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OSCILLATORY PROPERTIES OF SOLUTIONS OF CERTAIN ELLIPTIC EQUATIONS

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Certain elliptic equations of higher order are studied and a sufficient condition is given that every solution is oscillatory in an exterior domain. The principal tool is an averaging technique which enables one to reduce the n-dimensional problem to a one-dimensional problem.

Oscillation theory for higher order elliptic equations of the form $\Delta^m u + a_1\Delta^{m-1}u + \cdots + a_m u = 0$ (Δ is the Laplacian in $\mathbb{R}^n$) has been investigated by numerous authors. We refer the reader to [**1**, **4**] for $n = 3$, and to [**3**, **9**] for $n \geqslant 2$. In the case where $n = 3$, Górowski [**5**] obtained the oscillation results for the mth metaelliptic equation $\widetilde{L}^m u + a_1\widetilde{L}^{m-1}u + \cdots + a_m u = 0$, where $\widetilde{L} = \sum_{j,k=1}^{3} a_{jk}\left(\partial^2/\partial x_j \partial x_k\right)$ ($a_{jk} =$ constant).

We are concerned with the oscillatory behaviour of solutions of the elliptic equation

$$(1) \qquad \left(L^m + a_1 L^{m-1} + \cdots + a_{m-1}L + a_m\right)u(x) = 0, \qquad x \in \Omega,$$

where Ω is an exterior domain of $\mathbb{R}^n$ ($n \geqslant 2$), that is Ω contains the complement of some n-ball in $\mathbb{R}^n$. As usual, $x = (x_1, x_2, \ldots, x_n)$ denotes a point of $\mathbb{R}^n$. It is assumed that the coefficients a_j ($j = 1, 2, \ldots, m$) are real constants, L is the linear elliptic operator with constant coefficients

$$(2) \qquad L = \sum_{j,\, k=1}^{n} a_{jk}\frac{\partial^2}{\partial x_j \partial x_k},$$

where $a_{jk} = a_{kj}$ and (a_{jk}) is positive definite, and L^k is the kth iterate of L ($k = 1, 2, \ldots, m$).

The purpose of this paper is to present sufficient conditions for all solutions of (1) to be oscillatory in Ω. Our method is an adaptation of that used in [**3**].

DEFINITION: A function $u: \Omega \to \mathbb{R}^1$ is said to be *oscillatory* in Ω if u has a zero in $\{x \in \Omega : |x| > r\}$ for any $r > 0$, where $|x|$ denotes the Euclidean length of x.

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Since Ω is an exterior domain in $\mathbb{R}^n$, Ω contains $\mathbb{R}^n(r_0) \equiv \{x \in \mathbb{R}^n : |x| > r_0\}$ for some $r_0 > 0$. Let $x_0 = (x_1^0, x_2^0, \ldots, x_n^0)$ be a fixed point of $\mathbb{R}^n(r_0)$ and let $\rho(x)$ be defined by

$$\rho(x) = \left(\sum_{j,\,k=1}^{n} A_{jk}(x_j - x_j^0)(x_k - x_k^0) \right)^{1/2},$$

where (A_{jk}) denotes the inverse matrix of (a_{jk}). There is an $r_1 > 0$ such that $\{x \in \mathbb{R}^n : \rho(x) > r_1\} \subset \mathbb{R}^n(r_0)$. Associated with every function $u \in C(\Omega)$, we define the function $M[u](r)$ by

$$M[u](r) = \frac{1}{\sigma_n r^{n-1}} \int_{S_r} u \frac{d\sigma}{|\nabla \rho|}, \quad r > r_1, \tag{3}$$

where σ_n denotes the surface area of the unit sphere in $\mathbb{R}^n$ and $S_r = \{x \in \mathbb{R}^n : \rho(x) = r\}$.

Lemma 1. *If $u \in C^2(\Omega)$, then we obtain*

$$\frac{1}{\sigma_n r^{n-1}} \int_{S_r} Lu \frac{d\sigma}{|\nabla \rho|} = r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} M[u](r) \right), \quad r > r_1,$$

where L is given by (2).

Proof: It is easy to see that ${}^t(\nabla\rho)(A_{jk})(\nabla\rho) = 1$. Hence, the conclusion follows from a result of Suleĭmanov [8] (see also [12, Lemma 2.1]). ▯

Lemma 2. *If $u \in C^4(\Omega)$, then $M[u](r)$ satisfies*

$$\frac{1}{\sigma_n r^{n-1}} \int_{S_r} L^2 u \frac{d\sigma}{|\nabla \rho|} = r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr}\left(r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} M[u](r) \right)\right)\right), \; r > r_1. \tag{4}$$

Proof: Lemma 1 implies that

$$M[Lu](r) = r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} M[u](r) \right), \tag{5}$$

$$\frac{1}{\sigma_n r^{n-1}} \int_{S_r} L^2 u \frac{d\sigma}{|\nabla \rho|} = r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} M[Lu](r) \right). \tag{6}$$

Combining (5) with (6) yields the desired identity (4). ▯

Theorem. *Assume that the algebraic equation*

$$z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m = 0 \tag{7}$$

has simple roots only and has no real nonnegative root. Then every solution $u \in C^{2m}(\Omega)$ of (1) is oscillatory in Ω.

PROOF: Suppose to the contrary that there exists a solution $u \in C^{2m}(\Omega)$ of (1) which does not oscillate in Ω. Without loss of generality we may assume that $u > 0$ in $\mathbb{R}^n(r_2)$ for some $r_2 \geqslant r_1$. The hypothesis implies that

$$z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m = \prod_{k=1}^{p} \left(z^2 + 2b_k z + \left(b_k^2 + c_k^2\right)\right) \prod_{k=2p+1}^{m} \left(z + d_k^2\right),$$

where $c_k > 0\,(k = 1, 2, \ldots, p)$, $d_k > 0\,(k = 2p+1, 2p+2, \ldots, m)$, $-b_j \pm ic_j \neq -b_k \pm ic_k\left(j \neq k, i = \sqrt{-1}\right)$ and $d_j \neq d_k (j \neq k)$. Hence, (1) can be written in the form

$$\left(\prod_{k=1}^{p} \left(L^2 + 2b_k L + \left(b_k^2 + c_k^2\right)\right) \prod_{k=2p+1}^{m} \left(L + d_k^2\right)\right) u = 0.$$

It follows from a result of Wachnicki [**10**, Theorem 2] that there exists a unique system $\widetilde{u}_k(x) (k = 1, 2, \ldots, p)$, $u_k(x) (k = 2p+1, 2p+2, \ldots, m)$ such that

$$\left(L^2 + 2b_k L + \left(b_k^2 + c_k^2\right)\right)\widetilde{u}_k(x) = 0 \quad (k = 1, 2, \ldots, p),$$
$$\left(L + d_k^2\right) u_k(x) = 0 \quad (k = 2p+1, 2p+2, \ldots, m)$$

and

$$u(x) = \sum_{k=1}^{p} \widetilde{u}_k(x) + \sum_{k=2p+1}^{m} u_k(x) \tag{8}$$

(see [**4**, Lemma 4]). Then we easily obtain

$$M[u](r) = \sum_{k=1}^{p} M[\widetilde{u}_k](r) + \sum_{k=2p+1}^{m} M[u_k](r) \tag{9}$$

and we observe, using Lemmas 1 and 2, that

$$\begin{aligned} &r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} M[u_k](r)\right) + d_k^2 M[u_k](r) = 0, \\ &r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr}\left(r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} M[\widetilde{u}_k](r)\right)\right)\right) \\ &\qquad + 2b_k r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} M[\widetilde{u}_k](r)\right) + \left(b_k^2 + c_k^2\right) M[\widetilde{u}_k](r) = 0. \end{aligned} \tag{10}$$

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Using the same arguments as in [**3**, p.231], we see that

$$(11)\quad r^{(n-1)/2}M[u_k](r) \approx A_k \sin\left(\int_{\tilde{r}}^{r} \left(d_k^2 - (1-n^2)4^{-1}s^{-2}\right)^{1/2} ds + \theta_k\right) \quad (r \to \infty)$$

for some constants A_k and θ_k ($k = 2p+1, 2p+2, \ldots, m$). The following system

$$(12)\qquad y' = (A + V(r))y, \qquad y = {}^t(y_1, y_2, y_3, y_4),$$

is associated with (10), where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(b_k^2 + c_k^2) & 0 & -2b_k & 0 \end{pmatrix},$$

$$V(r) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\left(2b_k\frac{n-1}{r} - \frac{(n-1)(n-3)}{r^3}\right) & -\frac{(n-1)(n-3)}{r^2} & -\frac{2(n-1)}{r} \end{pmatrix}.$$

Since $\det(A - \lambda I) = \lambda^4 + 2b_k\lambda^2 + (b_k^2 + c_k^2)$, we find that the characteristic roots of A are $\pm\mu_1 \pm i\mu_2$, where $\mu_1 = 2^{-1/2}\left(-b_k + (b_k^2 + c_k^2)^{1/2}\right)^{1/2}$ and $\mu_2 = 2^{-1/2}\left(b_k + (b_k^2 + c_k^2)^{1/2}\right)^{1/2}$. It is easily seen that the characteristic polynomial for $A + V(r)$ is given by

$$\lambda^4 + \frac{2(n-1)}{r}\lambda^3 + \left(2b_k + \frac{(n-1)(n-3)}{r^2}\right)\lambda^2 + \left(2b_k\frac{n-1}{r} - \frac{(n-1)(n-3)}{r^3}\right)\lambda + b_k^2 + c_k^2.$$

Using Ferrari's formula (see [**11**, p.190]), we conclude that the characteristic roots $\lambda_j(r)$ of $A + V(r)$ can be written in the form

$$\lambda_j(r) = -\frac{n-1}{2r} + \mu_1(r) + (-1)^{j+1}i\mu_2(r) \qquad (j = 1, 2),$$
$$\lambda_j(r) = -\frac{n-1}{2r} - \mu_1(r) + (-1)^{j+1}i\mu_2(r) \qquad (j = 3, 4),$$

where $\lim_{r\to\infty} \mu_k(r) = \mu_k$ ($k = 1, 2$). We easily see that

$$\int_{r_1}^{\infty} |V'(r)|\, dr < \infty \quad \text{and} \quad \lim_{r\to\infty} V(r) = 0.$$

Then there are the solutions $\Phi_j(r)$ $(j = 1, 2, 3, 4)$ of (12) and $\tilde{r}$ $(r_1 \leqslant \tilde{r} < \infty)$ such that

$$\lim_{r\to\infty} \Phi_j(r)\exp\left(-\int_{\tilde{r}}^{r} \lambda_j(s)ds\right) = p_j \qquad (j = 1, 2, 3, 4),$$

where each p_j $(j = 1, 2, 3, 4)$ is a characteristic vector of A associated with $\mu_1 + (-1)^{j+1} i\mu_2$ $(j = 1, 2)$, $-\mu_1 + (-1)^{j+1} i\mu_2$ $(j = 3, 4)$ (see [**2**, p.92]). Hence, the following holds:

$$\Phi_j r^{(n-1)/2} \approx P_j \exp\left(\int_{\tilde{r}}^{r} \mu_1(s)ds\right)\left(\cos\int_{\tilde{r}}^{r} \mu_2(s)ds + (-1)^{j+1} i \sin\int_{\tilde{r}}^{r} \mu_2(s)ds\right)$$
$$(r \to \infty; j = 1, 2),$$
$$\Phi_j r^{(n-1)/2} \approx P_j \exp\left(-\int_{\tilde{r}}^{r} \mu_1(s)ds\right)\left(\cos\int_{\tilde{r}}^{r} \mu_2(s)ds + (-1)^{j+1} i \sin\int_{\tilde{r}}^{r} \mu_2(s)ds\right)$$
$$(r \to \infty; j = 3, 4),$$

where $P_j = K_j p_j$ for some constants $K_j \in \mathbb{R}^1$ $(j = 1, 2, 3, 4)$. Since $M[\tilde{u}_k](r)$ is a real-valued function and a linear combination of the first components of Φ_j $(j = 1, 2, 3, 4)$, we obtain

(13)
$$r^{(n-1)/2} M[\tilde{u}_k](r) \approx B_k \exp\left(\int_{\tilde{r}}^{r} \mu_1(s)ds\right) \sin\left(\int_{\tilde{r}}^{r} \mu_2(s)ds + \sigma_k\right)$$
$$+ C_k \exp\left(-\int_{\tilde{r}}^{r} \mu_1(s)ds\right) \sin\left(\int_{\tilde{r}}^{r} \mu_2(s)ds + \tau_k\right) \quad (r \to \infty)$$

for some constants B_k, C_k, σ_k and τ_k $(k = 1, 2, \ldots, p)$. Combining (9), (11) and (13) yields

(14)
$$r^{(n-1)/2} M[u](r) \approx \sum_{k=2p+1}^{m} A_k \sin\left(\int_{\tilde{r}}^{r} \left(d_k^2 - (1-n^2)4^{-1}s^{-2}\right)^{1/2} ds + \theta_k\right)$$
$$+ \sum_{k=1}^{p} B_k \exp\left(\int_{\tilde{r}}^{r} \mu_1(s)ds\right) \sin\left(\int_{\tilde{r}}^{r} \mu_2(s)ds + \sigma_k\right)$$
$$+ \sum_{k=1}^{p} C_k \exp\left(-\int_{\tilde{r}}^{r} \mu_1(s)ds\right) \sin\left(\int_{\tilde{r}}^{r} \mu_2(s)ds + \tau_k\right) \quad (r \to \infty).$$

Since $u > 0$ in $\mathbb{R}^n(r_2)$, the left hand side of (14) is positive for $r > r_2$. However, the right hand side of (14) changes sign in an arbitrary interval (r, ∞) (see [**4**, Lemma 6]).

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This is a contradiction. If (7) has only simple negative or only simple complex roots, then we replace (8) by

$$u(x) = \sum_{k=1}^{m} u_k(x), \qquad (L + d_k^2)u_k(x) = 0,$$

and

$$u(x) = \sum_{k=1}^{m/2} \widetilde{u}_k(x), \qquad (L^2 + 2b_k L + b_k^2 + c_k^2)\widetilde{u}_k(x) = 0,$$

respectively. Proceeding as above, we are led to a contradiction. The proof is complete. □

REMARK 1. In the case where $\Omega = \mathbb{R}^n$, $x_0 = 0$ and L is the Laplacian Δ in $\mathbb{R}^n$, we conclude that $\rho(x) = |x|$ and $|\nabla\rho| = 1$. Then, $M[u](r)$ given by (3) reduces to the spherical mean of u over $\{x \in \mathbb{R}^n : |x| = r\}$.

REMARK 2. In view of Lemma 1, we obtain

$$\frac{1}{\sigma_n r^{n-1}} \int_{S_r} L^k u \frac{d\sigma}{|\nabla\rho|} = \left(r^{1-n}\frac{d}{dr}r^{n-1}\frac{d}{dr}\right)^k M[u](r), \; (k = 1, 2, \ldots, m).$$

Hence, we can extend the results of Naito and Yoshida [7] to the more general elliptic equation

$$L^m u + a_1 L^{m-1} u + \cdots + a_m u + \Phi(x, u) = f(x),$$

where L is given by (2).

REMARK 3. If $u \in C^{2m}(\Omega)$ and u satisfies (1), then u is analytic in Ω (see [**6**, p.178]). Hence, the set of zeros of a nontrivial solution of (1) does not have interior points.

REMARK 4. Our theorem generalises a result of Górowski [**5**, Theorem 3]. If $L = \Delta$, our results reduce to the results of [**1**, **4**] for $n = 3$, and to the results of [**3**, **9**] for $n \geqslant 2$.

REFERENCES

[1] F. Barański, 'The mean value theorem and oscillatory properties of certain elliptic equations in three dimensional space', *Comment. Math. Prace Mat.* **19** (1976), 13–14.

[2] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations* (McGraw-Hill, New York, 1955).

[3] I.G. Dobrotvor, 'Oscillation properties of the solutions of equations with a polyharmonic operator in the space E^n', *Ukrainian Math. J.* **36** (1984), 230–232.

[4] V.I. Gorbaichuk and I.G. Dobrotvor, 'Conditions for the oscillation of solutions of a class of elliptic equations of high orders with constant coefficients', *Ukrainian Math. J.* **32** (1980), 385–391.

[5] J. Górowski, 'The mean value theorem and oscillatory behaviour for solutions of certain elliptic equations', *Math. Nachr.* **122** (1985), 259–265.

[6] L. Hörmander, *Linear partial differential operators* (Springer-Verlag, Berlin, 1963).

[7] M. Naito and N. Yoshida, 'Oscillation criteria for a class of higher order elliptic equations', *Math. Rep. Toyama Univ.* **12** (1989), 29–40.

[8] N.M. Suleĭmanov, 'On the behavior of solutions of second order nonlinear elliptic equations with linear principal part', *Soviet Math. Dokl.* **20** (1979), 815–819.

[9] E. Wachnicki, 'On the oscillatory properties of solutions of certain elliptic equations', *Comment. Math. Prace Mat.* **19** (1976), 165–169.

[10] E. Wachnicki, 'On boundary value problems for some partial differential equations of higher order', *Comment. Math. Prace Mat.* **20** (1977), 215–233.

[11] B.L. van der Waerden, *Algebra I* (Springer-Verlag, Berlin, 1960).

[12] N. Yoshida, 'Oscillation properties of solutions of second order elliptic equations', *SIAM J. Math. Anal.* **14** (1983), 709–718.

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