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On Certain Projective Configurations in Space of *n* Dimensions and a Related Problem in Arrangements.

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In the first part of this paper there are found the numbers of points, lines, etc., in a finite projective geometry of n dimensions. The substance of this has already been worked out by O. Veblen and W. H. Bussey.* The second part is concerned with the arrangements of the numbers representing the points in a finite projective plane desarguesian geometry.

I.

1. Consider an assemblage of points, lines, planes, 3-spaces, ... $(n-1)$-spaces in space of n dimensions R_n. With regard to these we shall make the assumptions :

(A) An $(r-1)$-space and a point belonging to the system and not in the $(r-1)$-space always determine an r-space belonging to the system.

(B) In an R_t a line cuts an R_{t-1} in a point belonging to the system.

From these assumptions it follows that

(A)′ an R_r and an R_s which both pass through an R_t $(r, s > t)$ always determine an R_{r+s-t} belonging to the system and containing the R_r and the R_s.

(B)′ in an R_t every R_s cuts every R_r $(r+s \geqq t)$ in an R_{r+s-t} which belongs to the system.

* "Finite projective geometries," *Amer. M. S. Trans.*, vii. (1906), 241–259. See also Whitehead, "The axioms of projective geometry," p. 13.

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We shall further assume

(C) that on each line there are the same number of points.

From (A), (B), and (C) it will be shown that

(C)′ in each R_t there are the same number of R_r's $(r<t)$; and through each R_s and lying in an R_t containing R_s there pass the same number of R_r's $(t>r>s)$; for all values of r, s, t subject to the given conditions. These numbers will all be expressed in terms of the number of points in a line.

2. Let the number of r-spaces which pass through, or lie in, an s-space (according as $r \gtrless s$) be denoted by p_{rs}; and the number of r-spaces containing a given s-space and contained in a given t-space be denoted by (r, s, t). In this notation $t>r>s$. If this condition is not satisfied, then for (r, s, t) we must substitute another expression, viz.,

$$(r, t, s) \text{ if } s>r>t;$$

$$p_{rs} \text{ if } t>s>r \text{ or } t<s<r;$$

$$p_{rt} \text{ if } s>t>r \text{ or } s<t<r.$$

Also if

$$s=t, \qquad (r, s, t)=p_{rs}=p_{rt};$$

$$r=s,\ t \text{ or } n, \quad (r, s, t)=1;$$

$$t=n, \qquad (r, s, t)=p_{rs},$$

$$s=n, \qquad (r, s, t)=p_{rt}.$$

3. *To express (r, s, t) in terms of the p's.*

In an R_t there are p_{rt} r-spaces; on each there lie p_{sr} s-spaces; therefore in R_t there are $p_{sr}p_{rt}$ s-spaces, each being counted (r, s, t) times. (This assumes that p_{sr} is the same for all the r-spaces, and (r, s, t) is the same for all the s-spaces).

Therefore $$p_{st}\,(r, s, t)=p_{sr}p_{rt}$$

and $$(r, s, t)=\frac{p_{sr}p_{rt}}{p_{st}} \qquad \text{- - - - - (1)}$$

Also, through an R_s there are p_{rs} r-spaces; through each there pass p_{tr} t-spaces, therefore through R_s there pass $p_{tr}p_{rs}$ t-spaces, each being counted (r, s, t) times. (With similar assumptions).

Therefore $$p_{ts}\,(r,\,s,\,t)=p_{rs}p_{tr},$$

and $$(r,\,s,\,t)=\frac{p_{rs}p_{tr}}{p_{ts}} \qquad (2)$$

From (1) and (2)

$$p_{rs}p_{st}p_{tr}=p_{rt}p_{ts}p_{sr} \qquad (3)$$

(3) is proved under the limitation $t>r>s$, but the symmetry of the result shows that it is independent of this assumption.

4. *To express p_{0t} in terms of p_{01}.*

Take an R_{t-1} and a point 0 outside it; then R_{t-1} and 0 determine an R_t (A). Join 0 to each of the $p_{0,\,t-1}$ points of R_{t-1} (A). On each of these lines there are p_{01} points (C), and we now have *all* the points in R_t, for if there were any other point 0′, 00′ cuts R_{t-1} in a point which has already been chosen (B)

Therefore $$p_{0t}=p_{0,\,t-1}(p_{01}-1)+1 \qquad (4)$$

(4) is a reduction formula, following from (A), (B) and (C) alone, by which p_{0t} may be expressed in terms of p_{01}. p_{0t} is therefore constant.

Let $p_{01}=p+1$, then from (4)

$$p_{0t}=p\cdot p_{0,\,t-1}+1.$$

Now $$p_{02}=p_{01}(p_{01}-1)+1.$$

$$=p^2+p+1=\frac{p^3-1}{p-1}.$$

Hence assuming $$p_{0,\,t-1}=\frac{p^t-1}{p-1}$$

we get $$p_{0t}=\frac{p^{t+1}-p}{p-1}+1=\frac{p^{t+1}-1}{p-1} \qquad (5)$$

From this it follows that

$$p_{0t}-p_{0s}=\frac{p^{t+1}-p^{s+1}}{p-1}.$$

We have now to express p_{rt} and $(r,\,s,\,t)$ in terms of p.

5. *To express p_{rt} $(r<t)$ in terms of p.*

An r-space is determined by $r+1$ points. Let us choose $r+1$ determining points in an R_t. The first point 0_1 can be chosen in p_{0t} ways, the second 0_2 in $p_{0t}-1$ ways. A third, 0_3, not in a line with these, can be chosen in $p_{0t}-p_{01}$ ways; a fourth, not in a plane with 0_1, 0_2, 0_3 in $p_{0t}-p_{02}$ ways, and so on. The number of ways of choosing the $r+1$ points is then

$$p_{0t}(p_{0t}-1)(p_{0t}-p_{01})(p_{0t}-p_{02})\ldots(p_{0t}-p_{0,r-1})/(r+1)!$$

Similarly, in this r-space we may choose $r+1$ determining points in

$$p_{0r}(p_{0r}-1)(p_{0r}-p_{01})(p_{0r}-p_{02})\ldots(p_{0r}-p_{0,r-1})/(r+1)!$$

ways. Hence

$$\begin{aligned} p_{rt} &= \frac{p_{0t}(p_{0t}-1)(p_{0t}-p_{01})\ldots(p_{0t}-p_{0,r-1})}{p_{0r}(p_{0r}-1)(p_{0r}-p_{01})\ldots(p_{0r}-p_{0,r-1})} \\ &= \frac{(p^{t+1}-1)(p^{t+1}-p)(p^{t+1}-p^2)\ldots(p^{t+1}-p^r)}{(p^{r+1}-1)(p^{r+1}-p)(p^{r+1}-p^2)\ldots(p^{r+1}-p^r)} \\ &= \frac{(p^{t+1}-1)(p^t-1)\ldots(p^{t-r+1}-1)}{(p^{r+1}-1)(p^r-1)\ldots(p-1)} \\ &= \frac{\Pi(p^{t+1}-1)}{\Pi(p^{r+1}-1)(p^{t-r}-1)}. \qquad (6) \end{aligned}$$

Hence $p_{r-1,t}=p_{t-r,t}$, a reciprocal relation between the number of $(r-1)$-spaces and the number of $(t-r)$-spaces in a t-space.

6. *To express* (r, s, t) *in terms of* p.

In R_t take an R_s. An R_r contained in R_t and passing through R_s requires $r+1$ points to determine it; $s+1$ of these are in the R_s. We can choose a first point, 0_1, outside R_s and in R_t in $p_{0t}-p_{0s}$ ways; a second, 0_2, not in the $(s+1)$-space determined by R_s and 0_1, in $p_{0t}-p_{0,s+1}$ ways, and so on. The number of ways of choosing the $r-s$ additional points is therefore

$$(p_{0t}-p_{0s})(p_{0t}-p_{0,s+1})\ldots(p_{0t}-p_{0,r-1})/(r-s)!$$

Similarly in the R_r we can choose the $r-s$ additional points which are required to determine it in

$$(p_{0r}-p_{0s})(p_{0r}-p_{0,s+1})\ldots(p_{0t}-p_{0,r-1})/(r-s)!$$

ways. Hence

$$\begin{aligned} (r, s, t) &= \frac{(p_{0t}-p_{0s})(p_{0t}-p_{0,s+1})\ldots(p_{0t}-p_{0,r-1})}{(p_{0r}-p_{0s})(p_{0r}-p_{0,s+1})\ldots(p_{0r}-p_{0,r-1})} \\ &= \frac{(p^{t-s}-1)(p^{t-s-1}-1)\ldots(p^{t-r+1}-1)}{(p^{r-s}-1)(p^{r-s-1}-1)\ldots(p-1)} \\ &= \frac{\Pi(p^{t-s}-1)}{\Pi(p^{r-s}-1)(p^{t-r}-1)} \qquad (7) \end{aligned}$$

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Hence $(r-1,\ s-1,\ t)=p_{t-r,\,t-s}$, a reciprocal relation between the number of $(r-1)$-spaces through an $(s-1)$-space and the number of $(t-r)$-spaces in a $(t-s)$-space in R_t.

Putting $t=n$ in (7) we find for $r>s$

$$p_{rs}=\frac{\Pi(p^{n-s}-1)}{\Pi(p^{r-s}-1)(p^{n-r}-1)} \quad . \quad . \quad . \quad . \quad (8)$$

7. All the numbers in the scheme have now been expressed in terms of p, using only the assumptions (A), (B), (C), hence they are all constant and determinate when p is given. We notice also that the configuration is reciprocal since

$$p_{r-1,\,t}=p_{t-r,\,t}$$

and $$(r-1,\ s-1,\ t)=p_{t-r,\,t-s}.$$

With the help of formulæ (6) and (7) the formulæ (1), (2), and (3) may now be verified. They have not been employed in the proofs of (6) and (7), and might therefore have been proved by means of (A), (B), and (C) alone.

8. The following correspondence may now be established in the case where p is a prime.

Construct an Abelian group of order p^{n+1} in which each operation is of order p. The number of subgroups of order p^{r+1} is p_{rn} in formula (6), the number of subgroups of order p^{r+1} contained in a given subgroup of order p^{t+1} is p_{rt}, and so on.*

Hence a correspondence is established between this group and *the configuration of points, lines, etc., in such a way that to a point* corresponds a subgroup of order p, to a line a subgroup of order p^2, and in general, to a t-space a subgroup of order p^{t+1}. For the connection with the Galois Field theory see Veblen and Bussey l.c.

II.

9. There is a problem of arrangements connected with these configurations. Confining our attention to a plane, consider the configuration with r points in each line and r^2-r+1 points altogether. Denoting the points by numbers, we can arrange the n numbers, each repeated r times, in n sets of r each, such that any pair of numbers occurs in one and only one set.

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* See Burnside, "Theory of Groups," p. 59.

For $r=4$, $n=13$ a possible arrangement is

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8

where each number occurs once in each row. We also observe that each complete row is obtained from the first by a cyclic permutation.

10. Let us assume the possibility of arranging the n numbers in n sets of r each, *i.e.*, in n r-ads, and investigate the nature of the arrangement when each row contains all the numbers. The first arrangement is possible whenever there is a finite projective desarguesian geometry in the plane with n points, and this happens whenever p or $r-1$ is a prime or a power of a prime. Let us assume therefore that the n numbers, each repeated r times, are disposed in r rows and n columns in such a way that each number occurs in each row and every pair of numbers occurs in one and only one column.

Let the substitutions by which the 2nd, 3rd, ..., rth rows are obtained from the first be denoted by (12), (13), ..., $(1r)$, and consider the r columns in which a specified number p occurs. When p is in the first row the other numbers in the same column are

$$p(12),\ p(13),\ \ldots,\ p(1r);$$

when p is in the mth row the other numbers are

$$p(1m)^{-1},\ p(1m)^{-1}(12),\ \ldots,\ p(1m)^{-1}(1r).$$

These numbers, for $m=2, 3, \ldots, r$, must be all different except p itself which occurs in each set.

Hence we get $r(r-1)+1=n$ different operations

$$1,\ (12),\ (13),\ \ldots,\ (1r),\ (12)^{-1},\ (12)^{-1}(13),\ \ldots,\ \ldots,\ (1r)^{-1}(1,\ r-1),$$

or, denoting $(1p)^{-1}(1q)$ by (pq), we have the n distinct operations

$$(pq) \qquad (p,\ q=1,\ 2,\ \ldots,\ r)$$

where $(pp)=1$ and $(pq)(qs)=(ps)$.

Starting with any number p, it is transformed by these substitutions into the n different numbers of the scheme. Let S, T be any two substitutions of the set, then corresponding to p in the first row we have pT in the tth row, say, so that corresponding to pS in the first row we have pST in the tth row. Therefore ST is a substitution of the set. Hence these operations form a group of order n.

Now each operation, except identity, changes all the symbols, and each changes a given symbol p into a different symbol, for if S and T both change p into q, then ST^{-1} leaves p unaltered, therefore $ST^{-1}=1$, or $S=T$. Again, since all the powers of S belong to the group, they must all change all the symbols, except that power which is the identical operation; hence each substitution must be regular.

Suppose now $n=ab$ and

$$S=(p_1 p_{a+1} p_{2a+1} \cdots p_{(b-1)a+1})(p_2 p_{a+2} \cdots p_{(b-1)a+2}) \cdots$$

then S is the ath power of

$$T=(p_1 p_2 \cdots p_a p_{a+1} p_{a+2} \cdots p_{2a} p_{2a+1} \cdots)$$

and the group is therefore the cyclical group generated by the single operation T of order n.

11. Now take any operation of order n of the group and denote it by 1 and its 2nd, 3rd, ... powers by 2, 3, ..., its nth power, which is identity, being denoted by 0. There is one set among the sets

$$(1q), (2q), \ldots, (rq) \qquad (q=1, 2, \ldots, r)$$

in which this operation occurs. Taking that set, denoted by the numbers, as the first column, the whole arrangement can be written down by writing the numbers in order in each row. Thus for $n=21$ a possible group makes (14) of order 21 and the powers of (14) are

0	1	2	3	4	5	6	7	8	9	10
1	(14)	(25)	(31)	(34)	(42)	(12)	(45)	(15)	(32)	(53)

11	12	13	14	15	16	17	18	19	20
(35)	(23)	(51)	(54)	(21)	(24)	(43)	(13)	(52)	(41)

The first column is then 0 1 6 8 18 so that we have the arrangement

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5
8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7
18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

12. There is in general considerable latitude in forming the group whose operations satisfy the given conditions. But if one arrangement has been obtained others may at once be obtained from it.

Any arrangement is completely determined when one column is given. Let the first column, expressed in terms of the operations of the group, be

$$1P_1P_2 \dots P_{r-1}$$

then forming the other columns which contain 1, the top row will be

$$1P_1^{-1}P_2^{-1} \dots P_{r-1}^{-1}$$

and if this is taken as the first column we get a new arrangement.* This gives then $2r$ different columns containing 1, r belonging to each arrangement. Again, if a is prime to n

$$1P_1^aP_2^a \dots P_{r-1}^a$$

gives another arrangement, for if S and T are any two numbers of the first arrangement S^a and T^a are distinct. If a is not prime to n, S^a may be equal to T^a without S being equal to T. If we take for a all the numbers less than n and prime to it, the resulting columns are in general all different. In particular if n be a prime, and

$$1PP^{a_1}P^{a_2} \dots P^{a_{r-2}}$$

is a column such that the columns formed by taking the powers of the elements are not all distinct, the only powers which give the same column are evidently $1, a_1, a_2, \dots, a_{r-2}$; hence these must be the $r-1$ numbers which appertain (mod n) to $r-1$ and its factors. If $r-1$ is even, one of these numbers is $n-1$ since $(n-1)^2 \equiv 1 (\text{mod } n)$, and we have seen above that 1, P and P^{n-1} cannot belong to the same column, so that in this case all the columns will be different. For $r=4$, $n=13$ the numbers a are 1, 3, 9, since $27 \equiv 1 (\text{mod } 13)$ and 0, 1, 3, 9 is a possible column.

13. When n is prime and r is odd it is easy to find at least a lower limit for the number of distinct arrangements. We get first $2r$ distinct columns; then by taking powers we get from any one column $n-1 = r(r-1)$ distinct columns, so that $k \cdot 2r = l \cdot r(r-1)$. The least value for l is 1 and the least value for k is $\frac{1}{2}(r-1)$, therefore the number of distinct arrangements is $r-1$ or a multiple of this. If r is even the set of numbers a can be found. If these form a possible column the number of distinct columns is

$$r + (l-1) \cdot r(r-1) = 2kr \text{ or } 2k = 1 + (l-1)(r-1).$$

* See, however, the end of § 14.

The least value for l is 2 and the least value for k is $\frac{1}{2}r$. If, however, the numbers do not form a possible column, the number of distinct columns is $l \,.\, r(r-1) = 2kr$ so that $l = 2$ and $k = (r-1)$. $2k$ or at least a multiple of $2k$ will be the number of distinct arrangements.

14. The following are the arrangements which I have obtained for $n = 3$, 7, 13, 21, 31. Each square represents the columns which contain 0, and can be read horizontally and vertically.

$n = 3$

0	2
1	0

$n = 7$

0	6	4
1	0	5
3	2	0

$n = 13$

0	12	9	7
1	0	10	8
4	3	0	11
6	5	2	0

0	12	10	4
1	0	11	5
3	2	0	7
9	8	6	0

$n = 21$

0	20	17	7	5
1	0	18	8	6
4	3	0	11	9
14	13	10	0	19
16	15	12	2	0

$n = 31$

0	30	28	23	19	13
1	0	29	24	20	14
3	2	0	26	22	16
8	7	5	0	27	21
12	11	9	4	0	25
18	17	15	10	6	0

0	30	28	21	17	5
1	0	29	22	18	6
3	2	0	24	20	8
10	9	7	0	27	15
14	13	11	4	0	19
26	25	23	16	12	0

0	30	27	25	18	10
1	0	28	26	19	11
4	3	0	29	22	14
6	5	2	0	24	16
13	12	9	7	0	23
21	20	17	15	8	0

0	30	27	21	19	14
1	0	28	22	20	15
4	3	0	25	23	18
10	9	6	0	29	24
12	11	8	2	0	26
17	16	13	7	5	0

0	30	23	20	18	14
1	0	24	21	19	15
8	7	0	28	26	22
11	10	3	0	29	25
13	12	5	2	0	27
17	16	9	6	4	0

The two arrangements which are obtained by reading the squares horizontally and vertically are really identical, differing only in notation. The symbols, written in cyclical order, being $0, 1, 2, 3, \ldots, n-1$, the substitution

$$(0)(1, n-1)(2, n-2)\ldots$$

simply reverses the cyclical order.

15. This problem is only one of a class of tactical problems connected with these configurations. E. H. Moore* has given a great many results relating to these arrangements. In his notation $S[k, l, m]$ represents a k-adic system in m letters of index 1 such that every l-ad of the system is incident with one and only one of the k-ads. The systems here considered have the index $l=2$ and are $S[r, 2, r^2-r+1]$. The general type of a tactical system to be considered here is the "finite geometry system," which may be denoted by FGS $[p_{lr}, *, p_{lt}]$ or FGS$p\,[(l, r, t), (t>r>l)$, *i.e.*, a p_{lr}-adic system in p_{lt} letters, with a reciprocal system FGS $[(l, r, t), *, p_{lt}]$ or FGS$c\,[l, r, t]\,(t>l>r)$, *i.e.*, an (l, r, t)-adic system in p_{lt} letters, where each p_{lr}-ad belonging to an FGS$p\,[(l, r, t)$ forms also an element in the FGS$p\,(r, s, t)$ and in the FGS$c\,[r, s, t]$ and each (l, r, t)-ad belonging to an FGS$c\,[l, r, t]$ forms also an element in the FGS$c\,[rst]$ and in the FGS$p\,[r, s, t]$. It may be further defined inductively thus. Every k-ad which is incident with a p_{lv}-ad belonging to an FGS$p\,[l, v, t]$ and not with a $p_{l,v-1}$-ad belonging to an FGS$p\,[l, v-1, t]$ is incident with (r, v, t) of the p_{lr}-ads; and, moreover, every k-ad of the p_{lr}-ads which is incident with an (r, v, t)-ad belonging to an FGS$c\,[r, v, t]$ and not with an $(r, v+1, t)$-ad belonging to an FGS$c\,[r, v+1, t]$ have in common a unique p_{lv}-ad.

The following is an FGS$p\,[0, 2, 3]$ with 15 $p_{02}\,(=7)$-ads :—

1	1	1	1	1	1	1	2	2	2	2	3	3	3	3
2	2	2	4	4	6	6	4	4	5	5	4	4	5	5
3	3	3	5	5	7	7	6	6	7	7	7	7	6	6
4	8	12	8	10	8	10	8	9	8	9	8	9	8	9
5	9	13	9	11	9	11	10	11	10	11	11	10	11	10
6	10	14	12	14	14	12	12	13	13	12	12	13	13	12
7	11	15	13	15	15	13	14	15	15	14	15	14	14	15

* "Tactical Memoranda," *Amer. J.*, XVIII. (1896), pp. 264–303.

where each of the 35 triads S [3, 2, 15] or FGSp [0, 1, 3].

1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3
2	4	6	8	10	12	14	4	5	8	9	12	13	4	5	8	9	12	13
3	5	7	9	11	13	15	6	7	10	11	14	15	7	6	11	10	15	14

4	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7
8	9	10	11	8	9	10	11	8	9	10	11	8	9	10	11
12	13	14	15	13	12	15	14	14	15	12	13	15	14	13	12

is incident with three of the 7-ads, each of the remaining 420 triads being incident with one and only one 7-ad. Also any pair of 7-ads have a unique triad in common, each pair of elements is incident with one and only one triad, and each element is incident with 7 of the 7-ads and with 7 of the triads.