1$.

For future reference, we make several observations concerning the structure of the B_{ij} sets. First, if there exist points z, w in some B_{ij} such that $[z, w] \not\subseteq B_{ij}$, then z and w lie in a segment in bdry B_{ij}. Second, using the fact that S has property $P(\text{m})$, it is easy to show that for any two distinct B_{ij} sets, say B_1 and B_2, and for any pair of nondegenerate segments s_1 and s_2 in B_1 and B_2, respectively, conv $(s_1 \cup s_2) \subseteq S$ only in case s_1 and s_2 are collinear, with $s_1 \subseteq \text{bdry } B_1$ and $s_2 \subseteq \text{bdry } B_2$. Finally, if V_{ij} is a maximal visually independent subset of B_{ij}, then since cl B_{ij} = cl int B_{ij} and cl B_{ij} is convex, to each point p of V_{ij} we may associate a segment s_p having endpoint p such that $s_p \subseteq B_{ij}$ and $s_p \not\subseteq \text{bdry } B_{ij}$. Then if B_{ij} is n_{ij}-convex, $1 \leqq j \leqq k_i - 1$, $1 \leqq i \leqq m - 1$, repeating this procedure for each B_{ij} set yields $\sum_i \sum_j (n_{ij} - 1)$ distinct (but not necessarily disjoint) segments, and by our comments above, no pair of these segments has its convex hull in S. Hence $\sum_i \sum_j (n_{ij} - 1) \leqq m - 1$.

Since cl B_{ij} is convex and all points of cl $B_{ij} \sim B_{ij}$ are in bdry B_{ij}, by [**1**, Lemma 1], B_{ij} is a union of max $(\text{n}_{ij} - 1, 3)$ or fewer convex sets. In fact, if $n_{ij} = 2$ or $n_{ij} > 3$, B_{ij} will be a union of $n_{ij} - 1$ or fewer convex sets. Assume that exactly r of the B_{ij} sets are 3-convex and are not a union of two convex sets. Then

$$S = \bigcup\{B_{ij}: 1 \leqq j \leqq k_i, 1 \leqq i \leqq m - 1\}$$

will be a union of $\sum_i \sum_j (n_{ij} - 1) + r$ or fewer convex sets, and we will show that

$$\sum_i \sum_j (n_{ij} - 1) + r \leqq m - 1.$$

For convenience of notation, let $\mathscr{B}$ denote the family of sets B_{ij} which are 3-convex and are not expressible as a union of two convex sets. Select B in $\mathscr{B}$. By the proof of [**1**, Lemma 1], without loss of generality we may assume that cl B is a convex polygon. Order the vertices of cl B in a clockwise direction along bdry B, letting p_i denote the i-th vertex in our ordering. Again by the proof of [**1**, Lemma 1], since any decomposition for B requires three convex sets, cl B has an odd number l of vertices, each vertex of cl B lies in B, and each edge of cl B contains some point not in B. Also, by the 3-convexity of B, $l > 3$. Hence the segments $[p_1, p_3]$, $[p_1, p_4]$, $[p_2, p_4]$ lie in B, no segment lies in bdry B, and no pair of these segments has its convex hull in B. Repeat the procedure for each B in $\mathscr{B}$, and let $\mathscr{L}_1$ denote the corresponding collection of $3r$ segments obtained. Clearly no pair of segments in $\mathscr{L}_1$ has its convex hull in S, since no segment in $\mathscr{L}_1$ lies in the boundary of any B_{ij} set.

Next, for each set B_{ij} not in $\mathscr{B}$, select $n_{ij} - 1$ visually independent points, and use a previous argument to choose a corresponding collection $\mathscr{L}_2$ of

$$\sum_i \sum_j (n_{ij} - 1) - \sum_i \sum_j \{(n_{ij} - 1): B_{ij} \in \mathscr{B}\} = \sum_i \sum_j (n_{ij} - 1) - 2r$$

segments in S, with no pair having its convex hull in S. Then $\mathscr{L}_1 \cup \mathscr{L}_2$ contains exactly $\sum_i \sum_j (n_{ij} - 1) + r$ segments, clearly no pair has its convex hull in S, and hence

$$\sum_i \sum_j (n_{ij} - 1) + r \leqq m - 1.$$

We conclude that S is a union of $m - 1$ or fewer convex sets, the desired result. Certainly the bound $m - 1$ is best possible, and the proof of Theorem 2 is complete.

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The University of Oklahoma,
Norman, Oklahoma