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RADIAL GROWTH AND EXCEPTIONAL SETS FOR CAUCHY–STIELTJES INTEGRALS

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This paper considers the radial and nontangential growth of a function f given by

$$f(z)=\int_{|\zeta|=1}\frac{1}{(1-\bar{\zeta}z)^{\alpha}}\,d\mu(\zeta)\quad\text{for}\quad |z|<1,$$

where $\alpha>0$ and μ is a complex-valued Borel measure on the unit circle. The main theorem shows how certain local conditions on μ near $e^{i\theta}$ affect the growth of $f(z)$ as $z\to e^{i\theta}$ in Stolz angles. This result leads to estimates on the nontangential growth of f where exceptional sets occur having zero β-capacity.

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1. Introduction

Let $\Delta=\{z:|z|<1\}$ and $\Gamma=\{z:|z|=1\}$. Let $\mathcal{M}$ denote the set of complex-valued Borel measures on Γ. For each $\alpha>0$ the family $\mathcal{F}_\alpha$ of analytic functions is defined as follows: $f\in\mathcal{F}_\alpha$ provided that there exists $\mu\in\mathcal{M}$ such that

$$f(z)=\int_{\Gamma}\frac{1}{(1-\bar{\zeta}z)^{\alpha}}\,d\mu(\zeta)\tag{1}$$

for $|z|<1$. (Here and throughout this paper every logarithm means the principal branch.)

Equation (1) is equivalent to

$$f(z)=\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,dg(t)\tag{2}$$

where g is a complex-valued function of bounded variation on $[-\pi,\pi]$. Throughout this paper we assume that every such g is extended to $(-\infty,\infty)$ by $g(t+\pi)-g(t-\pi)=g(\pi)-g(-\pi)$. Then (2) can be rewritten

$$f(z)=\int_{\theta-\pi}^{\theta+\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,dg(t)\tag{3}$$

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for each real number θ and g is of bounded variation on $[\theta-\pi, \theta+\pi]$.

We consider the effect of the differentiability of g, or of other local smoothness conditions at θ, on the radial growth of f in the direction θ. In particular, this yields a new proof of the result in [2, Theorem 7] that if $f \in \mathscr{F}_\alpha$ and $\alpha > 1$ then $\lim_{r\to 1-}(1-r)^{\alpha-1} f(re^{i\theta}) = 0$ for almost all θ in $[-\pi, \pi]$. If $f \in \mathscr{F}_\alpha$ then (1) implies

$$|f(z)| \leq \frac{\|\mu\|}{(1-|z|)^\alpha} = O\left[\frac{1}{(1-|z|)^\alpha}\right],$$

and this maximal growth is achieved, for example, by

$$f(z) = \frac{1}{(1-z)^\alpha}.$$

It was shown in [2. Theorem 11] that if $f \in \mathscr{F}_\alpha$ then

$$|f(re^{i\theta})| = o\left[\frac{1}{(1-r)^\alpha}\right]$$

as $r \to 1-$ except possibly for a set in θ which is countable. Also, when $\alpha > 1$ any growth smaller than

$$o\left[\frac{1}{(1-r)^{\alpha-1}}\right]$$

is achievable for some $f \in \mathscr{F}_\alpha$ and for *all* θ [2, Theorem 8]. Thus the growths

$$o\left[\frac{1}{(1-r)^\alpha}\right] \quad \text{and} \quad o\left[\frac{1}{(1-r)^{\alpha-1}}\right]$$

provide extreme cases for questions concerning exceptional sets.

Theorem 2 in this paper shows that certain growths between these two extremes are associated with exceptional sets whose β-capacity is zero. For example, it is proved that if $f \in \mathscr{F}_\alpha$, $0 < \beta < \alpha$ and $\beta < 1$ then

$$f(re^{i\theta}) = o\left[\frac{1}{(1-r)^{\alpha-\beta}}\right]$$

for all θ in $[-\pi, \pi]$ except possibly for a set whose β-capacity is zero.

The results obtained about radial growths are proved in more generality and are expressed in terms of suitable nontangential limits. Suppose that $-\pi \leqq \theta \leqq \pi$ and $0 \leqq \gamma \leqq \pi$. Let $S(\theta, \gamma)$ denote the closed Stolz angle having vertex $e^{i\theta}$ and opening γ. There are positive constants A and B depending only on γ such that if $z = re^{i\phi} \in S(\theta, \gamma)$ (and ϕ is chosen suitably) then

$$|z - e^{i\theta}| \leqq A(1 - |z|) \tag{4}$$

and

$$|\phi - \theta| \leqq B(1 - |z|). \tag{5}$$

A function f defined in Δ is said to have a nontangential limit at $e^{i\theta}$ provided that

$$\lim_{\substack{z \to e^{i\theta} \\ z \in S(\theta, \gamma)}} f(z) \quad \text{exists for every} \quad \gamma (0 \leqq \gamma < \pi).$$

The results in this paper complement facts about nontangential limits proved in [3]. In particular, it was shown in [3, Theorem 5] that if $f \in \mathscr{F}_\alpha$ for some α where $0 < \alpha < 1$ then f has a nontangential limit at $e^{i\theta}$ except possibly for a set of θ in $[-\pi, \pi]$ whose α-capacity is zero. The focus of this paper is primarily on the growth of f where $f \in \mathscr{F}_\alpha$ and $\alpha \geqq 1$.

2. Radial growth and exceptional sets of zero β-capacity

Theorem 1. *Let $\alpha > 0$ and for*

$$|z| < 1 \quad let \quad f(z) = \int_{-\pi}^{\pi} \frac{1}{(1 - e^{-it}z)^\alpha} \, dg(t),$$

where g is a complex-valued function of bounded variation on $[-\pi, \pi]$.

(a) *Suppose that $|g(t) - g(\theta)| = o(|t - \theta|^\beta)$ as $t \to \theta$, for some θ in $[-\pi, \pi]$ and for some $\beta > 0$. If $\beta < \alpha$ then $(1 - e^{-i\theta}z)^{\alpha - \beta} f(z)$ has the nontangential limit zero at $e^{i\theta}$. If $\beta = \alpha$ then*

$$\frac{f(z)}{\log\left(\dfrac{1}{1 - e^{-i\theta}z}\right)}$$

has the nontangential limit zero at $e^{i\theta}$.

(b) *Suppose that*

$$|g(t)-g(\theta)|=o\left[\frac{1}{\log\frac{1}{|t-\theta|}}\right] \text{ as } t\to\theta,$$

for some θ *in* $[-\pi,\pi]$. *Then*

$$\left[(1-e^{-i\theta})^{\alpha}\log\left(\frac{1}{1-e^{-i\theta}z}\right)\right]f(z)$$

has the nontangential limit zero at $e^{i\theta}$.

Proof. Equation (3) implies

$$f(z)=\int_{\theta-\pi}^{\theta+\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,d[g(t)-g(\theta)],$$

and an integration by parts gives

$$f(z)=\frac{g(\theta+\pi)-g(\theta-\pi)}{(1+e^{-i\theta}z)^{\alpha}}+i\alpha\int_{\theta-\pi}^{\theta+\pi}K(e^{-it}z)[g(t)-g(\theta)]\,dt$$

where

$$K(z)=\frac{z}{(1-z)^{\alpha+1}}.$$

Let $0\leqq\gamma<\pi$. If $z\in S(\theta,\gamma)$ then

$$|f(z)|\leqq C_1+\alpha\int_{\theta-\pi}^{\theta+\pi}\frac{|g(t)-g(\theta)|}{|1-e^{-it}z|^{\alpha+1}}\,dt,$$

where

$$C_1=|g(\theta+\pi)-g(\theta-\pi)|\sup_{z\in S(\theta,\gamma)}\left\{\frac{1}{|1+e^{-i\theta}z|^{\alpha}}\right\}<+\infty.$$

This can be written

$$|f(z)|\leqq C_1+\alpha\int_{-\pi}^{\pi}\frac{|g(\theta+t)-g(\theta)|}{|1-e^{-i(\theta+t)}z|^{\alpha+1}}\,dt. \tag{6}$$

Assume that $\beta>0$ and $|g(t)-g(\theta)|=o(|t-\theta|^{\beta})$ as $t\to\theta$. Let $\varepsilon>0$. There exists δ such that $0<\delta<1$ and $|g(\theta+t)-g(\theta)|\leqq\varepsilon|t|^{\beta}$ for $|t|<\delta$. Let A and B be the constants described by (4) and (5) and let $r=|z|$. There exists η such that $0<\eta\leqq 1/2$ and if $z\in S(\theta,\gamma)$ and $|z-e^{i\theta}|<\eta$ then $2B(1-r)<\delta$. For $|t|\leqq\pi$ and $|z|<1$ let

$$G(t,z)=\frac{|g(\theta+t)-g(\theta)|}{|1-e^{-i(\theta+t)}z|^{\alpha+1}}.$$

For $1\leqq n\leqq 5$ let $J_n=\int_{I_n}G(t,z)\,dt$ where

$$I_1=[-\pi,-\delta],\, I_2=[-\delta,-2B(1-r)],\, I_3=[-2B(1-r),2B(1-r)],$$

$$I_4=[2B(1-r),\delta],\text{ and } I_5=[\delta,\pi].$$

Clearly $G(t,z)$ is bounded for $t\in I_1$ and $z\in S(\theta,\gamma)$ and hence there is a constant C_2 such that $J_1\leqq C_2$ for $z\in S(\theta,\gamma)$. Likewise $J_5\leqq C_3$ for $z\in S(\theta,\gamma)$ where C_3 is some constant. For $z\in S(\theta,\gamma)$ and $|z-e^{i\theta}|<\eta$ we have

$$J_2=\int_{-\delta}^{-2B(1-r)}\frac{|g(\theta+t)-g(\theta)|}{|1-e^{-i(\theta+t)}z|^{\alpha+1}}\,dt\leqq\left(\frac{\pi}{\sqrt{2}}\right)^{\alpha+1}\varepsilon\int_{-\delta}^{-2B(1-r)}\frac{|t|^{\beta}}{|t-(\phi-\theta)|^{\alpha+1}}\,dt$$

$$\leqq 2^{\beta}\left(\frac{\pi}{\sqrt{2}}\right)^{\alpha+1}\varepsilon\int_{-\delta}^{-2B(1-r)}\frac{1}{|t-(\phi-\theta)|^{\alpha-\beta+1}}\,dt.$$

Suppose that $\beta<\alpha$. Then

$$\int_{-\delta}^{-2B(1-r)}\frac{1}{|t-(\phi-\theta)|^{\alpha-\beta+1}}\,dt=\frac{1}{\alpha-\beta}\left\{\frac{1}{[\phi-\theta+2B(1-r)]^{\alpha-\beta}}-\frac{1}{[\phi-\theta+\delta]^{\alpha-\beta}}\right\}$$

$$\leqq\frac{1}{(\alpha-\beta)[\phi-\theta+2B(1-r)]^{\alpha-\beta}}\leqq\frac{1}{(\alpha-\beta)B^{\alpha-\beta}(1-r)^{\alpha-\beta}},$$

because of (5). Hence

$$J_2\leqq\frac{(2B)^{\beta}}{\alpha-\beta}\left(\frac{\pi}{\sqrt{2}}\right)^{\alpha+1}\varepsilon\frac{1}{(1-r)^{\alpha-\beta}}.$$

The same inequality holds for J_4. Also,

$$J_3=\int_{-2B(1-r)}^{2B(1-r)}\frac{|g(\theta+t)-g(\theta)|}{|1-e^{-i(\theta+t)}z|^{\alpha+1}}\,dt\leqq\int_{-2B(1-r)}^{2B(1-r)}\frac{\varepsilon|t|^{\beta}}{(1-r)^{\alpha+1}}\,dt=\frac{2(2B)^{\beta+1}}{\beta+1}\frac{\varepsilon}{(1-r)^{\alpha-\beta}}.$$

Since (6) implies $|f(z)| \leqq C_1 + \alpha \sum_{n=1}^{5} J_n$ the estimates above yield

$$|f(z)| \leqq C_4 + \frac{C_5 \varepsilon}{(1-r)^{\alpha-\beta}}$$

for $z \in S(\theta, \gamma)$ and $|z - e^{i\theta}| < \eta$, where C_4 and C_5 are constants. This inequality and (4) imply that

$$\lim_{\substack{z \to e^{i\theta} \\ z \in S(\theta, \gamma)}} (1 - e^{-i\theta} z)^{\alpha-\beta} f(z) = 0.$$

This proves (a) in the case $\beta < \alpha$.

Next suppose that $\beta = \alpha$. Then the estimate for J_2 given above becomes

$$J_2 \leqq 2^{(\alpha-1)/2} \pi^{\alpha+1} \varepsilon \int_{-\delta}^{-2B(1-r)} \frac{1}{|t - (\phi - \theta)|} \, dt.$$

By considering the cases $\phi < \theta, \phi = \theta$ and $\phi > \theta$ we find that, in general,

$$\int_{-\delta}^{-2B(1-r)} \frac{1}{|t - (\phi - \theta)|} \, dt \leqq \log\left[\frac{1}{B(1-r)}\right].$$

Hence

$$J_2 \leqq 2^{(\alpha-1)/2} \pi^{\alpha+1} \varepsilon \log\left[\frac{1}{B(1-r)}\right].$$

The same inequality holds for J_4. Also,

$$J_3 \leqq \frac{2(2B)^{\alpha+1}}{\alpha+1} \varepsilon.$$

This yields

$$|f(z)| \leqq C_6 + C_7 \varepsilon \log\left(\frac{1}{1-r}\right)$$

for $z \in S(\theta, \gamma)$ and $|z - e^{i\theta}| < \eta$, where C_6 and C_7 are constants. Therefore

$$\lim_{\substack{z \to e^{i\theta} \\ z \in S(\theta,\gamma)}} \frac{f(z)}{\log\left(\dfrac{1}{1-r}\right)} = 0.$$

We are required to prove that

$$\lim_{\substack{z \to e^{i\theta} \\ z \in S(\theta,\gamma)}} \frac{f(z)}{\log\left(\dfrac{1}{1-e^{-i\theta}z}\right)} = 0.$$

Hence if suffices to show that

$$M \equiv \sup_{\substack{z \in S(\theta,\,\gamma) \\ |z|=r \\ 0 \leqq r < 1}} \frac{\log\left(\dfrac{1}{1-r}\right)}{\left|\log\left(\dfrac{1}{1-e^{-i\theta}z}\right)\right|} < +\infty. \tag{7}$$

Inequality (7) follows if it is shown that

$$\frac{\log\left(\dfrac{1}{1-r}\right)}{\log\left(\dfrac{1}{1-e^{-i\theta}z}\right)}$$

is bounded for $z \in S(\theta,\gamma)$, $|z|=r$ and r sufficiently near 1. If $z = re^{i\phi} \in S(\theta,\gamma)$ then (5) gives $|\phi-\theta| \leqq B(1-r)$ and hence there exists r_1 such that $0 < r_1 < 1$ and $1-\cos(\phi-\theta) \leqq (\phi-\theta)^2$ for $r_1 \leqq r < 1$. Thus

$$|1-e^{-i\theta}z|^2 = (1-r)^2 + 2r[1-\cos(\phi-\theta)] \leqq (1-r)^2$$
$$+2(\phi-\theta)^2 \leqq (1+2B^2)(1-r)^2.$$

Hence

$$\frac{1}{|1-e^{-i\theta}z|} \geqq \frac{1}{C(1-r)} \quad \text{for} \quad r_1 \leqq r < 1, \quad \text{where} \quad C = \sqrt{1+2B^2}.$$

There exists r_2 such that

$$r_1 \leqq r_2 < 1 \quad \text{and} \quad \frac{1}{C(1-r)} > 1 \quad \text{for} \quad r_2 \leqq r < 1.$$

Then

$$\left|\log\frac{1}{(1-e^{-i\theta}z)}\right| \geq \left|Re \log\frac{1}{(1-e^{-i\theta}z)}\right| = \left|\log\frac{1}{|(1-e^{-i\theta}z)|}\right| \geq \log\frac{1}{C(1-r)} \quad \text{for} \quad r_2 \leqq r < 1.$$

Therefore, if $z \in S(\theta, \gamma)$ and $r_2 \leqq r < 1$ then

$$\left|\frac{\log\dfrac{1}{1-r}}{\log\dfrac{1}{1-e^{-i\theta}z}}\right| \leqq \frac{\log\dfrac{1}{1-r}}{\log\dfrac{1}{C}+\log\dfrac{1}{1-r}} \equiv \sigma(r).$$

Since $\lim_{r\to 1^-} \sigma(r) = 1$ this proves (7). Hence the proof of (a) is complete.

The proof of (b) follows in a similar way. We have $0 < \delta < 1$ and

$$|g(\theta+t)-g(\theta)| \leqq \frac{\varepsilon}{\log\dfrac{1}{|t|}} \quad \text{for} \quad |t| < \delta.$$

The estimate on J_2 becomes

$$J_2 \leqq \left(\frac{\pi}{\sqrt{2}}\right)^{\alpha+1} \varepsilon \int_{-\delta}^{-2B(1-r)} \frac{1}{\left(\log\dfrac{1}{|t|}\right)|t-(\phi-\theta)|^{\alpha+1}}\, dt$$

$$\leqq \left(\frac{\pi}{\sqrt{2}}\right)^{\alpha+1} 2^{\alpha+1} \varepsilon \int_{-\delta}^{-2B(1-r)} \frac{1}{\left(\log\dfrac{1}{|t|}\right)|t|^{\alpha+1}}\, dt$$

$$= \left(\frac{\pi}{\sqrt{2}}\right)^{\alpha+1} 2^{\alpha+1} \varepsilon L(r),$$

where

$$L(r)=\int_{2B(1-r)}^{\delta}\frac{1}{t^{\alpha+1}\log\frac{1}{t}}\,dt.$$

Let

$$M(r)=\frac{1}{(1-r)^{\alpha}\log\left(\frac{1}{1-r}\right)}\quad\text{for}\quad 0<r<1.$$

Then $\lim_{r\to 1-}L(r)=+\infty$ and $\lim_{r\to 1-}M(r)=+\infty$ and l'Hospital's rule yields

$$\lim_{r\to 1-}\frac{L(r)}{M(r)}=\frac{1}{\alpha(2B)^{\alpha}}.$$

Hence then exists r_1 such that $0<r_1<1$ and

$$\frac{L(r)}{M(r)}\leqq\frac{2}{\alpha(2B)^{\alpha}}\quad\text{for}\quad r_1\leqq r<1.$$

Therefore there is η' such that $0<\eta'\leqq\eta$ and

$$J_2\leqq\frac{C_8\varepsilon}{(1-r)^{\alpha}\log\left(\frac{1}{1-r}\right)}\quad\text{for}\quad z\in S(\theta,\gamma)$$

and $|z-e^{i\theta}|<\eta'$, where C_8 is a constant. The same inequality holds for J_4.

Also

$$J_3\leqq\frac{\varepsilon}{(1-r)^{\alpha+1}}\int_{-2B(1-r)}^{2B(1-r)}\frac{1}{\log\frac{1}{|t|}}\,dt=\frac{2\varepsilon}{(1-r)^{\alpha+1}}\int_{0}^{2B(1-r)}\frac{1}{\log\frac{1}{t}}\,dt$$

$$\leqq\frac{2\varepsilon}{(1-r)^{\alpha+1}}\left\{\frac{1}{\log\frac{1}{2B(1-r)}}[2B(1-r)]\right\}=\frac{4B\varepsilon}{(1-r)^{\alpha}\log\frac{1}{2B(1-r)}}.$$

These estimates imply

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$$|f(z)| \leqq C_9 + \frac{C_{10}\varepsilon}{(1-r)^\alpha \log\left(\frac{1}{1-r}\right)} \quad \text{for} \quad z \in S(\theta,\gamma)$$

and $|z-e^{i\theta}|<\eta'$, where C_9 and C_{10} are constants. This proves that

$$\lim_{\substack{z \to e^{i\theta} \\ z \in S(\theta,\gamma)}} \left\{\left[(1-r)^\alpha \log \frac{1}{(1-r)}\right] f(z)\right\} = 0. \tag{8}$$

We have

$$\left|\log \frac{1}{(1-e^{-i\theta}z)}\right| \leqq \left|\log \frac{1}{(1-|z|)}\right| + \frac{\pi}{2} \leqq 2 \log\left(\frac{1}{1-r}\right) \quad \text{for} \quad |z| = r \geqq r_0$$

for a suitable r_0 $(0 < r_0 < 1)$. Thus (4) and (8) imply

$$\lim_{\substack{z \to e^{i\theta} \\ z \in S(\theta,\gamma)}} \left\{\left[(1-e^{-i\theta}z)^\alpha \log\left(\frac{1}{(1-e^{-i\theta}z)}\right)\right] f(z)\right\} = 0.$$

This completes the proof of (b). □

The notion of zero β-capacity for Borel subsets of $[-\pi,\pi]$ provides a useful measure of the fineness of exceptional sets for the radial growth of functions in $\mathscr{F}_\alpha$. The definition and properties of β-capacity are given in [1]. We note that 0-capacity corresponds to logarithmic capacity and that if the β-capacity of a set is zero for some $\beta (0 \leqq \beta < 1)$ then its Lebesgue measure is zero.

The next theorem uses the following lemma. It is proved in [4, Lemma 1] in the case $0 < \beta < 1$ and a similar argument proves the second assertion.

Lemma 1. *Suppose that g is a nondecreasing function on $[-\pi,\pi]$. If $0<\beta<1$ then $|g(t)-g(\theta)| = o(|t-\theta|^\beta)$ as $t \to \theta$ for all θ in $[-\pi,\pi]$ except possibly for a set whose β-capacity is zero. Also,*

$$|g(t)-g(\theta)| = o\left[\frac{1}{\log \frac{1}{|t-\theta|}}\right] \textit{ as } t \to \theta \quad \textit{for all} \quad \theta \textit{ in } [-\pi,\pi]$$

except possibly for a set whose logarithmic capacity is zero.

Theorem 2. *Suppose that $\alpha > 0$ and $f \in \mathscr{F}_\alpha$. If $0<\beta<1$ and $\beta<\alpha$ then*

$(1-e^{-i\theta}z)^{\alpha-\beta}f(z)$ has the nontangential limit zero at $e^{i\theta}$ for all θ in $[-\pi,\pi]$ except possibly for a set whose β-capacity is zero. Also,

$$\left[(1-e^{-i\theta}z)^{\alpha}\log\frac{1}{(1-e^{-i\theta}z)}\right]f(z)$$

has the notangential limit zero at $e^{i\theta}$ for all θ in $[-\pi,\pi]$ except possibly for a set whose logarithmic capacity is zero.

Proof. This is a direct consequence of Lemma 1 and Theorem 1.

3. Radial growth and differentiability

Theorem 3 below gives estimates on the radial growth of a function $f\in\mathscr{F}_{\alpha}$ in the direction θ when the function g representing f is differentiable at θ. The result is expressed in terms of suitable nontangential limits. This yields a new proof of a result in [2] about the radial growth of f off exceptional sets of measure zero. Also it is shown that Theorem 3 is sharp when $1\leqq\alpha\leqq 2$.

Theorem 3. *Suppose that $\alpha\geqq 1$, g is a complex-valued function of bounded variation on $[-\pi,\pi]$ and let*

$$f(z)=\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,dg(t) \tag{9}$$

for $|z|<1$. Assume that g is differentiable at some θ in $[-\pi,\pi]$. If $\alpha>1$ then $(1-e^{-i\theta}z)^{\alpha-1}f(z)$ has the nontangential limit zero at $e^{i\theta}$. If $\alpha=1$ then

$$\frac{f(z)}{\log\dfrac{1}{(1-e^{-i\theta}z)}}$$

has the nontangential limit zero at $e^{i\theta}$.

Proof. Suppose that (9) defines f where g is of bounded variation on $[-\pi,\pi]$ and assume that g is differentiable at θ. Define the function $\tilde{g}$ by $\tilde{g}(t)=g(t)-tg'(\theta)$ for $-\pi\leqq t\leqq\pi$. Since $g'(\theta)$ exists, we have $|\tilde{g}(t)-\tilde{g}(\theta)|=o(|t-\theta|)$ as $t\to\theta$. Also, because

$$\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,dt=2\pi$$

it follows that $f(z)=\tilde{f}(z)+2\pi g'(\theta)$ where

$$\tilde{f}(z)=\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,d\tilde{g}(t)\quad\text{for}\quad |z|<1.$$

Hence $\tilde{f}$ satisfies the assumptions of Theorem 1 and part (a) of that theorem where $\beta=1$ yields the conclusions. □

The following theorem was proved in [2, Theorem 7] using a different argument.

Theorem 4. *If* $\alpha>1$ *and* $f\in\mathcal{F}_{\alpha}$ *then* $(1-e^{-i\theta}z)^{\alpha-1}f(z)$ *has the nontangential limit zero at* $e^{i\theta}$ *for almost all* θ *in* $[-\pi,\pi]$.

Proof. Suppose that $\alpha>1$ and $f\in\mathcal{F}_{\alpha}$. There is a complex-valued function g of bounded variation on $[-\pi,\pi]$ such that (9) holds for $|z|<1$. Since a function of bounded variation is differentiable almost everywhere, there is a set $E\subset[-\pi,\pi]$ having Lebesgue measure 2π such that $g'(\theta)$ exists for $\theta\in E$. If $\theta\in E$ then Theorem 3 implies $(1-e^{-i\theta}z)^{\alpha-1}f(z)$ has the nontangential limit zero at $e^{i\theta}$. □

We add some remarks related to Theorem 3. Suppose that f is defined by (9) where g is of bounded variation and $g'(\theta)$ exists at some θ in $[-\pi,\pi]$. Now assume that $0<\alpha<1$. The differentiability of g at θ implies that there are numbers δ and C such that $0<\delta<\pi$, $C>0$ and

$$\left|\frac{g(\theta+t)-g(\theta)}{t}\right|\leqq C\quad\text{for}\quad 0<|t|\leqq\delta.$$

Since $\alpha<1$ this implies that

$$\int_{-\pi}^{\pi}\frac{|g(\theta+t)-g(\theta)|}{|t|^{\alpha+1}}\,dt<+\infty.$$

Therefore f has a nontangential limit at $e^{i\theta}$ (see [3, Theorems 2 and 4]).

When $\alpha=1$ the assumption that g is differentiable at θ does not imply that f has a radial limit in the direction θ. This fact is implied by (10) in Theorem 5 below. We see this implication, for example, by letting

$$\varepsilon(r)=\frac{1}{\sqrt{\log\left(\frac{1}{1-r}\right)}}\quad(0<r<1).$$

More generally, Theorem 5 asserts that Theorem 3 is sharp when $1\leqq\alpha\leqq2$. When $\alpha=1$ this shows that the argument given for Theorem 4 cannot be used to deduce the result: if $f\in\mathcal{F}_1$ then f has a nontangential limit in almost all directions. This result, of course, is derivable from known facts about functions in H^p spaces. We note that $\mathscr{F}_1 \subset H^p$ for $0<p<1$.

The proof of Theorem 5 uses the following lemma.

Lemma 2. *Let* $0<\beta\leqq 1$ *and let*

$$w=\frac{1}{(1-z)^\beta} \quad \textit{where} \quad z=re^{i\theta}.$$

If $0<r<1$ *and* $|\theta|\leqq 1-r$ *then*

$$Re\, w \geqq \frac{1}{2(1-r)^\beta}.$$

Proof.

$$|1-z|^\beta=\left[(1-r)^2+4r\sin^2\left(\frac{\theta}{2}\right)\right]^{\beta/2}\leqq[(1-r)^2+\theta^2]^{\beta/2}$$

$$\leqq 2^{\beta/2}(1-r)^\beta \leqq \sqrt{2}(1-r)^\beta.$$

We may assume that $\theta>0$. Then $0<\theta<1$ and this implies

$$\sin\theta+\cos\theta\leqq\frac{1}{1-\theta}.$$

Hence

$$\frac{r\sin\theta}{1-r\cos\theta}\leqq\frac{(1-\theta)\sin\theta}{1-(1-\theta)\cos\theta}\leqq 1,$$

and

$$\cos\left[\beta\arg\frac{1}{(1-z)}\right]=\cos\left[\beta\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right]$$

$$\geqq\cos[\beta\tan^{-1}(1)]\geqq\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}.$$

Therefore

$$Re\,w=\frac{1}{|1-z|^{\beta}}\cos\left[\beta\arg\frac{1}{(1-z)}\right]\geqq\frac{1}{\sqrt{2}(1-r)^{\beta}}\frac{\sqrt{2}}{2}$$
$$=\frac{1}{2(1-r)^{\beta}}.$$

Theorem 5. *Let* $1\leqq\alpha\leqq 2$ *and suppose that* ε *is a positive nonincreasing function on* $(0,1)$ *such that* $\lim_{r\to 1-}\varepsilon(r)=0$. *Then there is a function* f *given by*

$$f(z)=\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,dg(t)\quad\text{for}\quad |z|<1,$$

where g *is of bounded variation on* $[-\pi,\pi]$ *and* g *is differentiable at* 0. *Moreover, when* $\alpha=1$

$$\overline{\lim_{r\to 1-}}\frac{|f(r)|}{\varepsilon(r)\log\left(\frac{1}{1-r}\right)}=+\infty \tag{10}$$

and when $1\leqq\alpha\leqq 2$

$$\overline{\lim_{r\to 1-}}\frac{|f(r)|(1-r)^{\alpha-1}}{\varepsilon(r)}=+\infty. \tag{11}$$

Proof. The hypotheses on ε also hold for $\sqrt{\varepsilon}$. Hence it suffices to show such an f exists where (10) and (11) are respectively replaced by

$$\overline{\lim_{r\to 1-}}\frac{|f(r)|}{\varepsilon(r)\log\left(\frac{1}{1-r}\right)}\geqq 1 \tag{12}$$

and

$$\overline{\lim_{r\to 1-}}\frac{|f(r)|(1-r)^{\alpha-1}}{\varepsilon(r)}\geqq 1. \tag{13}$$

Let $\{x_n\}(n=1,2,\ldots)$ be a strictly decreasing sequence of real numbers such that $0<x_n<1$ and $\lim_{n\to\infty}x_n=0$ and let $r_n=1-x_n$. There is a strictly decreasing sequence $\{a_n\}$ of a positive real numbers such that $\lim_{n\to\infty}a_n=0$,

$$a_n\geqq\tfrac{1}{2}\varepsilon(r_n) \tag{14}$$

in the case $\alpha=1$ and

$$a_n \geqq (\alpha-1)\varepsilon(r_n) \tag{15}$$

when $\alpha>1$. A real-valued function h is defined on $[-\pi,\pi]$ as follows. Let $h(0)=0$ and require that h is odd and on $(0,\pi]$ let h be defined as described next. Let $x_n'=\frac{1}{2}(x_n+x_{n+1})$. For $x_1<x\leqq\pi$ let $h(x)=a_1$ and for $x_{n+1}<x<x_n'$ let $h(x)=a_{n+1}$. On each interval $[x_n',x_n]$ let h be linear and let $h(x_n')=a_{n+1}$ and $h(x_n)=a_n$. Then h is continuous on $(0,\pi]$ and on $[-\pi,0)$ and since $\lim_{n\to\infty}a_n=0$, h is continuous at 0.

For $-\pi\leqq t\leqq\pi$ let $k(t)=\int_{-\pi}^{t}h(s)ds$. Then k is of bounded variation on $[-\pi,\pi]$ and $k'(t)=h(t)$ for $-\pi<t<\pi$. Let f be defined by

$$f(z)=\int_{-\pi}^{\pi} K(e^{-it}z)\,dk(t) \tag{16}$$

for $|z|<1$, where

$$K(z)=\frac{z}{(1-z)^{\alpha}}.$$

It is not difficult to show that (16) can be expressed

$$f(z)=\int_{-\pi}^{\pi}\frac{1}{(1-e^{-it}z)^{\alpha}}\,dg(t) \tag{17}$$

for $|z|<1$, where g is a complex-valued function which is of bounded variation on $[-\pi,\pi]$ and is differentiable at 0.

Equation (16) is the same as $f(z)=\int_{-\pi}^{\pi}K(e^{-it}z)h(t)\,dt$. Since h is odd and $K(e^{-it}r)=\overline{K(e^{it}r)}$ for $0<r<1$ it follows that

$$f(r)=-2i\int_{0}^{\pi} Im[K(e^{it}r)]h(t)\,dt. \tag{18}$$

If $0\leqq t\leqq\pi$ and $0<r<1$ then $Im\,K(e^{it}r)\geqq 0$ and $h(t)\geqq 0$. Thus, (18) implies

$$|f(r)|\geqq 2\int_{0}^{\pi} Im[K(e^{it}r)]h(t)\,dt \geqq 2\int_{x_n}^{\pi} Im[K(e^{it}r)]h(t)\,dt$$

$$\geqq 2a_n\int_{x_n}^{\pi} Im\,K(e^{it}r)\,dt.$$

This inequality is the same as

$$|f(r)| \geqq 2a_n \, Im\left\{\int_{x_n}^{\pi} K(e^{it}r)\,dt\right\}. \tag{19}$$

Suppose that $\alpha = 1$. Then

$$Im\left\{\int_{x_n}^{\pi} K(e^{it}r_n)\,dt\right\} = Im\left\{\int_{x_n}^{\pi} i\frac{d}{dt}\log(1-e^{it}r_n)\,dt\right\}$$

$$= \log\frac{1}{|1-e^{ix_n}r_n|} + \log(1+r_n) \geqq \log\frac{1}{|1-e^{ix_n}r_n|}$$

$$= \frac{1}{2}\log\left[\frac{1}{(1-r_n)^2 + 4r_n\sin^2\left(\frac{x_n}{2}\right)}\right]$$

$$\geqq \frac{1}{2}\log\left[\frac{1}{(1-r_n)^2 + x_n^2}\right] = \log\frac{1}{(1-r_n)} - \frac{1}{2}\log 2.$$

Therefore, (19) and (14) imply

$$|f(r_n)| \geqq \varepsilon(r_n)\left\{\log\frac{1}{(1-r_n)} - \frac{1}{2}\log 2\right\}.$$

This proves (12).

Suppose that $1 < \alpha \leqq 2$. Then Lemma 2 implies

$$Im\left\{\int_{x_n}^{\pi} K(e^{it}r_n)\,dt\right\} = Im\left\{\int_{x_n}^{\pi} \frac{i}{-\alpha+1}\frac{d}{dt}\left[(1-e^{-it}r_n)^{-\alpha+1}\right]dt\right\}$$

$$= \frac{1}{\alpha-1}\left\{Re\frac{1}{(1-e^{ix_n}r_n)^{\alpha-1}} - \frac{1}{(1+r_n)^{\alpha-1}}\right\}$$

$$\geqq \frac{1}{\alpha-1}\left\{\frac{1}{2(1-r_n)^{\alpha-1}} - \frac{1}{(1+r_n)^{\alpha-1}}\right\}.$$

Therefore, (19) and (15) imply

$$|f(r_n)| \geqq 2\varepsilon(r_n)\left\{\frac{1}{2(1-r_n)^{\alpha-1}} - \frac{1}{(1+r_n)^{\alpha-1}}\right\}.$$

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This proves (13).

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