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LINEAR DETERMINING EQUATIONS FOR DIFFERENTIAL CONSTRAINTS

OLEG V. KAPTSOV and ALEXEY V. SCHMIDT
Institute of Computing Modelling RAS, Akademgorodok, 660036, Krasnoyarsk, Russia
e-mail: kaptsov@ksc.krasn.ru

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Abstract. A construction of differential constraints compatible with the Gibbons-Tsarev equation is considered. Certain linear determining equations with parameters are used to find such differential constraints. They generalize the classical determining equations that are used in searching for Lie operators. We introduce the notion of an invariant solution under an involutive distribution and give sufficient conditions for existence of such solutions.

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1. Introduction. As is well known, one can produce many exact solutions of partial differential equations by means of additional constraints [**1**], [**2**]. Differential constraints arose originally in the theory of partial differential equations of the first order. Lagrange in particular used differential constraints to find the total integral of the nonlinear equation

$$F(x, y, u, u_x, u_y) = 0.$$

Darboux [**3**] applied differential constraints to integrate partial differential equations of second order. The detailed description of the Darboux method can be found in [**1**], [**4**].

The general formulation of the method of differential constraints requires that the original system of partial differential equations

$$F^1 = 0, \dots, \quad F^m = 0 \tag{1.1}$$

be enlarged by appending additional differential equations (differential constraints)

$$h_1 = 0, \dots, \quad h_p = 0, \tag{1.2}$$

such that the over-determined system (1.1), (1.2) satisfies some conditions of compatibility.

The theory of over-determined systems was developed by Delassus, Riquier, Cartan, Ritt, Kuranishi, Spencer and others. One can find references in the book of Pommaret [**5**]. Now the applications of over-determined systems include such diverse fields as differential geometry, continuum mechanics and nonlinear optics. Unfortunately the problem of finding all differential constraints compatible with certain equations can be more complicated than the investigation of the original equations. Therefore it is better to content oneself with finding constraints in some classes, and these classes must be chosen using additional considerations.

Recently a new method was proposed for finding differential constraints, which uses linear determining equations. These equations are more general than the classical determining equations for Lie generators [**6**] and depend on some parameters. Given an evolution equation

$$u_t = F(t, x, u, u_1, \ldots, u_n), \tag{1.3}$$

where $u_k = \frac{\partial^k u}{\partial x^k}$, then according to [**7**] the linear determining equation corresponding to (1.3) is of the form

$$D_t(h) = \sum_{i=0}^{n} \sum_{k=0}^{i} b_{ik} D_x^{i-k}(F_{u_{n-k}}) D_x^{n-i}(h), \qquad b_{ik} \in R. \tag{1.4}$$

Here, and throughout, D_t, D_x are the operators of total differentiation with respect to t and x. Equality (1.4) must hold for all solutions of (1.3). The function h may depends on $t, x, u, u_1, \ldots, u_p$. The number p is called the order of the solution of the equation (1.4). If we have some solution h, the corresponding differential constraint is

$$h = 0. \tag{1.5}$$

It was also shown in [**7**] that the equations (1.4) and (1.5) constitute a compatible system. Applications of this approach to diffusion equations can be found in [**8**].

The organization of this chapter is as follows. In section 2 we focus on solutions of second and third order to the linear determining equation for the Gibbons-Tsarev equation [**9**]

$$u_{tt} = u_x u_{tx} - u_t u_{xx} + 1. \tag{1.6}$$

This gives the corresponding differential constraints and allow us to find some exact solutions of (1.6). In the final section, the invariant solutions under an involutive distribution are discussed. We consider the problem of finding involutive distributions that enable us to obtain invariant solutions to evolution equations.

2. Gibbons-Tsarev equation. In this section we will consider the Gibbons-Tsarev equation [**9**]

$$z_{xx} + z_y z_{xy} - z_x z_{yy} + 1 = 0, \tag{2.1}$$

which arises in reductions of the Benney equation.

The linear determining equation has the form

$$D_x^2 h + z_y D_x D_y h - z_x D_y^2 h + b_1 z_{yy} D_x h + b_2 z_{xy} D_y h = 0, \tag{2.2}$$

the constants b_1 and b_2 are to be determined together with the function h. It can be shown that the equation (2.2) has a solution of the form

$$h = z_{yy} + g(x, y, z, z_x, z_y, z_{xx})$$

if and only if the function g is independent of z_{xx}. Therefore we shall start with solutions of the second order

$$h = z_{yy} + g(x, y, z, z_x, z_y). \tag{2.3}$$

Substituting (2.3) into (2.2) leads to an equation which includes derivatives of the third order. We can express all mixed derivatives by means of (2.1). Setting the coefficients of z_{xxx} and z_{yyy} equal to zero we obtain:

$$b_1 = 1, \qquad b_2 = -1.$$

The left-hand side of (2.2) is a polynomial with respect to z_{xx} and z_{yy}. This polynomial must identically vanish. Collecting similar terms we have the following equations:

$$pg_{pp} + qg_{pq} - g_{qq} + 2g_p = 0, \tag{2.4}$$

$$\begin{aligned}&(-q^2 - 2p)g_{pp} + 2g_{qq} + q^2 g_{xp} + q(q^2 + 2p)g_{yp} + q^2(q^2 + 3p)g_{zp}\\ &- 2qg_{xq} - q^2 g_{yq} - q(q^2 + 2p)g_{zq} + qg_y + q^2 g_z - 4g_p = 0,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&2p^2 g_{pp} - (q^2 + 2p)g_{qq} + pq^2 g_{xp} - 2p^2 qg_{yp} - p^2 q^2 g_{zp}\\ &+ q(q^2 + 2p)g_{xq} - pq^2 g_{yq} + 2p^2 qg_{zq} + q^2 g_x - pqg_y + 4pg_p = 0,\end{aligned} \tag{2.6}$$

$$\begin{aligned}&pg_{pp} - g_{pq} - g_{qq} + q^2 g_{xp} - 2pqg_{yp} - pq^2 g_{zp} + 2qg_{xq} + q^2 g_{yq} + q(q^2 + 2p)g_{zq}\\ &- q^2(q^2 + 2p)g_{xz} + pq^3 g_{yz} - p^2 q^2 g_{zz} + 2g_p - q^2 g_{xx} - q^3 g_{xy} + pq^2 g_{yy} - qg_y = 0,\end{aligned} \tag{2.7}$$

where $p = z_x$ and $q = z_y$.

It is possible to show that the general solution of the equations (2.4)–(2.7) is

$$\begin{aligned}h = z_{yy} &+ c_1\big(z_y^4 + (3z_x + 4x)z_y^2 + 3yz_y + (z_x + 2x)^2 + 2z\big)\\ &+ c_2\big(z_y^3 + (2z_x + 3x)z_y + 2y\big) + c_3\big(z_y^2 + z_x + 2x\big) + c_4 z_y + c_5.\end{aligned}$$

Hence the differential constraint $h = 0$ is compatible with the Gibbons-Tsarev equation (2.1). In the case $c_1 = c_2 = c_3 = 0$ we obtain the differential constraint

$$z_{yy} + c_4 z_y + c_5 = 0. \tag{2.8}$$

From (2.8) we find the following representation

$$z = a_1 \exp(-c_4 y) - c_5 y/c_4 + a_2, \tag{2.9}$$

where a_1 and a_2 depend on x. Substituting (2.9) into (2.1) we derive two ordinary differential equations

$$a_2'' + 1 = 0, \qquad a_1'' + c_5 a_1' - c_4^2 a_1 a_2' = 0.$$

The first equation has the solution

$$a_2 = -x^2/2 + c_6 x + c_7, \qquad c_6, c_7 \in R.$$

In this case the second equation is

$$a_1'' + c_5 a_1' + c_4^2(x - c_6)a_1 = 0.$$

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Setting $a_1 = \exp(-c_5 x/2) v(x)$ we obtain the equation

$$v'' + (A + Bx)v = 0, \qquad A, B \in R.$$

According to [10] the solutions of the last equation can be expressed in terms of Airy functions.

It can be shown that the linear determining equation (2.2) has the following solution of third order

$$\begin{aligned} h = z_{yyy} &+ c_1\big(3z_y^5 + (10z_x + 12x)z_y^3 + 6yz_y^2 + \big(6z_x^2 + 18xz_x + 2z + 12x^2\big)z_y + 4yz_x + 6xy\big) \\ &+ c_2\big(5z_y^4 + (12z_x + 15x)z_y^2 + 6yz_y + 3z_x^2 + 10xz_x + 15/2x^2 + z\big) \\ &+ c_3\big(2z_y^3 + (3z_x + 4x)z_y + y\big) + c_4\big(3z_y^2 + 2z_x + 3x\big) + c_5 z_y + c_6. \end{aligned}$$

The corresponding constants b_1 and b_2 in (2.2) are given by

$$b_1 = 2, \qquad b_2 = -2.$$

In the case $c_1 = c_2 = c_3 = c_6 = 0$ and $c_5 = -1$ the function h gives the differential constraint

$$z_{yyy} - z_y = 0. \tag{2.10}$$

From (2.10) we obtain the following representation

$$z = s_1(x) + s_2(x)e^y + s_3(x)e^{-y}.$$

The functions $s_1(x)$, $s_2(x)$ and s_3 must satisfy the equations

$$s_2'' - s_1' s_2 = 0, \quad s_1'' - 2s_3 s_2' - 2s_2 s_3' + 1 = 0, \quad s_3'' - s_1' s_3 = 0.$$

If $s_3 = as_2$ then the last system reduces to the two equations

$$s_2'' - s_1' s_2 = 0, \qquad s_1'' - 4as_2 s_2' + 1 = 0, \quad a \in R. \tag{2.11}$$

Integrating the second equation, we find that

$$s_1' = -x - b + 2as_2^2, \qquad b \in R.$$

We can insert this expression in (2.11) and obtain the second-order equation

$$s_2'' + \big(x + b - 2as_2^2\big)s_2 = 0.$$

Using the transformations $t_1 = x + b$ and $w = \sqrt{a}s_2$, we take the equation in s_2 to the second Painlevé equation [11]

$$w'' = 2w^3 - t_1 w.$$

The differential constraint

$$z_{yyy} = 0$$

leads to a solution of (2.1) which is expressed in terms of elementary functions.

3. Invariant solutions under involutive distributions. In this section we introduce invariant solutions under involutive distributions. Suppose that a collection of p vector fields

$$X_s = \sum_{i=1}^{n} \xi_s^i(x)\partial_{x_i}$$

is given on an open set $U \subset R^n$. If this collection is linearly disconnected, i.e., the rank of the matrix $|\xi_s^i(x)|$ equals p for all $x \in U$ and satisfies the involution condition

$$[X_i, X_j] = \sum_{k=1}^{p} c_{ij}^k(x) X_k, \qquad \forall\, 1 \leq i, j \leq p, \tag{3.1}$$

where c_{ij}^k are smooth functions, then this collection generates an involutive p-dimensional distribution D_p. A collection of vector fields with these properties is called an involutive basis or just a basis. It is well known that a distribution D_p is involutive if and only if it possesses at least one involutive basis.

Definition. A solution $u = \varphi$ to a system of partial differential equation E is invariant under an involutive distribution D_p if D_p is tangent to the manifold $S = \{(x, u) : u = \varphi(x)\}$. Obviously, the invariance of a solution under D_p amounts to its invariance under the operators of an arbitrary involutive basis for D_p.

Now, consider the system of evolution equations

$$u_t^i = F^i(t, x, u, u_\alpha), \quad i = 1, \dots, m, \tag{3.2}$$

where t and $x = (x_1, \dots, x_n)$ are independent variables, $u^1, \dots, u^m$ are functions, $u = (u^1, \dots, u^m)$, and u_α stands for various partial derivatives with respect to $x_1, \dots, x_n$. Denote the total derivatives with respect to t and x_i by the symbols D_t and D_{x_i}.

Let $J^k(U, R^m)$ be the space of k-jets on $U \subset R^n$. Recall that a manifold $H \subset J^k(R^{n+1}, R^m)$, defined by the equations

$$h^j(t, x, u, u_\beta) = 0, \quad j = 1, \dots, s, \tag{3.3}$$

is an invariant manifold for (3.2) if the following identity holds on the set $[E] \cap [H]$:

$$D_t h^j = 0.$$

Here $[E]$ and $[H]$ stand for the differential consequences of (3.2) and (3.3) with respect to $x_1, \dots, x_n$. Denote the involutive distribution generated by vector fields $X_1, \dots, X_r$ by $\langle X_1, \dots, X_r \rangle$.

Lemma 1. *Suppose that vector fields*

$$X_k = \sum_{i=1}^{n} \xi_k^i(t, x, u)\partial_{x_i} + \sum_{j=1}^{m} \eta_k^j(t, x, u)\partial_{u^j}, \quad k = 1, \dots, n, \tag{3.4}$$

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generate an involutive distribution and that $det(\xi^i_k) \neq 0$. *If the manifold defined by the equations*

$$h^j_k = \sum_{i=1}^{n} \xi^i_k u^j_{x_i} - \eta^j_k = 0, \quad 1 \leq j \leq m, 1 \leq k \leq n, \tag{3.5}$$

is invariant with respect to (3.2) *then system* (3.2) *has invariant solutions under this involutive distribution.*

Proof. Write down the collection of fields $X_1, \ldots, X_n$ in vector form as follows:

$$X = \xi \partial_x + \eta \partial_u.$$

Acting by the matrix ξ^{-1} on X, we obtain the involutive collection

$$Z = \partial_x + \tilde{\eta} \partial_u,$$

where $\tilde{\eta} = \xi^{-1}\eta$. The distribution $\langle Z_1, \ldots, Z_n \rangle$ is involutive.

The invariant solutions under $\langle X_1, \ldots, X_n \rangle$ must satisfy (3.5). The invariant solutions under $\langle Z_1, \ldots, Z_n \rangle$ must satisfy the equations

$$u^j_{x_k} = \tilde{\eta}^j_k(t, x, u). \tag{3.6}$$

Obviously, (3.5) and (3.6) have the same solutions. Since Z is an involution distribution, the Poisson bracket $[Z_i, Z_k]$ vanishes. Consequently, we have

$$Z_i\big(\tilde{\eta}^j_k\big) = Z_k\big(\tilde{\eta}^j_i\big),$$

which means that the consistency conditions for (3.6) are satisfied.

Using (3.6) and inserting the derivatives of the functions u^j with respect to x_k in the right-hand side of (3.2), we come to the system

$$u^j_t = G^j(t, x, u), \qquad j = 1, \ldots, m. \tag{3.7}$$

By the Frobenius theorem, the system of (3.6) and (3.7) is compatible if the relations

$$D_{x_k} G^j = D_t \tilde{\eta}^j_k, \qquad j = 1, \ldots, m; \; k = 1, \ldots, n \tag{3.8}$$

are valid by virtue of (3.7) and (3.8). Validity of these conditions follows from the invariance of (3.5) with respect to (3.2). Indeed, this invariance means that

$$D_t\big(u^j_{x_k} - \tilde{\eta}^j_k\big) = D_{x_k} F^j - D_t \tilde{\eta}^j_k = 0. \tag{3.9}$$

Inserting the derivatives with respect to x_k in (3.9), we see that (3.9) coincides with (3.8). □

Remark. If an involutive distribution is generated by analytic vector fields $X_1, \ldots, X_p$, where $p < n$, (3.2) is a system of first-order equations with analytic right-hand sides, and the rank of the matrix (ξ^i_k) equals p, then (3.2) has an invariant solution relative to $X_1, \ldots, X_p$. The proof is carried out by the above scheme, but instead of the Frobenius theorem we should use the Riquier theorem on the existence of analytic solutions to an autonomous system with analytic right-hand sides [**5**].

To exemplify the application of a distribution to constructing solutions, consider the equation

$$u_t = \Delta \ln u, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{3.10}$$

which arises in various application [**12**, **13**] and possesses an infinite-dimensional algebra of point symmetries [**14**]. Some exact solutions to this equation can be found in [**15**, **16**]. We give a solution to this equation which is invariant relative to the pair of commuting operators

$$X_1 = \partial_x - (u^2 + (t\,u^2 - x\,u^2 + u)\tan(t))\partial_u,$$
$$X_2 = \partial_y - (t\,u^2 + u - x\,u^2)\partial_u.$$

The corresponding manifold for these vector fields is

$$u_x + u^2 + (t\,u^2 - x\,u^2 + u)\tan(t) = 0, \tag{3.11}$$
$$u_y + t\,u^2 + u - x\,u^2 = 0. \tag{3.12}$$

It is easy to verify that this is an invariant manifold for (3.10). Note that the vector fields X_1 and X_2 do not belong to the algebra of symmetries of (3.10). The general solution to (3.11), (3.12) and (3.10) has the form

$$u = \frac{1}{A[\exp((x-t)\tan(t) + y)]\cos t + x - t}, \qquad A \in R.$$

To use vector fields and distributions, we need a method for finding them. The classical approach to constructing vector fields relative to which the given differential equations are invariant was proposed by S. Lie. A modern exposition with many examples and new results was given by L. V. Ovsyannikov [**6**].

A determining equation enables us to find differential constraints compatible with the original equation. In the case of differential equations in more than two independent variables, we can propose systems of defining equations which would enable us to find involutive distributions.

Consider the system of involution equations (3.2) and the manifold in $J^1(U, R^m)$ defined by

$$h^i_j = u^i_{x_j} + g^i_j(t, x, u) = 0, \tag{3.13}$$

where $i = 1, \dots, m$, and $j = 1, \dots, n$.

THEOREM. *Suppose that the manifold* (3.13) *is invariant under the system* (3.2) *whose right-hand sides are polynomials in derivatives whose coefficients depend on* $t, x_1, \dots, x_n$ *and* $u^1, \dots, u^m$. *Then the functions* h^i_j *satisfy the following system:*

$$D_t h^i_j + m_{ij}(h)|_{[E]} = 0, \qquad 1 \le i \le m, 1 \le j \le n. \tag{3.14}$$

Here $m_{ij}(h)$ *is some operator representing a polynomial in* $h^k_l, D_{x_1}h^k_l, \dots, D_{x_n}h^k_l, \dots, D^\alpha h^k_l$ $(k = 1, \dots, m, l = 1, \dots, n)$. *The operators* $m_{ij}(h)$ *vanish whenever all* h^k_l *are zero.*

Proof. We first show that the total derivative of h^i_j with respect to t is representable as

$$D_t h^i_j = m_{ij}(h) + \gamma_{ij}, \tag{3.15}$$

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where m_{ij} are operators whose shape is described in the theorem and γ_{ij} are functions which may depend only on t, x and u.

The following identities are valid on $[E]$:

$$D_t h^i_j = D_{x_j} F^i + \frac{\partial g^i_j}{\partial t} + \sum_{k=1}^{m} F^k \frac{\partial g^i_j}{\partial u^k}. \tag{3.16}$$

Let $\frac{\partial^{|s|} u^k}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}}$ be a derivative of maximal order on the right-hand side of (3.16) and $s_p \neq 0$ for some p. By (3.16) and the assumptions of the theorem, this derivative enters (3.16) polynomially. Using (3.13), we can write down this derivative as follows:

$$D_{x_1}^{s_1} \cdots D_{x_p}^{s_p - 1} \cdots D_{x_n}^{s_n} \left(h^k_p\right) - D_{x_1}^{s_1} \cdots D_{x_p}^{s_p - 1} \cdots D_{x_n}^{s_n} \left(g^k_p\right).$$

Note that the second summand involves no derivatives of order $|s|$ and is a polynomial in derivatives. Thus, all derivatives of maximal order on the right-hand side of (3.16) can be expressed in terms of the total derivatives of the functions $h^r_q (r = 1, \dots, m$, and $q = 1, \dots, n)$. Afterwards, it is possible to express the derivatives of order $|s| - 1$, etc. down to the first-order derivatives.

We are left with demonstrating that the functions γ_{ij} in (3.15) are all zero. By the conditions of the theorem, the manifold (3.13) is an invariant manifold for (3.2). Consequently, the following identity holds on $[E] \cap [H]$:

$$m_{ij}(h) + \gamma_{ij} = D_t h^i_j = 0.$$

Since the m_{ij}'s vanish on $[H]$, the functions γ_{ij} are zero on $[E] \cap [H]$. Once the m_{ij}'s are independent of the derivatives of the functions u^k, all m_{ij} are identically zero. □

REMARK. As we see from the proof of the theorem, the choice of the operators m_{ij} is not uniquely defined.

For example, consider the second-order equation in three independent variables:

$$u_t = G \equiv F^1 u_{xx} + F^2 u_{yy} + F^3 u_x^2 + F^4 u_y^2 + F^5, \tag{3.17}$$

where F^i are some functions depending on u. Suppose that

$$h_1 \equiv u_x + g_1(t, x, y, u) = 0, \quad h_2 \equiv u_y + g_2(t, x, y, u) = 0 \tag{3.18}$$

define an invariant manifold for (3.17). To derive a system of determining equations like (3.14), we express the derivatives $D_t h_1$ and $D_t h_2$ in terms of h_i, $D_x h_i$, $D_y h_i$, $D_x^2 h_i$, $D_x D_y h_i$, and $D_y^2 h_i$ $(i = 1, 2)$. By (3.17), the following holds:

$$D_t h_1 = D_x G + \frac{\partial g_1}{\partial t} + \frac{\partial g_1}{\partial u} G.$$

It is easy to verify that the right-hand side of the last equality is representable as

$$\begin{aligned} m_{11}(h_1, h_2) = {} & G_{u_{xx}} D_x^2 h_1 + G_{u_{yy}} D_y^2 h_1 + \left[G_{u_x} + D_x\left(G_{u_{xx}}\right)\right] D_x h_1 + G_{u_y} D_y h_1 \\ & + D_x\left(G_{u_{yy}}\right) D_y h_2 + \left[G_u - D_x^2\left(G_{u_{xx}}\right) - D_y^2(G_{u_{yy}}) + r_1\right] h_1 + s_1 h_2 + \gamma_1, \end{aligned} \tag{3.19}$$

where r_1, s_1, and γ_1 are functions depending on h_1, h_2, and G. Since (3.18) is an invariant manifold, the function γ_1 equals 0. Consequently, the first defining equation has the

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form

$$D_t h_1 = m_{11}(h_1, h_2).$$

To obtain the second defining equation

$$D_t h_2 = m_{12}(h_1, h_2),$$

we should replace h_1 in (3.12) with h_2, x with y, r_1 with r_2, and s_1 with s_2. The following lemma asserts that, under some conditions, solutions to equations like (3.13) enable us to construct differential constraints compatible with the system of evolution equations (3.2). It is worth noting that the form of the operators m_{ij} is unimportant, provided that only $m_{ij}(0) = 0$.

LEMMA 2. *Suppose that the functions*

$$h_j^i = \sum_{s=1}^{n} \xi_j^s(t, x, u)u_{x_s}^i - g_j^i(t, x, u)$$

satisfy a system like (3.14) *on* $[E]$ *with* $m_{ij}(0) = 0$. *If the vector fields*

$$X_j = \sum_{s=1}^{n} \xi_j^s \partial_{x_s} + \sum_{i=1}^{m} g_j^i \partial_{u_i}, \qquad j = 1, \dots, n$$

generate an involutive distribution and $\det(\xi_j^s) \neq 0$ *then there is a solution to the system of* (3.2) *and the equations*

$$h_j^i = 0, \qquad i = 1, \dots, m, \qquad j = 1, \dots, n. \tag{3.20}$$

Proof. Since the functions h_j^i satisfy (3.14), in view of $m_{ij}(0) = 0$ (3.20) defines an invariant manifold for (3.2). To complete the proof, it suffices to refer to Lemma 1. □

Finding solutions to general nonlinear equations (3.14) might represent a very complicated problem. To simplify the problem, we remove all terms nonlinear in h_l^k from the operators m_{ij} as was done above in the case of an evolution equation with one space variable. In result, we obtain some linear equation

$$D_t h_j^i + l_{ij}(h) = 0.$$

As we have done above, multiply the coefficients of the operators l_{ij} by undetermined constants and write down the resultant equations as

$$D_t h_j^i + L_{ij}(h) = 0 \tag{3.21}$$

calling them linear determining equations (LDEs). For example, the LDEs for (3.17) have the form

$$\begin{aligned} D_t h_1 = L_{11}(h_1, h_2) \equiv {} & a_1 G_{u_{xx}} D_x^2 h_1 + a_2 G_{u_{yy}} D_y^2 h_1 + \left[a_3 G_{u_x} + a_4 D_x\left(G_{u_{xx}}\right)\right] D_x h_1 + a_5 G_{u_y} D_y h_1 \\ & + a_6 D_x\left(G_{u_{yy}}\right) D_y h_2 + \left[a_7 G_u + a_8 D_x^2\left(G_{u_{xx}}\right) + a_9 D_y^2\left(G_{u_{yy}}\right)\right] h_1, \\ D_t h_2 = {} & L_{12}(h_1, h_2), \end{aligned} \tag{3.22}$$

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where $L_{12}(h_1, h_2)$ is obtained from $L_{11}(h_1, h_2)$ by replacing h_1 with h_2, x with y, and a_i with b_i.

Although the above arguments were for systems of evolution equations, we can try to extend them to a more general situation. Assume given a system

$$n_i(u) = F^i(t, x, u, u_\alpha), \qquad i = 1, \ldots, m,$$

where n_i are linear differential operators with constant coefficients and the right-hand sides are similar to those in the evolution systems (3.2). To find the functions h^i_j, we suggest using the following equation in place of (3.21):

$$N_i\big(h^i_j\big) + L_{ij}(h) = 0, \tag{3.23}$$

where the operators N_i are obtained from n_i by replacing partial derivatives with total derivatives. Alongside (3.23), it is useful to introduce the following analog of B-defining equations [**17**]:

$$N_i\big(h^i_j\big) + L_{ij}(h) + \sum_{\substack{1\le l\le m\\ 1\le k\le n}} b^{k\,i}_{l\,j} h^l_k = 0, \tag{3.24}$$

where $1 \le i \le m$, $1 \le j \le n$, and $b^{k\,i}_{l\,j}$ are functions that may depend on t, x, and u.

We call equations of the form (3.23) quasilinear determining equations (QDEs). We exhibit an example of QDEs in finding involutive distributions. Consider one of the nonlinear dispersion models describing the propagation of long two-dimensional waves [22]:

$$\eta_{tt} = gd\Delta\eta + \frac{d^2}{3}\Delta\eta_{tt} + \frac{3}{2}g\Delta\eta^2,$$

where $\eta(t, x, y)$ is the deviation of a fluid from an equilibrium state, d is the depth of the unperturbed fluid, and g is the free fall acceleration. By translations and dilations, we can reduce this equation to the form

$$u_{tt} - \Delta(u_{tt}) - u\Delta u - (\nabla u)^2 = 0. \tag{3.25}$$

In accordance with the above method, the QLEs for (3.25) have the form

$$D_t^2 h_1 - D_t^2 D_x^2 h_1 - D_t^2 D_y^2 h_1 + a_1 u\big(D_x^2 h_1 + D_y^2 h_1\big) + a_2 u_x D_x h_1 + a_3 u_y D_y h_1 \\ + a_4 u_x D_y h_2 + (a_5 \Delta u + a_6 u_{xx} + a_7 u_{yy} + r_1) h_1 + q_1 h_2 = 0, \tag{3.26}$$

$$D_t^2 h_2 - D_t^2 D_x^2 h_2 - D_t^2 D_y^2 h_2 + b_1 u\big(D_x^2 h_2 + D_y^2 h_2\big) + b_2 u_y D_y h_2 + b_3 u_x D_x h_2 \\ + b_4 u_y D_x h_1 + (b_5 \Delta u + b_6 u_{xx} + b_7 u_{yy} + r_2) h_2 + q_2 h_1 = 0, \tag{3.27}$$

where a_i and b_i are constants, and r_j and q_j are functions which may depend on t, x, y, and u and which should be found together with h_1 and h_2. The scheme for solving (3.26) and (3.27) is completely analogous to the standard scheme of group analysis of differential equations [**6**, **18**]. For this reason, we omit all intermediate computations and set forth only the final results.

If h_1 and h_2 are sought in the form corresponding to the point symmetries

$$h_1 = \xi^1_1 u_t + \xi^1_2 u_x + \xi^1_3 u_y + \eta^1,$$
$$h_2 = \xi^2_1 u_t + \xi^2_2 u_x + \xi^2_3 u_y + \eta^2,$$

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where ξ^i and η^j are functions of t, x, y, and u, then under the condition $(\xi_1^1)^2 + (\xi_3^1)^2 + (\xi_1^2)^2 + (\xi_2^2)^2 \neq 0$ equations (3.26) and (3.27) can be shown to have solutions leading only to admissible operators for (3.25). There appear new solutions only when

$$h_1 = u_x + g_1(t, x, y, u), \quad h_2 = u_y + g_2(t, x, y, u).$$

The final form of g_1 and g_2 is as follows:

$$g_1 = s_1 x + s_2 y + s_3, \quad g_2 = s_2 x + s_4 y + s_5.$$

Moreover, the functions s_i $(i = 1, \dots, 5)$ depend only on t and satisfy the following system of five second-order differential equations:

$$\begin{aligned}
s_1'' &+ 3s_1^2 + s_1 s_4 + 2s_2^2 = 0,\\
s_2'' &+ 3s_1 s_2 + 3s_2 s_4 = 0,\\
s_3'' &+ 3s_1 s_3 + 2s_2 s_5 + s_3 s_4 = 0,\\
s_4'' &+ s_1 s_4 + 2s_2^2 + 3s_4^2 = 0,\\
s_5'' &+ s_1 s_5 + 2s_2 s_3 + 3s_4 s_5 = 0.
\end{aligned}$$

For completeness of exposition, we write down the constants a_i and b_i $(i = 1, \dots, 7)$ and the functions r_j and q_j $(j = 1, 2)$ in (3.26) and (3.27) corresponding to g_1 and g_2:

$$\begin{aligned}
&a_1 = b_1 = a_4 = b_4 = -1, \quad a_2 = b_2 = a_3 = b_3 = -3,\\
&a_5 = a_6 = a_7 = b_5 = b_6 = b_7 = 0,\\
&r_1 = 3s_1 + s_4, r_2 = s_1 + 3s_4, \quad q_1 = 2s_1, \quad q_2 = 2s_2.
\end{aligned}$$

The functions h_1 and h_2 generate the differential constraints

$$\begin{aligned}
u_x &+ s_1 x + s_2 y + s_3 = 0,\\
u_y &+ s_2 x + s_4 y + s_5 = 0.
\end{aligned}$$

These constraints enable us to find the following representation for a solution to (3.25):

$$u = \frac{-s_1 x^2}{2} - s_2 x y - \frac{s_4 y^2}{2} - s_3 x - s_5 y + s_6.$$

Inserting this in (3.25), we obtain the following equation for s_6:

$$s_6'' = 3s_1^2 + 2s_1 s_4 - s_1 s_6 + 4s_2^2 + s_3^2 + 3s_4^2 - s_4 s_6 + s_5^2.$$

The system of the six differential equations in the six functions s_i deserves further study. For example, it would be interesting to find a solution expressible via elementary functions.

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