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DISCRETE SPECTRUM OF MANY BODY SCHRÖDINGER OPERATORS WITH NON-CONSTANT MAGNETIC FIELDS I

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1. Introduction

In this paper we discuss the discrete spectrum of the Schrödinger operator $H_{N,Z}(b)$, defined as below, for an atomic system in a magnetic field. Let $x = (x^1, \ldots, x^N) \in \mathbf{R}^{3N}$, where x^j is a point in $\mathbf{R}^3 (1 \leq j \leq N)$, and ∇_j be the gradient in $\mathbf{R}^3$ with respect to $x^j (1 \leq j \leq N)$. Then we consider the following operator:

$$H_{N,Z}(b) = \sum_{j=1}^{N} \left(T_j(b)^2 - \frac{Z}{|x^j|} \right) + \sum_{1 \leq i < j \leq N} \frac{1}{|x^i - x^j|} \tag{1.1}$$

defined on $C_0^\infty(\mathbf{R}^{3N})$, where $Z > 0$, $N \in \mathbf{N}$, $b \in C^1(\mathbf{R}^3)^3$ being real-valued and

$$T_j \equiv T_j(b) = -i\nabla_j - b(x^j) \quad (1 \leq j \leq N). \tag{1.2}$$

For a vector potential $b \in C^1(\mathbf{R}^3)^3$, the vector field $\vec{B}(y) \equiv \nabla \times b(y)$ $(y \in \mathbf{R}^3)$ is called the *magnetic field*. By [11] (p 190) or [12] (Chap. 9), the operator $H_{N,Z}(b)$ is essentially self-adjoint in $L^2(\mathbf{R}^{3N})$, so we denote its self-adjoint extension by the same notation $H_{N,Z}(b)$, which we study in this paper. This operator $H_{N,Z}(b)$ is the atomic Hamiltonian with a nucleus, that is assumed to be infinitely heavy, of charge Z and N electrons of charge -1 and mass $1/2$, and with the magnetic vector potential b. The eigenvalues and the eigenfunctions of Schrödinger operators are often called *energy levels* and *bound states*, respectively.

The problem is the finiteness or the infiniteness of the discrete spectrum of $H_{N,Z}(b)$, which is one of the characteristic spectral properties. This problem in the case that $b = 0$ was studied by Zhislin [17], [18], Jafaev [10], Uchiyama [15] and others. Zhislin treated the case $Z \neq N - 1$ in [17] and [18], and thereafter Jafaev [10] treated the delicate case $Z = N - 1$. The following theorem, which is obtained by combining [17], [18] with [10], gives the necessary and sufficient

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condition for the finiteness of the discrete spectrum of $H_{N,Z}(0)$.

THEOREM 0.1 ([10], [17], [18]). *The number of the discrete spectrum of* $H_{N,Z}(0)$ *is finite if and only if* $Z \leq N - 1$.

On the other hand, in the case of constant magnetic fields, Avron-Herbst-Simon [4] gave a necessary condition for the finiteness of the discrete spectrum of the atomic Hamiltonians, and Vugal'ter-Zhislin [16] proved that it is also sufficient. In fact, Avron-Herbst-Simon [4] only proved that once negatively charged ion has infinitely many bound states. It seems to be natural that any neutral atom and any positively charged ion also have infinitely many bound states. This is not trivial but easily seen.

THEOREM 0.2 ([4],[16]). *The number of the discrete spectrum of* $H_{N,Z}(b_c)$ *is finite if and only if* $Z < N - 1$.

Here $b_c(y) = (0,0, B/2) \times y (y \in \mathbf{R}^3$, B is a positive constant). This gives the constant magnetic field $\nabla \times b_c = (0,0, B)$, which we have only to consider by the change of coordinates. We remark that, comparing Theorem 0.2 with Theorem 0.1, the difference between the presence and the absence of constant magnetic fields appears only in the delicate case $Z = N - 1$.

Then our concern is the case of non-constant magnetic fields. There are not many works about this problem both for atomic Hamiltonians and for many-body Schrödinger operators with short-range scalar potentials (for example Zhislin [19]). Some different phenomena are expected to occur in non-constant magnetic fields. This is true. In fact, we have the following theorems, which are our main results of this paper.

THEOREM 1.1. *For any positive number* ε, *there exists a vector potential* $b_\varepsilon \in C^1(\mathbf{R}^3)^3$, *which gives a perturbed constant magnetic field and which is independent of* N *and* Z, *such that the number of the discrete spectrum of* $H_{N,Z}(b_\varepsilon)$ *is finite for* $N \geq 2$ *and* $Z \geq \varepsilon$.

In other words, any atomic system has only finitely many bound states, corresponding to the discrete spectrum, in a suitable magnetic field. Also the finiteness or the infiniteness of the number of bound states generically depends on magnetic fields.

As stated as above we construct the vector potential b_ε in Theorem 1.1 as a perturbation of constant magnetic fields. Not adhering to it, we can extend the result to the case $N=1$.

THEOREM 1.2. *For any positive number* ε, *there exists a vector potential* $\tilde{b}_\varepsilon \in C^1(\mathbf{R}^3)^3$, *which is independent of* N *and* Z, *such that the number of the discrete spectrum of* $H_{N,Z}(\tilde{b}_\varepsilon)$ *is finite for* $N \geq 1$ *and* $Z \geq \varepsilon$.

For studying the above problem, geometric methods, that make explicit use of the geometry of the phase space, have been used effectively. Agmon [2] developed geometric methods for studying the exponential decay of eigenfunctions of Schrödinger operators with non-isotropic potentials. In the lecture note [2] (see also [1]) he characterized the infimum of the essential spectrum and constructed Agmon's K-function which is useful to show the HVZ theorem. In §2 by using this function we prove the HVZ theorem for many-body Schrödinger operators with perturbed constant magnetic fields. In relation to Agmon's works, Evans-Lewis-Saitō [7] gave a sufficient condition, which are represented by Agmon's function, for the finiteness and the infiniteness of the discrete spectrum of those operators. In addition, they also reprove Theorem 0.1 except the case $Z=N-1$ by using this result ([8]). In §3 we extend Evans-Lewis-Saitō's result to the general magnetic case. We do it in the same but slightly simplified way as in [7]. In §4 we introduce some magnetic vector potentials, which are used in the proof of Theorem 1.1, and study the essential spectrum of the atomic Hamiltonians with these vector potentials. At the end, in §5 we prove our main results Theorems 1.1 and 1.2.

2. Preliminaries

In this section we prepare the IMS-localization formula, the HVZ theorem and related facts, which play a basic role in the proof of Theorems 1.1 and 1.2.

We consider the following operator:

$$H \equiv H_N = \sum_{j=1}^{N} (T_j(b)^2 + V_{0j}(x^j)) + \sum_{1 \leq i<j \leq N} V_{ij}(x^j - x^i) \tag{2.1}$$

in $L^2(\mathbf{R}^{3N})$, where we assume

$$\begin{cases} V_{ij} \in L^2_{\mathrm{loc}}(\mathbf{R}^3),\ V_{ij}(y) \to 0 \text{ as } |y| \to \infty\ (0 \leq i < j \leq N) \\ V_{0j}(y) \leq 0 (1 \leq j \leq N),\ V_{ij}(y) \geq 0 (1 \leq i < j \leq N). \end{cases} \tag{2.2}$$

The operator $H_{N,Z}(b)$ in §1 is defined by (2.1) with

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(2.3) $$V_{0j}(y) = -\frac{Z}{|y|}\ (1 \leq j \leq N),\ V_{ij}(y) = \frac{1}{|y|}\ (1 \leq i < j \leq N).$$

We denote the spectrum, the discrete spectrum, the essential spectrum of H by $\sigma(H)$, $\sigma_d(H)$, $\sigma_e(H)$, respectively, and the cardinal number of a set Y by $\# Y$.

At first we define the quadratic form

$$q_H[\phi, \psi] = \sum_{j=1}^{N} \int_{\mathbf{R}^{3N}} T_j\phi \cdot \overline{T_j\psi}\, dx + \int_{\mathbf{R}^{3N}} V\phi\bar{\psi}\, dx,$$
$$q_H[\phi] = q_H[\phi, \phi]$$

for ϕ, $\psi \in C_0^\infty(\mathbf{R}^{3N})$, where

(2.4) $$V(x) = \sum_{j=1}^{N} V_{0j}(x^j) + \sum_{1 \leq i < j \leq N} V_{ij}(x^j - x^i).$$

Let

(2.5) $$\Lambda(H) = \inf\{q_H[\phi]\,;\, \phi \in C_0^\infty(\mathbf{R}^{3N}),\ \|\phi\|_{L^2} = 1\},$$

(2.6) $$\Sigma(H) = \sup_{E:\text{compact}} \inf\{q_H[\phi]\,;\, \phi \in C_0^\infty(\mathbf{R}^{3N} \setminus E),\ \|\phi\|_{L^2} = 1\}.$$

We remark that under the assumption (2.2) the scalar potential $V(x)$ (especially the negative part $V_-(x) = \max\{-V(x), 0\}$) is $\sum_{j=1}^{N} T_j^2$-form bounded with the bound zero. In fact, $V(x)$ is in the Kato class (see §3) and the functions in the Kato class have the above property (Lemma 3.1 in §3, [5] (Chap. 1) and [12] (Chap. 9)). Also we remark that each $V_{0j}(x^j)$ is T_j^2-form bounded with the bound zero. So $\Lambda(H) > -\infty$ follows. Then we can show the following lemma in the same way as in [1] (Theorem 1.6) or [2] (Chap. 3).

LEMMA 2.1.

$$\Lambda(H) = \inf \sigma(H) \text{ and } \Sigma(H) = \inf \sigma_e(H).$$

The following formula holds as in the case without magnetic fields.

LEMMA 2.2 (IMS-localization formula). *For a smooth partition of unity* $\{J_\beta\}_\beta$ *such that* $\sum_\beta J_\beta^2(x) = 1$, *the following equality holds*:

$$H = \sum_\beta (J_\beta H J_\beta - |\nabla J_\beta|^2) \text{ in the form sense,}$$

that is,

$$q_H[\phi] = \sum_\beta (q_H[J_\beta\phi] - (|\nabla J_\beta|^2\phi, \phi)_{L^2}) \text{ for } \phi \in C_0^\infty(\mathbf{R}^{3N}).$$

Furthermore, let Ω *be an open subset in* $\mathbf{R}^{3N}$. *Then*

$$\int_\Omega \{\sum_{j=1}^{N} |T_j\phi|^2 + V|\phi|^2\}dx = \sum_\beta \int_\Omega \{\sum_{j=1}^{N} |T_jJ_\beta\phi|^2 + V|J_\beta\phi|^2\}dx - \sum_\beta \int_\Omega |\nabla J_\beta|^2 |\phi|^2 dx$$

for $\phi \in C_0^\infty(\mathbf{R}^{3N})$.

One can show the above lemma in the same way as in the case that $b=0$ ([5] (p.28), [7], [13]) because of the fact that the commutator

$$[T_j, \phi] = -i(\nabla_j\phi), \text{ for } \phi \in C^1(\mathbf{R}^3),$$

is independent of b.

Next we define Agmon's K-function to derive the HVZ theorem for some cases. We note that the HVZ theorem for the case of constant magnetic fields is shown in [14].

Definition (Agmon's K-function). Let

$$S^{3N-1} = \{\omega = (\omega^1, \ldots, \omega^N) \in \mathbf{R}^{3N}; |\omega| = 1\}.$$

For a subset $U \subset S^{3N-1} (U \neq \emptyset)$ and for positive numbers R and δ, we put

$$(2.7)\qquad \begin{cases} U_\delta = \{\omega \in S^{3N-1}; \mathrm{dist}(\omega, U) < \delta\}, \\ \Gamma(U_\delta, R) = \{x \in \mathbf{R}^{3N}; x/|x| \in U_\delta, |x| > R\}, \\ K(U_\delta, R; H) = \inf\{q_H[\phi]; \phi \in C_0^\infty(\Gamma(U_\delta, R)), \|\phi\|_{L^2} = 1\}, \\ K(U; H) = \lim_{\delta \downarrow 0} \lim_{R \uparrow \infty} K(U_\delta, R; H), \\ M \equiv M(N) = \{\omega \in S^{3N-1}; K(\omega; H) = \inf_{\omega \in S^{3N-1}} K(\omega; H)\}, \end{cases}$$

where $K(\omega; H) = K(\{\omega\}; H)$.

Here the set function $K(\cdot; H)$ and the set M are called *Agmon's* K-*function* and the *minimizing set for* H, respectively. The following properties of K can be shown in the same way as in the case that $b=0$ ([1] (§5,6), [2] (Chap. 2) and [7]).

Lemma 2.3. *The function* K *has the following properties.*

(i) *The value of* $K(U; H)$ *is the same regardless of the order of the limits.*

(ii) *The function* $K(\omega; H)$ *is lower semi-continuous on* S^{3N-1}.

(iii) $\Sigma(H) = \min_{\omega \in S^{3N-1}} K(\omega; H)$.

(iv) $K(U; H) = K(\bar{U}; H) = \inf_{\omega \in \bar{U}} K(\omega; H)$ *for* $U \subset S^{3N-1}$.

As is clear from the proof in [1], [2], [7], Lemmas 2.1, 2.2, 2.3 hold also for the operator

$$H = \sum_{j=1}^{N} \tilde{T}_j^2 + V, \tag{2.8}$$

where $\tilde{T}_j = -i\nabla_j - b_j(x^j)$, $b_j \in C^1(\mathbf{R}^3)^3$ which is real-valued $(1 \leq j \leq N)$.

For the statement of the HVZ theorem we usually use the subsystem of H. For $\omega = (\omega^1, \ldots, \omega^N) \in S^{3N-1}$, let

$$H_\omega = \sum_{j=1}^{N} T_j(b)^2 + V_\omega \text{ in } L^2(\mathbf{R}^{3N}) \text{ and} \tag{2.9}$$
$$V_\omega(x) = \sum_{\omega^j=0} V_{0j}(x^j) + \sum_{\omega^i=\omega^j, 1\leq i<j\leq N} V_{ij}(x^j - x^i).$$

This operator H_ω is called the *subsystem of* H with respect to $\omega \in S^{3N-1}$. Letting $x^0 = \omega^0 = 0$, we can write

$$V_\omega(x) = \sum_{\omega^i=\omega^j, 0\leq i<j\leq N} V_{ij}(x^j - x^i).$$

Now we shall show the HVZ theorem only in the case that

$$\begin{cases} b(y) = b_M(y) + b_p(y), \\ b_M(y) = f(\vec{\rho})(-y_2, y_1, 0), \\ b_p(y), \operatorname{div} b_p(y) \to 0 \text{ as } |y| \to \infty, \end{cases} \tag{2.10}$$

where $y = (y_1, y_2, y_3) \in \mathbf{R}^3$, $\vec{\rho} = (y_1, y_2)$ and $f(\vec{\rho}) \in C^1(\mathbf{R}^2)$. We notice that (2.10) includes both the case without magnetic fields and the case of constant magnetic fields. Also we remark that

$$\Sigma(T_j(b_M)^2) = \Lambda(T_j(b_M)^2) \ (1 \leq j \leq N). \tag{2.11}$$

In this case we have to modify the subsystem of H as follows.

DEFINITION. For $\omega = (\omega^1, \ldots, \omega^N) \in S^{3N-1}$, let

$$H^\omega = \sum_{\omega^j=0} T_j(b)^2 + \sum_{\omega^j\neq 0} T_j(b_M)^2 + V_\omega(x) \text{ in } L^2(\mathbf{R}^{3N}). \tag{2.12}$$

Note that in the case that $b_p = 0$ the *modified subsystem* H^ω is equal to the usual subsystem H_ω. Then the HVZ theorem for the case of (2.10) is stated as follows.

THEOREM 2.4 (HVZ theorem). *For the case of* (2.10) *, we have*

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$$\Sigma(H) = \min_{\omega \in S^{3N-1}} \Lambda(H^{\omega}). \tag{2.13}$$

Before going into the proof of Theorem 2.4, we prepare several lemmas.

LEMMA 2.5. *Let H_ω and H^ω be as in* (2.9) *and* (2.12), *respectively. Then*

$$K(\omega\,; H) = K(\omega\,; H_\omega) = K(\omega\,; H^\omega) \text{ for } \omega \in S^{3N-1}.$$

Proof of Lemma 2.5. Let $\omega = (\omega^1, \ldots, \omega^N) \in S^{3N-1}$ and $x^0 = \omega^0 = 0$. Then

$$H - H_\omega = \sum_{\omega^i \neq \omega^j, 0 \leq i < j \leq N} V_{ij}(x^j - x^i).$$

Let ω_δ be the neighbourhood of $\{\omega\}$. If $x \in \Gamma(\omega_\delta, R)$, then $| x^j/|x| - \omega^j | < \delta$ for $1 \leq j \leq N$, which implies

$$| x^j - x^i | > (| \omega^j - \omega^i | - 2\delta) \, | x | \text{ if } \omega^i \neq \omega^j.$$

Let

$$\alpha_1 = \frac{1}{4} \min_{\omega^i \neq \omega^j, 0 \leq i < j \leq N} | \omega^j - \omega^i |.$$

Then for $\delta \leq \alpha_1$, it follows that $| x^j - x^i | > 2\alpha_1 R$ if $\omega^i \neq \omega^j$. This implies that

$$V_{ij}(x^j - x^i) \to 0 \text{ as } R \to \infty \text{ if } \omega^i \neq \omega^j.$$

Hence for any $\varepsilon_0 > 0$ there exists $R_0 > 0$ such that

$$q_{H_\omega}[\phi] - \varepsilon_0 \| \phi \|_{L^2}^2 \leq q_H[\phi] \leq q_{H_\omega}[\phi] + \varepsilon_0 \| \phi \|_{L^2}^2$$

for $\delta \leq \alpha_1$, $R \geq R_0$ and $\phi \in C_0^\infty(\Gamma(\omega_\delta, R))$, which implies

$$K(\omega\,; H) = K(\omega\,; H_\omega).$$

Next we show the second equality. By a simple calculation,

$$H^\omega - H_\omega = \sum_{\omega^j \neq 0} (T_j(b_M)^2 - T_j(b)^2) = \sum_{\omega^j \neq 0} (2b_p \cdot T_j(b_M) - i \operatorname{div} b_p(x^j) - | b_p(x^j) |^2).$$

Letting

$$\beta(x) = - \sum_{\omega^j \neq 0} (i \operatorname{div} b_p(x^j) + | b_p(x^j) |^2),$$

we have

$$| q_{H^\omega}[\phi] - q_{H_\omega}[\phi] | \leq \sum_{\omega^j \neq 0} (\varepsilon_1 \| T_j(b_M)\phi \|_{L^2}^2 + \varepsilon_1^{-1} \| b_p(x^j)\phi \|_{L^2}^2) + (\beta\phi, \phi)_{L^2}$$

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$$\leq \varepsilon_1\{q_{H^\omega}[\phi] - \sum_{\omega^j=0} (\| T_j(b)\phi \|_{L^2}^2 + (V_{0j}(x^j)\phi, \phi)_{L^2})\} + ((\varepsilon_1^{-1} \sum_{\omega^j\neq 0} | b_p(x^j) |^2 + \beta)\phi, \phi)_{L^2}$$

for any $\varepsilon_1 > 0$ and $\phi \in C_0^\infty(\mathbf{R}^{3N})$. Here we have used the positivity of V_{ij}. By using the fact that V_{0j} is T_j^2-form bounded with the bound zero (stated as before in this section, see also §3), we have

$$(- V_{0j}(x^j)\phi, \phi)_{L^2} \leq \| T_j(b)\phi \|_{L^2}^2 + C_j \| \phi \|_{L^2}^2 \ (1 \leq j \leq N)$$

for some positive constant C_j. So,

$$| q_{H^\omega}[\phi] - q_{H_\omega}[\phi] | \leq \varepsilon_1 q_{H^\omega}[\phi] + \varepsilon_1 C_0 \| \phi \|_{L^2}^2 + (\beta_{\varepsilon_1}(x)\phi, \phi)_{L^2},$$

where

$$\beta_{\varepsilon_1}(x) = \varepsilon_1^{-1} \sum_{\omega^j\neq 0} | b_p(x^j) |^2 + \beta(x) \text{ and } C_0 = \sum_{j=1}^N C_j.$$

Now let $x \in \Gamma(\omega_\delta, R)$ and $\delta \leq \min_{\omega^j\neq 0} | \omega_j | /4 \equiv \alpha_2$. Then $| x^j | \geq 2\alpha_2 R$ if $\omega^j \neq 0$, which implies $\beta(x) \to 0$, $\beta_{\varepsilon_1}(x) \to 0$ as $R \to \infty$ by (2.10). Hence we obtain

$$(1 - \varepsilon_1)K(\omega\,; H^\omega) - \varepsilon_1 C_0 \leq K(\omega\,; H_\omega) \leq (1 + \varepsilon_1)K(\omega\,; H^\omega) + \varepsilon_1 C_0,$$

which implies

$$K(\omega\,; H^\omega) = K(\omega\,; H_\omega). \qquad \square$$

LEMMA 2.6. *For* $\omega = (\omega^1, \ldots, \omega^N) \in S^{3N-1}$ *such that* $\omega^j = (0,0, z^j)$ $(1 \leq j \leq N)$, *the following equalities hold.*

$$K(\omega\,; H) = K(\omega\,; H^\omega) = \Sigma(H^\omega) = \Lambda(H^\omega).$$

Proof of Lemma 2.6. For the above $\omega \in S^{3N-1}$ it is easy to see by (2.10) that

$$\begin{cases} V_\omega(x + t\omega) = V_\omega(x), \\ b(x^j + t\omega^j) = b(x^j) \text{ if } \omega^j = 0, \\ b_M(x^j + t\omega^j) = b_M(x^j) \text{ if } \omega^j \neq 0 \ (t \in \mathbf{R},\ x = (x^1, \ldots, x^N) \in \mathbf{R}^{3N}). \end{cases}$$

Hence, for any $\phi \in C_0^\infty(\mathbf{R}^{3N})$, letting $\phi_t(x) = \phi(x + t\omega)$ $(t \in \mathbf{R})$, we have

$$q_{H^\omega}[\phi] = q_{H^\omega}[\phi_t] \ \ (t \in \mathbf{R}),$$

which implies

$$K(\omega\,; H^\omega) = \Sigma(H^\omega) = \Lambda(H^\omega)$$

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by the same method as in [1] (§6) or [2] (Chap. 4). Here we use Lemmas 2.1 and 2.3. □

LEMMA 2.7. *For any $\omega \in S^{3N-1}$ there exists $\omega_r = (\omega_r^1, \ldots, \omega_r^N) \in S^{3N-1}$ such that*

$$\omega_r^j = (0,0, z_r^j) \ (1 \leq j \leq N) \text{ and } H^\omega = H^{\omega_r}.$$

Proof of Lemma 2.7. Letting $\omega = (\omega^1, \ldots, \omega^N)$, we define

$$z^i = 0 \text{ if } \omega^i = 0.$$

Picking up $i_1 \in \{1, \ldots, N\} \setminus \{i ; \omega^i = 0\} (\neq \emptyset)$, we define

$$z^i = 1 \text{ if } \omega^i = \omega^{i_1}.$$

If we can pick up $i_2 \in \{1, \ldots, N\} \setminus \{i ; \omega^i = 0 \text{ or } \omega^i = \omega^{i_1}\}$, we define

$$z^i = 2 \text{ if } \omega^i = \omega^{i_2}.$$

If it is not the case, this operation ends. Continue this operation till the end, and let

$$\tilde{\omega} = (\tilde{\omega}^1, \ldots, \tilde{\omega}^N) \in \mathbf{R}^{3N}, \ \tilde{\omega}^j = (0, 0, z^j) \ (1 \leq j \leq N)$$

and

$$\omega_r = \tilde{\omega} / |\tilde{\omega}| \in S^{3N-1}.$$

This ω_r satisfies $H^\omega = H^{\omega_r}$. □

LEMMA 2.8. *Recall* (2.7). *If $\omega \in M$, then*

$$\Sigma(H) = K(\omega ; H) = K(\omega ; H^\omega) = \Sigma(H^\omega) = \Lambda(H^\omega).$$

Proof of Lemma 2.8 By Lemmas 2.1, 2.3 and 2.5 it is easy to see that

$$K(\omega ; H) = K(\omega ; H^\omega) \geq \min_{\omega' \in S^{3N-1}} K(\omega'; H^\omega) = \Sigma(H^\omega) \geq \Lambda(H^\omega). \tag{2.14}$$

For $\omega \in M$, picking up ω_r in Lemma 2.7, we have

$$\begin{aligned} K(\omega ; H) &= \Sigma(H) = \min_{\omega \in S^{3N-1}} K(\omega ; H) \\ &\leq K(\omega_r ; H) = K(\omega_r ; H^{\omega_r}) = \Sigma(H^{\omega_r}) = \Lambda(H^{\omega_r}) = \Lambda(H^\omega). \end{aligned}$$

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Here we have used Lemmas 2.3 and 2.6. Summing up, we have desired equalities. □

Combining these lemmas, we prove the HVZ theorem.

Proof of Theorem 2.4. By (2.14) and Lemma 2.3,

$$\Sigma(H) \geq \min_{\omega \in S^{3N-1}} \Lambda(H^{\omega}).$$

Now we define the set $\tilde{M}$ by

$$\tilde{M} \equiv \tilde{M}(N) = \{\omega \in S^{3N-1}; \Lambda(H^{\omega}) = \min_{\omega \in S^{3N-1}} \Lambda(H^{\omega})\}. \tag{2.15}$$

For $\omega \in \tilde{M}$, pick up $\omega_r \in \tilde{M}$ in Lemma 2.7. Then

$$\min_{\omega \in S^{3N-1}} \Lambda(H^{\omega}) = \Lambda(H^{\omega_r}) = K(\omega_r; H) \geq \Sigma(H)$$

by Lemmas 2.3 and 2.6. Summing up we obtain

$$\Sigma(H) = \min_{\omega \in S^{3N-1}} \Lambda(H^{\omega}).$$ □

Next we study the minimizing set M. We remark that M is a closed set in S^{3N-1} because of the lower semi-continuity of $K(\omega; H)$. The following lemma asserts the relation between M and $\tilde{M}$.

LEMMA 2.9. $M \subset \tilde{M}$.

Proof of Lemma 2.9. For $\omega \in M$, it follows from (2.14) that

$$\Sigma(H) = K(\omega; H) \geq \Lambda(H^{\omega}).$$

Hence by using the HVZ theorem we have $\omega \in \tilde{M}$. □

From now on we consider H_N defined by (2.1), where V_{0j} and V_{ij} are assumed that

$$V_{0j}(y) = V_0(y) \ (1 \leq j \leq N), \ V_{ij}(y) = V_1(y) \ (1 \leq i < j \leq N),$$

and to satisfy the condition (2.2). We note that (2.3) satisfies this assumption. In this case we have the following proposition, roughly characterizing M (and $\tilde{M}$).

PROPOSITION 2.10. *Let* $N \geq k+1 \geq 2$ *and* $\sigma_d(H_{N-k}) \neq \emptyset$. *Then*

$$M(N) \subset \tilde{M}(N) \subset \bigcup_{i_1,\ldots,i_k} M_{i_1,\ldots,i_k}, \tag{2.16}$$

where

$$M_{i_1,\ldots,i_k} = \{\omega = (\omega^1,\ldots,\omega^N) \in S^{3N-1}\,; \omega^j = 0 \ if j \notin \{i_1,\ldots,i_k\}\}$$

for $\{i_1,\ldots,i_k\} \subset \{1,\ldots,N\}$.

Proof of Proposition 2.10. Let $\omega = (\omega^1,\ldots,\omega^N) \in S^{3N-1} \setminus \bigcup_{i_1,\ldots,i_k} M_{i_1,\ldots,i_k}$. Then $\#\{i\,; \omega^i \neq 0\} \geq k+1$. Therefore we can assume without loss of generality that

$$\omega^p \neq 0 (p = 1,\ldots,l),\ \omega^p = 0 (p = l+1,\ldots,N),\ l \geq k+1.$$

Then

$$\begin{aligned} H^\omega &= \sum_{p=1}^{l} T_p(b_M)^2 + \sum_{p=l+1}^{N} T_p(b)^2 + \sum_{p=l+1}^{N} V_0(x^p) + \sum_{\omega^i=\omega^j} V_1(x^j - x^i) \\ &\geq lB + 1 \otimes H_{N-l} \end{aligned}$$

in the form sense, where $B = \Lambda(T_j(b_M)^2)$ which is independent of j, and we have dropped $V_1(x^j - x^i)$ for $1 \leq i \leq p$ or $1 \leq j \leq p$. From the above inequality it follows that

$$\begin{aligned} \Lambda(H^\omega) &\geq lB + \Lambda(H_{N-l}) \\ &= (l-1)B + \Lambda(T_1(b_M)^2 \otimes 1 + 1 \otimes H_{N-l}). \end{aligned}$$

Since $T_1(b_c)^2 \otimes 1 + 1 \otimes H_{N-l}$ acting on $\mathbf{R}^{3(N-l+1)}$ is one of the modified subsystem of H_{N-l+1}, we have by HVZ theorem

$$\begin{aligned} lB + \Lambda(H_{N-l}) &\geq (l-1)B + \Sigma(H_{N-l+1}) \\ &\geq (l-1)B + \Lambda(H_{N-l+1}) \\ &\geq \ldots \geq kB + \Sigma(H_{N-k}). \end{aligned}$$

Now, if $\sigma_d(H_{N-k}) \neq \emptyset$, that is $\Lambda(H_{N-k}) < \Sigma(H_{N-k})$, then

$$\begin{aligned} \Lambda(H^\omega) &> kB + \Lambda(H_{N-k}) \geq \ldots \\ &\geq B + \Lambda(H_{N-1}) = \Lambda(T_j(b_M)^2 \otimes 1 + 1 \otimes H_{N-1}), \end{aligned}$$

where j is a suitable number. Since $T_j(b_M)^2 \otimes 1 + 1 \otimes H_{N-1}$ is one of the modified subsystem of H_N with respect to $\omega \in \bigcup_{i=1}^{N} M_i$, it follows that

$$\Lambda(H^{\omega}) > \Sigma(H),$$

which implies, by the HVZ theorem, $\omega \notin \tilde{M}$. Thus we obtain (2.16). □

Remark 2.11. From the proof of Proposition 2.10, it follows that

$$NB \geq (N-1)B + \Lambda(H_1) \geq \ldots \geq B + \Lambda(H_{N-1}). \tag{2.17}$$

for $\omega \in S^{3N-1}$, there exists a number $l \in \{1, \ldots, N\}$ such that

$$\Lambda(H_N^{\omega}) \geq lB + \Lambda(H_{N-l}).$$

Here let $H_0 (H_n \text{ with } n = 0) = 0$. Hence, by (2.17) and HVZ theorem,

$$\Sigma(H_N) = \min_{\omega \in S^{3N-1}} \Lambda(H_N^{\omega}) \geq B + \Lambda(H_{N-1}).$$

Since $B + \Lambda(H_{N-1}) = \Lambda(H_N^{\omega'})$ for $\omega' \in \cup_{i=1}^{N} M_i$, we have

$$\bigcup_{i=1}^{N} M_i \subset \tilde{M}, \tag{2.18}$$

in other words,

$$\Sigma(H_N) = B + \Lambda(H_{N-1}) (\leq kB + \Lambda(H_{N-k}), 1 \leq k \leq N). \tag{2.19}$$

3. Finiteness of discrete spectrum

In [7] Evans-Lewis-Saitō give a sufficient condition for the finiteness of the discrete spectrum of Schrödinger operators with non-isotropic scalar potentials and without magnetic fields. In this section we extend their result to the case that $b \neq 0$, that is useful to derive the finiteness of the discrete spectrum in the proof of Theorems 1.1 and 1.2.

To state the theorem we make some preparations. Let

$$\mathscr{K}(\mathbf{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \, ; \lim_{r \downarrow 0} \sup_{x^0 \in \mathbf{R}^n} \int_{|x - x^0| \leq r} |f(x)| \, |x - x^0|^{2-n} dx = 0\} \ (n \geq 3),$$

which is called the *Kato class* (see [2](Chap. 0), [5] (Chap. 1), [7], [8]). We remark that $V(x)$ in (2.4) and $V_{ij}(y)$ in (2.2) belong to $\mathscr{K}(\mathbf{R}^{3N})$ and $\mathscr{K}(\mathbf{R}^3)$, respectively. Only in this section we consider the operator:

$$H = \sum_{j=1}^{N} T_j(b)^2 + V \text{ in } L^2(\mathbf{R}^{3N}), \tag{3.1}$$

including (1.1) and (2.1), where $T_j(b)$ is defined by (1.2) for $b \in C^1(\mathbf{R}^3)^3$ and

$$\begin{cases} V \in L^1_{\text{loc}}(\mathbf{R}^{3N}) \text{ and} \\ V_-(x) \equiv \max\{-V(x), 0\} \in \mathscr{K}(\mathbf{R}^{3N}). \end{cases} \tag{3.2}$$

DEFINITION. We recall (2.7). For given $\delta > 0$ and $R > 0$, let

$$\Delta \equiv \Delta(\delta, R) = \overline{\Gamma(M_\delta, R)} \setminus \Gamma(M_{\delta/2}, 2R)$$

and χ_Δ be a characteristic function of Δ, where M is the minimizing set of H defined by (2.7) and M_δ is U_δ in (2.7) with $U = M$. Then we define the operator, for $\alpha > 0$,

$$H_\alpha = H - \alpha |x|^{-2}\chi_\Delta \text{ in } L^2(\mathbf{R}^{3N}). \tag{3.3}$$

In addition, we define the quadratic form

$$q_H[\phi, \psi] = \sum_{j=1}^N \int_{\mathbf{R}^{3N}} T_j\phi \cdot \overline{T_j\psi} dx + \int_{\mathbf{R}^{3N}} V\phi\bar{\psi} dx,$$
$$q_H[\phi] = q_H[\phi, \phi],$$

for $\phi, \psi \in C_0^\infty(\mathbf{R}^{3N})$.

We remark that the self-adjointness of H and H_α is guaranteed in [11] (Chap. X) or [12] (Chap. 8 and 9), by using the following property of the Kato class.

LEMMA 3.1 ([2] (Chap. 0), [12] (Chap. 9)). *If $f \in \mathscr{K}(\mathbf{R}^{3N})$, then f is $\sum_{j=1}^N T_j^2$-form bounded with the bound zero. Namely, for any $\varepsilon > 0$ there exists a positive constant C_ε such that*

$$(|f|\phi, \phi)_{L^2} \leq \varepsilon \sum_{j=1}^N \| T_j\phi \|_{L^2}^2 + C_\varepsilon \| \phi \|_{L^2}^2 \text{ for } \phi \in C_0^\infty(\mathbf{R}^{3N}).$$

For a quadratic form q on $C_0^\infty(\mathbf{R}^{3N})$, we denote its closure in $L^2(\mathbf{R}^{3N})$ by $\tilde{q}$, and for an essentially self-adjoint operator A on $C_0^\infty(\mathbf{R}^{3N})$ we denote its self-adjoint extension in $L^2(\mathbf{R}^{3N})$ by the same notation A. We remark that H is associated with $\tilde{q}_H$.

Now our aim in this section is to prove the following theorem, which is in the same form of Evans-Lewis-Saitō's result in [7].

THEOREM 3.2 (*The case that $b \neq 0$*). *Recall* (2.6) *and* (2.7). *Suppose that there exist $\delta_0 > 0$, $R_0 > 0$ and $\alpha > 0$ such that $M_{\delta_0} \neq S^{3N-1}$ and*

$$K(M_{\delta_0}, R_0; H_\alpha) = \Sigma(H). \tag{3.4}$$

Then

$$\#\{\sigma_d(H) \cap (-\infty, \Sigma(H))\} < +\infty.$$

Remark 3.3. Also for (3.1) Lemmas 2.1, 2.2 and 2.3 hold.

Remark 3.4. For any self-adjoint operator H' in $L^2(\mathbf{R}^{3N})$ such that $H_\alpha \geq H'$, if $K(M_{\delta_0}, R_0; H') = \Sigma(H)$, then $K(M_{\delta_0}, R_0; H_\alpha) = \Sigma(H)$. Hence, we can replace H_α in (3.4) by H' in Theorem 3.2, for example,

$$H' = H - \alpha |x|^{-2} \chi_{B(1)^c}.$$

Here $B(r) = \{x \in \mathbf{R}^{3N}; |x| < r\}$, $B(r)^c = \mathbf{R}^{3N} \setminus B(r)$ $(r > 0)$ and χ_D denotes the characteristic function of a set D.

Now we go into the proof of Theorem 3.2, which is in the same but slightly simplified way as in [7]. Also the structure of the proof is due to [7]. At first, suppose that $M_{\delta_0} \neq S^{3N-1}$ (δ_0 is in Theorem 3.2). We prepare two partitions of unity in order to define the weight function $w(x)$ and the related operator. The following lemma is shown in [7].

LEMMA 3.5 ([7]). *For* M, δ_0, R_0 *in Theorem* 3.2, *there exist two partitions of unity* $\{J_0, J_1, J_2\}$ *and* $\{I_1, I_2\}$ *satisfying*

$$\begin{cases} J_i \in C^\infty(\mathbf{R}^{3N}),\ 0 \leq J_i \leq 1 \text{ on } \mathbf{R}^{3N}\ (i = 0,1,2), \\ \sum_{i=0}^{2} J_i^2(x) \equiv 1\ \ (x \in \mathbf{R}^{3N}), \\ \operatorname{supp} J_0 \subset B(1),\ \operatorname{supp} J_1 \subset \Gamma(M_{\delta_0}, 1/2) \\ \operatorname{supp} J_2 \subset (\Gamma(M_{\delta_0/2}, 0) \cup B(1/2))^c, \\ J_1 \text{ and } J_2 \text{ are homogeneous of degree zero in } B(1)^c, \end{cases}$$

and

$$\begin{cases} I_1 \text{ and } I_2 \text{ are functions of } |x|, \\ I_j \in C^\infty(\mathbf{R}^{3N}),\ 0 \leq I_j \leq 1 \text{ on } \mathbf{R}^{3N}\ (j = 1,2), \\ \sum_{i=1}^{2} I_j^2(x) \equiv 1\ \ (x \in \mathbf{R}^{3N}), \\ \operatorname{supp} I_1 \subset B(R_0)^c,\ \operatorname{supp} I_2 \subset B(2R_0). \end{cases}$$

By using Lemma 3.5 we define

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(3.5) $$w = (J_0^2 + I_2^2 J_1^2 + J_2^2)^{1/2},$$

which satisfies

(3.6) $$\begin{cases} 0 \leq w \leq 1 \text{ on } \mathbf{R}^{3N}, \\ w = 0 \text{ on } \Gamma(M_{\delta_0/2}, 2R_0), \\ w = 1 \text{ on } \mathbf{R}^{3N} \setminus \Gamma(M_{\delta_0}, R_0). \end{cases}$$

DEFINITION. Using the above partitions of unity, we define the quadratic form

(3.7) $$\begin{cases} Q[\phi, \psi] = \int_{|x|<R_0} \sum_{j=1}^{N} T_j\phi \cdot \overline{T_j\psi} dx \\ \qquad + \int_{|x|\geq R_0} \sum_{j=1}^{N} \{T_j I_2 J_1 \phi \cdot \overline{T_j I_2 J_1 \psi} + T_j J_2 \phi \cdot \overline{T_j J_2 \psi}\} dx + (Vw\phi, w\psi)_{L^2}, \\ Q[\phi] = Q[\phi, \phi] \qquad \text{for } \phi, \psi \in C_0^\infty(\mathbf{R}^{3N}). \end{cases}$$

For the sake of convenience we put

(3.8) $$\tau[\phi] = \int_{|x|<R_0} \sum_{j=1}^{N} |T_j\phi|^2 dx + \int_{|x|\geq R_0} \sum_{j=1}^{N} \{|T_j I_2 J_1 \phi|^2 + |T_j J_2 \phi|^2\} dx.$$

Then

(3.9) $$Q[\phi] = \tau[\phi] + (Vw\phi, w\phi)_{L^2}.$$

LEMMA 3.6. *The following equality holds.*

(3.10) $$Q[\phi, \psi] = q_H[J_0\phi, J_0\psi] + q_H[I_2 J_1 \phi, I_2 J_1 \psi] + q_H[J_2\phi, J_2\psi] - \int_{|x|<R_0} (|\nabla J_0|^2 + |\nabla J_1|^2 + |\nabla J_2|^2) \phi\bar{\psi} w^2 dx$$

for $\phi, \psi \in C_0^\infty(\mathbf{R}^{3N})$.

Proof of Lemma 3.6. As is easily seen, for $\phi, \psi \in C_0^\infty(\mathbf{R}^{3N})$,

$$\begin{aligned} (\#) &\equiv q_H[J_0\phi, J_0\psi] + q_H[I_2 J_1\phi, I_2 J_1\psi] + q_H[J_2\phi, J_2\psi] \\ &= Q[\phi, \psi] - \int_{|x|<R_0} \sum_{j=1}^{N} T_j\phi \cdot \overline{T_j\psi} dx \\ &\quad + \int_{|x|<R_0} \sum_{j=1}^{N} \{T_j J_0\phi \cdot \overline{T_j J_0 \psi} + T_j I_2 J_1 \phi \cdot \overline{T_j I_2 J_1 \psi} + T_j J_2 \phi \cdot \overline{T_j J_2 \psi}\} dx. \end{aligned}$$

Since $I_2 = 1$, $w = 1$ on $B(R_0)$ and $J_0^2 + J_1^2 + J_2^2 = 1$ on $\mathbf{R}^{3N}$, by a straightforward calculation we have

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$$
\begin{aligned}
&T_jJ_0\phi\cdot\overline{T_jJ_0\phi}+T_jI_2J_1\phi\cdot\overline{T_jI_2J_1\phi}+T_jJ_2\phi\cdot\overline{T_jJ_2\phi}\\
&=(J_0^2+J_1^2+J_2^2)T_j\phi\cdot\overline{T_j\phi}+(|\nabla J_0|^2+|\nabla J_1|^2+|\nabla J_2|^2)\phi\bar{\phi}\\
&\quad -i/2\nabla_j(J_0^2+J_1^2+J_2^2)\cdot(\overline{T_j\phi})\phi+i/2\nabla_j(J_0^2+J_1^2+J_2^2)\cdot(T_j\phi)\bar{\phi}\\
&=T_j\phi\cdot\overline{T_j\phi}+(|\nabla J_0|^2+|\nabla J_1|^2+|\nabla J_2|^2)\phi\bar{\phi}w^2
\end{aligned}
$$

on $B(R_0)$. Hence we have

$$(\#)=Q[\phi,\phi]+\int_{|x|<R_0}(|\nabla J_0|^2+|\nabla J_1|^2+|\nabla J_2|^2)\phi\bar{\phi}w^2dx,$$

which implies (3.10). □

LEMMA 3.7. *The quadratic form Q is densely defined, bounded below and closable in $L^2(\mathbf{R}^{3N};w^2dx)$.*

Here we denote the weighted L^2-space with the weight w^2 by $L^2(\mathbf{R}^{3N};w^2dx)$, and its inner product and its norm are denoted by $(\cdot,\cdot)_{w^2}$ and $\|\cdot\|_{w^2}$, respectively.

Proof of Lemma 3.7. First we show that Q is bounded below in $L^2(\mathbf{R}^{3N};w^2dx)$. By Lemma 3.1, for any $\varepsilon_1\in(0,1)$ there is a positive constant C_{ε_1} such that

$$
\begin{aligned}
(3.11)\quad (V_-w\phi,w\phi)_{L^2}&=(V_-J_0\phi,J_0\phi)_{L^2}+(V_-I_2J_1\phi,I_2J_1\phi)_{L^2}+(V_-J_2\phi,J_2\phi)_{L^2}\\
&\le\varepsilon_1\sum_{j=1}^N\{\|T_jJ_0\phi\|_{L^2}^2+\|T_jI_2J_1\phi\|_{L^2}^2+\|T_jJ_2\phi\|_{L^2}^2\}+C_{\varepsilon_1}\|w\phi\|_{L^2}^2
\end{aligned}
$$

for $\phi\in C_0^\infty(\mathbf{R}^{3N})$. By the proof of Lemma 3.6

$$
\begin{aligned}
(3.12)\quad &\int_{B(R_0)}(|T_jJ_0\phi|^2+|T_jI_2J_1\phi|^2+|T_jJ_2\phi|^2)dx\\
&\le\int_{B(R_0)}|T_j\phi|^2dx+d_0\int_{B(R_0)}|w\phi|^2dx,
\end{aligned}
$$

where $d_0=\sup_{x\in B(R_0)}\{|\nabla J_0|^2+|\nabla J_1|^2+|\nabla J_2|^2\}<+\infty$. Combining (3.11) with (3.12) we have

$$(3.13)\qquad (V_-w\phi,w\phi)_{L^2}\le\varepsilon_1\tau[\phi]+(C_{\varepsilon_1}+d_0)\|w\phi\|_{L^2}^2$$

for $\phi\in C_0^\infty(\mathbf{R}^{3N})$. Hence we have

$$
\begin{aligned}
Q[\phi]&=\tau[\phi]+(V_+w\phi,w\phi)_{L^2}-(V_-w\phi,w\phi)_{L^2}\\
&\ge(1-\varepsilon_1)\tau[\phi]+(V_+w\phi,w\phi)_{L^2}-(C_{\varepsilon_1}+d_0)\|w\phi\|_{L^2}^2
\end{aligned}
$$

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$$\geq -(C_{\varepsilon_1}+d_0)\|\phi\|_{w^2}^2 \text{ for } \phi\in C_0^\infty(\mathbf{R}^{3N}),$$

where $V_+(x)=\max\{V(x),0\}$. Further there is a positive constant $\gamma=\gamma(\varepsilon_1)$ satisfying

$$Q[\phi]+\gamma\|\phi\|_{w^2}^2\geq(1-\varepsilon_1)\tau[\phi]+(V_+w\phi,w\phi)_{L^2}+\|\phi\|_{w^2}^2 \tag{3.14}$$

for $\phi\in C_0^\infty(\mathbf{R}^{3N})$. This implies the boundedness from below.

Next we show that Q is closable in $L^2(\mathbf{R}^{3N};w^2dx)$. Suppose that $\phi\in C_0^\infty(\mathbf{R}^{3N})$, $\{\psi_j\}_{j\in\mathbf{N}}\subset C_0^\infty(\mathbf{R}^{3N})$ and $\psi_j\to 0$ strongly in $L^2(\mathbf{R}^{3N};w^2dx)$. Then

$$J_0\psi_j,\ I_2J_1\psi_j,\ J_2\psi_j\to 0 \text{ strongly in } L^2(\mathbf{R}^{3N}).$$

It easily follows that

$$\begin{aligned}&\left|\int_{B(R_0)}(|\nabla J_0|^2+|\nabla J_1|^2+|\nabla J_2|^2)w\phi\overline{w\psi_j}dx\right|\\ &\leq\sup_{x\in B(R_0)}(|\nabla J_0|^2+|\nabla J_1|^2+|\nabla J_2|^2)\left\{\int_{B(R_0)}|w\phi|^2dx\right\}^{1/2}\left\{\int_{B(R_0)}|w\psi_j|^2dx\right\}^{1/2}\\ &\leq(\text{constant})\cdot\|\psi_j\|_{w^2}\to 0.\end{aligned}$$

Therefore, by (3.10) and an integration by parts,

$$\begin{aligned}Q[\phi,\psi_j]&=(HJ_0\phi,J_0\psi_j)_{L^2}+(HI_2J_1\phi,I_2J_1\psi_j)_{L^2}+(HJ_2\phi,J_2\psi_j)_{L^2}\\ &\quad-\int_{B(R_0)}(|\nabla J_0|^2+|\nabla J_1|^2+|\nabla J_2|^2)w\phi\overline{w\psi_j}dx\to 0 \text{ as } j\to\infty.\end{aligned}$$

This implies the closability of Q. □

By Lemma 3.7 we can define the self-adjoint operator (denoted by P) in $L^2(\mathbf{R}^{3N};w^2dx)$, which is associated with the closure $\tilde{Q}$, that is,

$$\tilde{Q}[\phi,\psi]=(P\phi,\psi)_{w^2} \text{ for } \phi\in D(P),\ \psi\in D(\tilde{Q}).$$

Here we denote by $D(\cdot)$ the domain of an operator or a quadratic form. For $\gamma=\gamma(1/2)$ in (3.14) ($\varepsilon_1=1/2$) we define the inner product and the norm in $D(\tilde{Q})$:

$$(\phi,\psi)_\gamma=\tilde{Q}[\phi,\psi]+\gamma(\phi,\psi)_{w^2},\quad \|\phi\|_\gamma=(\tilde{Q}[\phi]+\gamma\|\phi\|_{w^2}^2)^{1/2},$$

then $D(\tilde{Q})$ is a Hilbert space with the above inner product and norm, and $D(P)$ is dense in $D(\tilde{Q})$.

LEMMA 3.8. *Let*

$$l_k\equiv l_k(P)=\inf\{Q[\phi];\phi\in C_0^\infty(|x|\geq k),\|\phi\|_{w^2}=1\}\quad(k\in\mathbf{N})$$

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and

$$l \equiv l(P) = \lim_{k\to\infty} l_k = \sup_k l_k.$$

Then

$$\inf \sigma_e(P) = l.$$

Proof of Lemma 3.8. At first we prove that $l \leq \inf \sigma_e(P)$. For any $\lambda \in \sigma_e(P)$ there is a sequence $\{\psi_n\}_n \subset D(P)$ such that $\| \psi_n \|_{w^2} = 1$ and

$$\psi_n \to 0 \text{ weakly, } (P - \lambda I)\psi_n \to 0 \text{ strongly in } L^2(\mathbf{R}^{3N}; w^2 dx).$$

It follows that

$$\psi_n \to 0 \text{ weakly in } D(\tilde{Q}) \text{ and } \tilde{Q}[\psi_n] \to \lambda. \tag{3.15}$$

In fact, for $\phi \in D(\tilde{Q})$,

$$\begin{aligned}(\psi_n, \phi)_\gamma &= (P\psi_n, \phi)_{w^2} + \gamma(\psi_n, \phi)_{w^2} \\ &= ((P - \lambda I)\psi_n, \phi)_{w^2} + (\lambda + \gamma)(\psi_n, \phi)_{w^2} \to 0 \text{ as } n \to \infty,\end{aligned}$$

and

$$\tilde{Q}[\psi_n] = ((P - \lambda I)\psi_n, \psi_n)_{w^2} + \lambda \| \psi_n \|_{w^2}^2 \to \lambda \text{ as } n \to \infty.$$

Since $C_0^\infty(\mathbf{R}^{3N})$ is dense in $D(\tilde{Q})$, there is a sequence $\{\phi_n\}_n \subset C_0^\infty(\mathbf{R}^{3N})$ such that $\| \psi_n - \phi_n \|_\gamma \to 0$ as $n \to \infty$. Hence by (3.15) we have

$$\begin{cases} \| \phi_n \|_{w^2} \to 1, \; Q[\phi_n] \to \lambda \text{ and} \\ \phi_n \to 0 \text{ weakly in } D(\tilde{Q}) \text{ as } n \to \infty, \\ \text{in paricular, } \phi_n \to 0 \text{ weakly in } L^2(\mathbf{R}^{3N}; w^2 dx) \text{ as } n \to \infty. \end{cases} \tag{3.16}$$

Letting $k \geq 2R_0$, we pick up a function $\theta(x) \in C_0^\infty(B(k+1))$ such that $\theta = 1$ on $B(k)$ and $0 \leq \theta \leq 1$. Then, by the definition of l_k,

$$l_k(\| \phi_n \|_{w^2} \mp \| \theta\phi_n \|_{w^2})^2 \leq l_k \| (1-\theta)\phi_n \|_{w^2}^2 \leq Q[(1-\theta)\phi_n], \tag{3.17}$$

where we choose the sign in the left-hand side whether l_k is positive or negative, respectively. We estimate $Q[(1-\theta)\phi_n]$. Notice that $(1-\theta)^2 \leq 1$, $\operatorname{supp}(1-\theta) \subset B(k)^c$, $J_0 = 0$ on $\operatorname{supp}(1-\theta)$ and

$$T_j(1-\theta)u\phi = (1-\theta)T_j u\phi + i(\nabla_j\theta)u\phi \text{ for } u \in C^1(\mathbf{R}^{3N}).$$

Then, for $\phi \in C_0^\infty(\mathbf{R}^{3N})$,

$$(3.18)\qquad Q[(1-\theta)\phi] = \sum_{j=1}^{N}\int_{B(R_0)^c}\{|T_j(1-\theta)I_2J_1\phi|^2 + |T_j(1-\theta)J_2\phi|^2\}dx$$
$$+\int_{\mathbf{R}^{3N}} V|(1-\theta)w\phi|^2 dx$$
$$\leq \sum_{j=1}^{N}\int_{B(R_0)^c}(1+\delta_1)(1-\theta)^2\{|T_jI_2J_1\phi|^2 + |T_jJ_2\phi|^2\}dx$$
$$+\sum_{j=1}^{N}\int_{B(R_0)^c}(1+\delta_1^{-1})|\nabla_j\theta|^2|w\phi|^2dx + \int_{\mathbf{R}^{3N}}(V_+ - V_-(1-\theta)^2)|w\phi|^2dx$$
$$\leq \sum_{j=1}^{N}\int_{B(R_0)^c}\{|T_jI_2J_1\phi|^2 + |T_jJ_2\phi|^2\}dx + \delta_1\sum_{j=1}^{N}\int_{B(R_0)^c}\{|T_jI_2J_1\phi|^2 + |T_jJ_2\phi|^2\}dx$$
$$+\int_{\mathbf{R}^{3N}}\{V|w\phi|^2 + \eta V_-|w\phi|^2\}dx + C(\delta_1)\int_{B(k+1)}|w\phi|^2dx$$
$$\leq Q[\phi] + \delta_1\tau[\phi] + \int_{\mathbf{R}^{3N}}\eta V_-|w\phi|^2dx + C(\delta_1)\int_{B(k+1)}|w\phi|^2dx,$$

where $\delta_1(\in(0,1))$ is arbitrary, $C(\delta_1)$ is some positive constant depending on δ_1, and $\eta = 1-(1-\theta)^2 \in C_0^\infty(B(k+1))$. Here we have used the positivity of $\int_{B(R_0)}|T_j\phi|^2dx$.

We estimate the third term of the last line of (3.18) as follows. By (3.13),

$$(3.19)\qquad (\eta V_-w\phi, w\phi)_{L^2} \leq (V_-w\phi, w\phi)_{L^2}^{1/2}\cdot(V_-w\eta\phi, w\eta\phi)_{L^2}^{1/2}$$
$$\leq \frac{C_1}{\sqrt{2}}(\tau[\phi] + \|w\phi\|_{L^2}^2)^{1/2}\cdot\left\{\frac{\delta_2}{4}\tau[\eta\phi] + (C_{\delta_2/4}+d_0)\|\eta w\phi\|_{L^2}^2\right\}^{1/2}$$

for any $\delta_2 > 0$ and some positive constant C_1. Here the constant $C_{\delta_2/4}+d_0$ appears in (3.13) and C_1 is independent of δ_1 and δ_2. Since

$$|T_j\eta u\phi|^2 \leq 2|\eta T_ju\phi|^2 + 2|\nabla_j\eta|^2\cdot|u\phi|^2 \leq 2|T_ju\phi|^2 + d_1|u\phi|^2\chi_{\mathrm{supp}\,\eta}$$

for $u\in C^1(\mathbf{R}^{3N})$ and some positive constant d_1, we have

$$(3.20)\qquad \tau[\eta\phi] \leq 2\tau[\phi] + d_1\|w\phi\|_{L^2(B(k+1))}^2.$$

By combining (3.19) with (3.20) we have

$$(3.21)\qquad (\eta V_-w\phi, w\phi)_{L^2} \leq \frac{C_1}{\sqrt{2}}(\tau[\phi] + \|w\phi\|_{L^2}^2)^{1/2}$$
$$\times\left\{\frac{\delta_2}{2}\tau[\phi] + C(\delta_2)\int_{B(k+1)}|w\phi|^2dx\right\}^{1/2},$$

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where $C(\delta_2)$ is some positive constant depending on δ_2. Now we recall (3.14) with $\varepsilon_1 = 1/2$:

$$(3.22)\qquad \begin{aligned} \|\phi\|_\gamma^2 &\geq \tau[\phi]/2 + (V_+ w\phi, w\phi)_{L^2} + \|w\phi\|_{L^2}^2 \\ &\geq (\tau[\phi] + \|w\phi\|_{L^2}^2)/2 \geq \tau[\phi]/2. \end{aligned}$$

By (3.21) and (3.22) we have

$$(3.23)\qquad (\eta V_- w\phi, w\phi)_{L^2} \leq C_1 \|\phi\|_\gamma \left\{\delta_2 \|\phi\|_\gamma^2 + C(\delta_2) \int_{B(k+1)} |w\phi|^2 dx\right\}^{1/2}.$$

By combining (3.18) with (3.23) and by using (3.22) again for the second term of the last line of (3.18), we have

$$(3.24)\qquad \begin{aligned} Q[(1-\theta)\phi] \leq Q[\phi] + 2\delta_1 \|\phi\|_\gamma^2 + C_1 \|\phi\|_\gamma \left\{\delta_2 \|\phi\|_\gamma^2 + C(\delta_2) \int_{B(k+1)} |w\phi|^2 dx\right\}^{1/2} \\ + C(\delta_1) \int_{B(k+1)} |w\phi|^2 dx. \end{aligned}$$

Now we shall prove

$$(3.25)\qquad \int_{B(k+1)} |w\phi_n|^2 dx \to 0 \text{ and } \|\theta\phi_n\|_{w^2} \to 0 \text{ as } n \to \infty.$$

As is easily seen,

$$\|\nabla_j \phi\|_{L^2(B(k+1))}^2 \leq 2(\|T_j\phi\|_{L^2(B(k+1))}^2 + \|b(x')\phi\|_{L^2(B(k+1))}^2) \text{ for } \phi \in C_0^\infty(\mathbf{R}^{3N}).$$

Hence, by using the fact that $\|\phi_n\|_\gamma < +\infty$, we have

$$\begin{aligned} &\|J_0\phi_n\|_{H^1(B(k+1))}^2 + \|I_2 J_1 \phi_n\|_{H^1(B(k+1))}^2 + \|J_2\phi_n\|_{H^1(B(k+1))}^2 \\ &\leq \int_{B(k+1)} \left\{2\sum_{j=1}^N (|T_j J_0\phi_n|^2 + |T_j I_2 J_1 \phi_n|^2 + |T_j J_2 \phi_n|^2) + (2\sum_{j=1}^N |b(x')|^2 + 1)|w\phi_n|^2\right\} dx \\ &\leq 2\tau[\phi_n] + (\text{constant}) \cdot \|\phi_n\|_{w^2}^2 < +\infty \end{aligned}$$

Hence there is a subsequence $\{\phi_{n_j}\}_j \subset \{\phi_n\}$ such that

$$\begin{cases} J_0\phi_{n_j} \to \Phi_0,\ I_2 J_1 \phi_{n_j} \to \Phi_1,\ J_2\phi_{n_j} \to \Phi_2 \\ \text{strongly in } L^2(B(k+1)), \end{cases}$$

where $\Phi_j (j = 0,1,2)$ is some function in $L^2(B(k+1))$. By (3.16)

$$\begin{aligned} 0 \leq \int_{B(k+1)} J_2^2 |\Phi_2|^2 dx \leq \int_{B(k+1)} w^2 |\Phi_2|^2 dx &= (\Phi_2, \Phi_2 \chi_{B(k+1)})_{w^2} \\ &= \lim_{j\to\infty} (J_2\phi_{n_j}, \Phi_2 \chi_{B(k+1)})_{w^2} = 0, \end{aligned}$$

which implies

$$\int_{B(k+1)} |J_2\phi_n|^2 dx \to 0 \text{ as } n \to \infty.$$

In the same way as above we have

$$\int_{B(k+1)} |J_0\phi_n|^2 dx \to 0, \int_{B(k+1)} |I_2 J_1 \phi_n|^2 dx \to 0 \text{ as } n \to \infty.$$

Summing up we have (3.25).

By (3.24) with $\phi = \phi_n$, (3.16), (3.17) and (3.25), letting $n \to \infty$, we have

$$l_k \leq \lambda + 2\delta_1(\lambda + \gamma) + C_1(\lambda + \gamma)^{1/2}\{\delta_2(\lambda + \gamma)\}^{1/2}.$$

Since δ_1 and δ_2 are arbitrary, this implies $l \leq \lambda$. Thus we obtain

$$l \leq \inf \sigma_e(P).$$

Next we show the reverse inequality in the same way as in [6] (Theorem 10). Taking $\mu < \inf \sigma_e(P)$, we can put

$$(-\infty, \mu) \cap \sigma(P) = \{\lambda_j\}_{j=1}^m \text{ or } \emptyset.$$

If $(-\infty, \mu) \cap \sigma(P) = \emptyset$, then it is clear that $l \geq \inf \sigma_e(P)$. We assume $(-\infty, \mu) \cap \sigma(P) = \{\lambda_j\}_{j=1}^m$ and we denote by $\{\psi_j\}_{j=1}^m$ the orthonormal eigenfunctions corresponding to $\{\lambda_j\}_{j=1}^m$. Let E_λ denote the spectral projection of P. Then for $\phi \in D(P)$,

$$\begin{aligned} \tilde{Q}[\phi] &= \sum_{j=1}^m \lambda_j |(\phi, \psi_j)_{w^2}|^2 + \int_\mu^{+\infty} \lambda d(E_\lambda \phi, \phi)_{w^2} \\ &\geq \sum_{j=1}^m \lambda_j |(\phi, \psi_j)_{w^2}|^2 + \mu \|\phi\|_{w^2}^2 - \mu \int_{-\infty}^\mu d(E_\lambda \phi, \phi)_{w^2} \\ &= \sum_{j=1}^m (\lambda_j - \mu) |(\phi, \psi_j)_{w^2}|^2 + \mu \|\phi\|_{w^2}^2. \end{aligned} \tag{3.26}$$

By the definition of l_k, there exists $\{\phi_i\}_i \subset C_0^\infty(\mathbf{R}^{3N})$ such that

$$\operatorname{supp} \phi_j \cap \operatorname{supp} \phi_k = \emptyset \text{ if } j \neq k, \|\phi_i\|_{w^2} = 1 \text{ and } Q[\phi_i] \to l. \tag{3.27}$$

By (3.26) with $\phi = \phi_i$ and (3.27), letting $i \to \infty$, we have $l \geq \mu$ which implies

$$l \geq \inf \sigma_e(P). \qquad \square$$

Lemma 3.9. *For $x \in \mathbf{R}^{3N}$ and $R > 0$, we put*

$$\Lambda_R(x\,;H)=\inf\{q_H[\phi]\,;\phi\in C_0^\infty(B(x\,;R)),\,\|\phi\|_{L^2}=1\},$$

where $B(x\,;R)=\{y\in\mathbf{R}^{3N}\,;|y-x|<R\}$. *Then, for any* $\varepsilon>0$ *there exists* $R(\varepsilon)>0$ *such that*

$$q_H[\phi]\geq\int_{\mathbf{R}^{3N}}(\Lambda_R(x\,;H)-\varepsilon)\,|\phi|^2dx \text{ for } R\geq R(\varepsilon) \text{ and } \phi\in C_0^\infty(\mathbf{R}^{3N}).$$

One can show the above lemma in the same way as in [2] (p 33).

LEMMA 3.10. *The following inequality holds.*

$$K(S^{3N-1}\setminus M_{\delta_0/4}\,;H)\leq\inf\sigma_e(P). \tag{3.28}$$

Proof of Lemma 3.10. By Lemma 3.9, for any $\varepsilon>0$ there exists $r_1\equiv R(\varepsilon)>0$ such that

$$Q[\phi]=q_H[I_2J_1\phi]+q_H[J_2\phi]\geq\int_{\mathbf{R}^{3N}}(\Lambda_{r_1}(x\,;H)-\varepsilon)\,|w\phi|^2dx$$

for $\phi\in C_0^\infty(B(2R_0)^c)$. For $k>2R_0$,

$$\begin{aligned} l_k&\geq\inf\Big\{\int_{\mathbf{R}^{3N}}(\Lambda_{r_1}(x\,;H)-\varepsilon)\,|w\phi|^2dx\,;\phi\in C_0^\infty(|x|\geq k),\,\|w\phi\|_{L^2}=1\Big\}\\ &\geq\inf\{\Lambda_{r_1}(x\,;H)\,;x\in\operatorname{supp}w\cap B(k)^c\}-\varepsilon.\end{aligned}$$

Hence there exists a sequence $\{x_k\}_k\subset\operatorname{supp}w\cap B(k)^c$ such that

$$l_k\geq\Lambda_{r_1}(x_k\,;H)-2\varepsilon \text{ for } k\geq 2R_0. \tag{3.29}$$

Since $\{x_k/|x_k|\}_k\subset S^{3N-1}\setminus M_{\delta_0/2}$, there are $k_1(\geq 2R_0)$ and a sequence $\{R_k\}_k\subset\mathbf{R}$ such that

$$B(x_k\,;r_1)\subset\Gamma(S^{3N-1}\setminus M_{\delta_0/4},\,R_k) \text{ for } k\geq k_1 \tag{3.30}$$

and $R_k\uparrow\infty$ as $k\to\infty$ (for example $R_k=|x_k|-r_1$, $k>>1$). Hence, by (3.29) and (3.30),

$$l_k\geq K(S^{3N-1}\setminus M_{\delta_0/4},\,R_k\,;H)-2\varepsilon \text{ for } k\geq k_1.$$

By letting $k\to+\infty$, we have

$$l\geq K(S^{3N-1}\setminus M_{\delta_0/4}\,;H)-2\varepsilon,$$

which implies that

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$$l \geq K(S^{3N-1} \setminus M_{\delta_0/4}; H).$$

By Lemma 3.8 we obtain (3.28). □

Now we estimate the error term in the localization formula, which is obtained by the lemma on $\{J_0, J_1, J_2\}$ and $\{I_1, I_2\}$ in [7].

Definition. For two partitions of unity $\{J_0, J_1, J_2\}$ and $\{I_1, I_2\}$ in Lemm 3.6, we define

$$A_J(x) = \{\sum_{i=1}^{2} |(\nabla J_i)(x/|x|)|^2\}^{1/2}, \; A_I(x) = \{\sum_{j=1}^{2} |(\nabla I_i)(x)|^2\}^{1/2}.$$

The following lemma appears in Evans-Lewis-Saitō [7].

Lemma 3.11 ([7]). *For sufficiently small $\varepsilon' > 0$ there is a positive constant $C_{\varepsilon'}$ satisfying*

$$\sum_{i=1}^{2} |\nabla J_i(x)|^2 \leq (\varepsilon' J_1^2(x) + C_{\varepsilon'} J_2^2(x)) |x|^{-2} A_J(x),$$
$$\sum_{i=1}^{2} |\nabla I_j(x)|^2 \leq (\varepsilon' I_1^2(x) + C_{\varepsilon'} I_2^2(x)) |x|^{-2} A_I(x)$$

for $|x| \geq 1$.

Then, by using the above lemma, we have the following one, which is the estimation of the error of the localization.

Lemma 3.12. *For sufficiently small $\varepsilon' > 0$ there exist positive constants $C(\varepsilon')$ and C_0 such that*

$$|\nabla(I_1 J_1)|^2 + |\nabla(I_2 J_1)|^2 + |\nabla J_2|^2 \leq C_0 \varepsilon' I_1^2 J_1^2 |x|^{-2} \chi_\Delta + C(\varepsilon') w^2 |x|^{-2} \chi_{B(R_0)^c}$$

for $|x| \geq 1$, where C_0 is independent of ε'.

Proof of Lemma 3.12. We note that $w = (I_2^2 J_1^2 + J_2^2)^{1/2}$ on $|x| \geq 1$ and $\operatorname{supp}\{\nabla(w^2)\} = \Delta$ (Δ appears in the definition of H_α). By a simple inequality,

$$|\nabla(I_2 J_1)|^2 \leq 2(I_2^2 |\nabla J_1|^2 + J_1^2 |\nabla I_2|^2)\chi_\Delta,$$
$$|\nabla(I_1 J_1)|^2 \leq 2(I_1^2 |\nabla J_1|^2 + J_1^2 |\nabla I_1|^2)\chi_\Delta.$$

Summing up and using Lemma 3.11, we have

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$$|\nabla(I_1J_1)|^2+|\nabla(I_2J_1)|^2+|\nabla J_2|^2\le 2\{|\nabla J_1|^2+|\nabla J_2|^2+J_1^2(|\nabla I_1|^2+|\nabla I_2|^2)\}\chi_\Delta$$
$$\le \{2(\varepsilon' J_1^2+C_{\varepsilon'}J_2^2)|x|^{-2}A_J+2J_1^2(\varepsilon' I_1^2+C_{\varepsilon'}I_2^2)|x|^{-2}A_I\}\chi_\Delta$$
$$\le C_0\varepsilon' I_1^2J_1^2|x|^{-2}\chi_\Delta+C(\varepsilon')w^2|x|^{-2}\chi_{B(R_0)^c}.$$

Here we have used $J_1^2=(I_1^2+I_2^2)J_1^2$ and $C_0=2\sup_{x\in\mathbf{R}^{3N}}(A_I(x)+A_J(x))$, which is independent of ε', and

$$C(\varepsilon')=2\sup_{x\in\mathbf{R}^{3N}}((\varepsilon'+C_{\varepsilon'})A_J(x)+C_{\varepsilon'}A_I(x)).\qquad\square$$

LEMMA 3.13. *For α in* (3.4) *there exists a positive constant C_α such that the following inequality holds.*

$$q_H[\phi]\ge\Sigma(H)\|I_1J_1\phi\|_{L^2}^2+Q[\phi]-\int_{B(R_0)^c}C_\alpha|x|^{-2}|w\phi|^2dx \tag{3.31}$$

for $\phi\in C_0^\infty(\mathbf{R}^{3N})$.

Remark 3.14 The above inequality holds for $\phi\in D(\tilde{q_H})$ by a limiting method, that is,

$$\tilde{q_H}[\phi]\ge\Sigma(H)\|I_1J_1\phi\|_{L^2}^2+\tilde{Q}[\phi]-\int_{B(R_0)^c}C_\alpha|x|^{-2}|w\phi|^2dx \tag{3.32}$$

for $\phi\in D(\tilde{q_H})\subset D(\tilde{Q})$.

Proof of Lemma 3.13. By using Lemmas 2.2 with $\Omega=B(R_0)^c$ and by Lemma 3.12, for $\phi\in C_0^\infty(\mathbf{R}^{3N})$ and sufficiently small $\varepsilon'>0$, we have

$$\begin{aligned}q_H[\phi]&=Q[\phi]+\int_{B(R_0)^c}\sum_{j=1}^N(|T_jI_1J_1\phi|^2+V|I_1J_1\phi|^2)dx\\&\quad-\int_{B(R_0)^c}(|\nabla(I_1J_1)|^2+|\nabla(I_2J_1)|^2+|\nabla J_2|^2)|\phi|^2dx\\&\ge Q[\phi]+\int_{B(R_0)^c}\sum_{j=1}^N(|T_jI_1J_1\phi|^2+V|I_1J_1\phi|^2)dx\\&\quad-\int_{B(R_0)^c}C_0\varepsilon'|x|^{-2}\chi_\Delta|I_1J_1\phi|^2dx-\int_{B(R_0)^c}C(\varepsilon')|x|^{-2}|w\phi|^2dx,\end{aligned}$$

where $C(\varepsilon')$ is a positive constant in Lemma 3.12. Choosing $\varepsilon'\le\alpha/C_0$ and using the fact that $I_1J_1\phi\in C_0^\infty(\Gamma(M_{\delta_0},R_0))$, we have

$$q_H[\phi]\ge Q[\phi]+K(M_{\delta_0},R_0;H_\alpha)\|I_1J_1\phi\|_{L^2}^2-\int_{B(R_0)^c}C_\alpha|x|^{-2}|w\phi|^2dx,$$

where $C_\alpha = C(\varepsilon')$. By the hypothesis of Theorem 3.2, that is, $K(M_{\delta_0}, R_0; H_\alpha) = \Sigma(H)$, we obtain (3.31). □

LEMMA 3.15 *We define quadratic forms:*

$$v[\phi] = -\int_{B(R_0)^c} C_\alpha |x|^{-2} |w\phi|^2 dx \text{ and } Q'[\phi] = Q[\phi] + v[\phi] \quad (\phi \in C_0^\infty(\mathbf{R}^{3N})),$$

then we have the self-adjoint operator in $L^2(\mathbf{R}^{3N}; w^2 dx)$ (*denoted by* P'), *which is associated with* $\tilde{Q}'$, *and*

$$\inf \sigma_e(P) = \inf \sigma_e(P'). \tag{3.33}$$

Proof of Lemma 3.15. The first part is easily seen. We show only (3.33). Let

$$V' = V - C_\alpha |x|^{-2} \chi_{B(R_0)^c}.$$

Then it is easy to see that $(V')_- \in \mathcal{K}(\mathbf{R}^{3N})$, so Lemma 3.8 holds also for P' and Q'. The following inequality is easily obtained:

$$l_k(P) \geq l_k(P') \geq l_k(P) - C_\alpha k^{-2}.$$

Letting $k \to \infty$, we have $l(P) = l(P')$ which implies (3.33). □

Now, by using lemmas prepared in this section, we prove Theorem 3.2.

Proof of Theorem 3.2. Lemmas 3.10 and 3.15 imply

$$\Sigma(H) < K(S^{3N-1} \setminus M_{\delta_0/4}; H) \leq \inf \sigma_e(P) = \inf \sigma_e(P').$$

Hence we have

$$\#(\sigma_d(P') \cap (-\infty, \Sigma(H))) < +\infty.$$

Let $E_1 = \{\psi_1, \dots, \psi_m\}$ $(m < +\infty)$ be orthonormal eigenfunctions corresponding to the eigenvalues $\{\sigma_d(P') \cap (-\infty, \Sigma(H))\}$. Let $E_2 = \{w^2\psi_1, \dots, w^2\psi_m\}$. If $\phi \in D(\tilde{q_H})$ satisfies $\phi \perp E_2$ in $L^2(\mathbf{R}^{3N})$, then $\phi \in D(\tilde{Q}')$ and $\phi \perp E_1$ in $L^2(\mathbf{R}^{3N}; w^2 dx)$. Hence we have

$$\tilde{Q}'[\phi] \geq \Sigma(H) \|\phi\|_{w^2}^2 \text{ for } \phi \in D(\tilde{q_H}) \text{ satisfying } \phi \perp E_2 \text{ in } L^2(\mathbf{R}^{3N}). \tag{3.34}$$

By Remark 3.14 and (3.34), we have

$$\tilde{q_H}[\phi] \geq \Sigma(H) \|I_1 J_1 \phi\|_{L^2}^2 + \tilde{Q}'[\phi] \geq \Sigma(H) \|\phi\|_{L^2}^2$$

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for $\phi \in D(\tilde{q_H})$ satisfying $\phi \perp E_2$ in $L^2(\mathbf{R}^{3N})$. This implies

$$\#(\sigma_d(H) \cap (-\infty, \Sigma(H))) < +\infty. \qquad \square$$

4. Some vector potentials

In this section we introduce some magnetic vector potentials, which are used in the proof of Theorem 1.1.

For $a \in (0,1)$, we define a function $f \in C^1([0, \infty))$ by

$$f(s) = \begin{cases} 1/2 + (2-a)^{-1}s^{-a} & (s \geq 2), \\ 1/2 + (2-a)^{-1}2^{-a-2}(4 + 2as - as^2) & (1 < s < 2), \\ 1/2 + (2-a)^{-1}2^{-a-2}(4+a) & (0 \leq s \leq 1). \end{cases} \tag{4.1}$$

Furthermore, letting

$$f_t(s) = t^{-2}f(s/t) \tag{4.2}$$

for a parameter $t > 1$, we consider a vector potential:

$$b_t(y) = f_t(\rho)(-y_2, y_1, 0), \tag{4.3}$$

where $y = (y_1, y_2, y_3) \in \mathbf{R}^3$, $\vec{\rho} = (y_1, y_2)$ and $\rho = |\vec{\rho}|$. This gives a perturbed constant magnetic field. The following lemma, which also follows from Theorem 2.9 in [3], appears in [9].

LEMMA 4.1 ([9]). *In the case that $b(y) = g(y)(-y_2, y_1, 0)$, where $g \in C^1(\mathbf{R}^3)$, the following inequality holds:*

$$(T(b)^2\phi, \phi)_{L^2(\mathbf{R}^3)} \geq \int_{\mathbf{R}^3} \left\{\rho \frac{\partial}{\partial \rho} g + 2g(y)\right\} |\phi|^2 dx \tag{4.4}$$

for $\phi \in C_0^\infty(\mathbf{R}^3)$, where $T(b) = -i\nabla_y - b(y)$.

By an elementary manipulation, using (4.1), we have

$$\rho \frac{\partial}{\partial \rho} f_t + 2f_t \geq t^{-2} + \min\{c_1(t), c_2(t)\rho^{-a}\}$$

for some positive constants $c_1(t)$ and $c_2(t)$ depending on t. Therefore by Lemma 4.1 we have the following lemma.

LEMMA 4.2 (key inequality).

(4.5) $$(T(b_t)^2\phi,\ \phi)_{L^2(\mathbf{R}^3)} \geq t^{-2}\|\phi\|^2_{L^2(\mathbf{R}^3)} + \int_{\mathbf{R}^3} \min\{c_1(t),\ c_2(t)\rho^{-a}\}\,|\phi|^2 dy$$

for $\phi \in C_0^\infty(\mathbf{R}^3)$.

Now letting

(4.6) $$H_t = T(b_t)^2 - \frac{Z}{|y|} \text{ in } L^2(\mathbf{R}^3_y),$$

we study the essential spectrum of H_t and $H_{N,Z}(b_t)$. First we want to show the following proposition.

PROPOSITION 4.3. $\sigma_e(H_t) = [t^{-2}, \infty)$.

To prove Proposition 4.3, we prepare the following two lemmas.

LEMMA 4.4. *For fixed* $t > 1$, *let*

(4.7) $$\begin{aligned}\phi^t_m(\vec{\rho}) &= \beta^t_m e^{im\theta}\rho^m \exp\Big(-\int_0^\rho f_t(s)sds\Big)\\ &= \beta^t_m(y_1 + iy_2)^m \exp\Big(-\int_0^\rho f_t(s)sds\Big) \quad (m \in \mathbf{N}),\end{aligned}$$

where β^t_m *is a normalizing constant in* $L^2(\mathbf{R}^2_\rho)$ *and we use the polar coordinate* $(\rho,\ \theta)$ *in* $(y_1,\ y_2)$*-space. Then the following equality holds.*

(4.8) $$P_t\phi^t_m \equiv (T(b_t)^2 - (-\partial^2/\partial y_3^2))\phi^t_m = \{2f_t(\rho) + f_t'(\rho)\rho\}\phi^t_m \quad (m \in \mathbf{N}).$$

Proof of Lemma 4.4. Let $\nabla_2 = (\partial/\partial y_1,\ \partial/\partial y_2)$. We remark that $\operatorname{div} b_t = 0$ and

$$\begin{cases}\dfrac{\partial\phi_m}{\partial\rho} = (m/\rho - f_t(\rho)\rho)\phi_m,\\ \dfrac{\partial\phi_m}{\partial\theta} = im\phi_m.\end{cases}$$

By the equality

$$\begin{aligned}P_t &= -\nabla_2\cdot\nabla_2 + 2ib_t\cdot\nabla_2 + i\operatorname{div} b_t + |b_t|^2\\ &= -\nabla_2\cdot\nabla_2 + 2ib_t\cdot\nabla_2 + |b_t|^2\end{aligned}$$

and by a straightforward calculation, we easily obtain (4.8). □

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LEMMA 4.5. *For any $R > 0$ there is a positive constant C_R^t, which is independent of m, such that*

$$(\beta_m^t)^2 \leq C_R^t R^{-(2m+1)} \quad (m \in \mathbf{N}). \tag{4.9}$$

Proof of Lemma 4.5. Letting $g_t(s) = f_t(s) - t^{-2}/2$, by the definition of β_m^t we have

$$\begin{aligned}(\beta_m^t)^{-2} &= 2\pi \int_0^{+\infty} \rho^{2m+1} \exp(-t^{-2}\rho^2/2)\exp\Big(-2\int_0^\rho g_t(s)sds\Big)d\rho \\ &\geq 2\pi \int_R^{+\infty} \rho^{2m+1} \exp\{-\rho^2(t^{-2}/2 + \gamma_1(t))\}d\rho\end{aligned}$$

for some constant $\gamma_1(t) \geq 0$. Then, putting

$$C_R^t = \Big(2\pi \int_R^{+\infty} \exp\{-\rho^2(t^{-2}/2 + \gamma_1(t))\}d\rho\Big)^{-1},$$

we obtain (4.9). □

Now we go into the proof of Proposition 4.3.

Proof of Proposition 4.3. Since $\Sigma(H_t) = \Sigma(T(b_t)^2) \geq t^{-2}$ by Lemma 2.1 and (4.5), it suffices to show that

$$[t^{-2}, \infty) \subset \sigma_e(H_t).$$

Picking up $\eta_0 \in C_0^\infty(1 \leq |y_3| \leq 2)$ normalized as $\|\eta_0\|_{L^2(\mathbf{R}^1)} = 1$, we define

$$\eta_m(y_3) = m^{-1/2}\eta_0(y_3/m) \quad (m \in \mathbf{N})$$

and

$$\psi_m^t(y) = \eta_m(y_3)e^{i\lambda y_3}\phi_m^t(\vec{\rho}) \quad (m \in \mathbf{N},\ \lambda \geq 0,\ t > 1).$$

Since supp $\psi_m^t \subset \{y \in \mathbf{R}^3 ; |y| \geq m\}$, it is easy to see that

$$-\frac{Z}{|y|}\psi_m^t \to 0 \text{ strongly in } L^2(\mathbf{R}^3) \text{ as } m \to \infty. \tag{4.10}$$

By a straightforward calculation,

$$\Big(\frac{\partial^2}{\partial y_3^2} + \lambda^2\Big)\psi_m^t \to 0 \text{ strongly in } L^2(\mathbf{R}^3) \text{ as } m \to \infty. \tag{4.11}$$

Letting

$$h_t(\rho) = 2(f_t(\rho) - t^{-2}/2) + f_t'(\rho)\rho,$$

we will show that

(4.12) $$h_t(\rho)\,\psi_m^t \to 0 \text{ strongly in } L^2(\mathbf{R}^3) \text{ as } m \to \infty.$$

If this is proved, we have by (4.8), (4.10) and (4.11)

$$\{H_t - (t^{-2} + \lambda^2)\}\psi_m^t \to 0 \text{ strongly in } L^2(\mathbf{R}^3) \text{ as } m \to \infty,$$

which implies $t^{-2} + \lambda^2 \in \sigma_e(H_t)$, hence $[t^{-2}, \infty) \subset \sigma_e(H_t)$.

We will show (4.12) to complete the proof. It easily follows from (4.1) and (4.2) that for any $\varepsilon > 0$ there exists $R(t) > 0$ such that

$$|\,h_t(\rho)\,|^2 < \varepsilon \text{ for } \rho \geq R(t),$$

so we have

$$\int_{\rho > R(t)} |\,h_t(\rho)\,|^2\,|\,\psi_m^t\,|^2 dy \leq \varepsilon.$$

On the other hand,

$$\begin{aligned}\int_{\rho \leq R(t)} |\,h_t(\rho)\,|^2\,|\,\psi_m^t\,|^2 dy &\leq \sup_{0 \leq \rho \leq R(t)} |\,h_t(\rho)\,|^2 \int_{\rho \leq R(t)} |\,\psi_m^t\,|^2 dy \\ &\leq d_1(t)\,(2\pi)\,(\beta_m^t)^2 \int_0^{R(t)} \rho^{2m+1} \exp\left\{-2\int_0^{\rho} f_t(s)\,s ds\right\} d\rho \\ &\leq 2\pi d_1(t)\,(\beta_m^t)^2 R(t)^{2m+1} d_2(t)\end{aligned}$$

where

$$d_1(t) = \sup_{0 \leq \rho \leq R(t)} |\,h_t(\rho)\,|^2 \text{ and } d_2(t) = \int_0^{R(t)} \exp\left\{-2\int_0^{\rho} f_t(s)\,s ds\right\} d\rho.$$

By using Lemma 4.5 with $R = 2R(t)$, we have

$$(\beta_m^t)^2 R(t)^{2m+1} \to 0 \text{ as } m \to \infty.$$

Thus we have (4.10). □

Next we study the essential spectrum of $H_{N,Z}(b_t)$. For the sake of convenience, we denote $H_{N,Z}(b_t)$ and $T_j(b_t)$ by H_N and T_j, respectively, for fixed $Z > 0$ and fixed $t > 1$. Now we want to prove the following proposition.

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Proposition 4.6.

$$\sigma_e(H_N) = [\Sigma(H_N), \infty). \tag{4.13}$$

Before the proof of the above proposition we prepare the following lemma.

Lemma 4.7 (uncertainty principle lemma). *Let* $T(b) = -i\nabla_y - b(y)$ *for* $b \in C^1(\mathbf{R}^3)^3$ *which is real-valued, and let* $k \in \mathbf{R}^3$. *Then we have the following inequality*:

$$(T(b)^2\phi, \phi)_{L^2(\mathbf{R}^3)} \geq 1/4(|y-k|^{-2}\phi, \phi)_{L^2(\mathbf{R}^3)} \text{ for } \phi \in C_0^\infty(\mathbf{R}^3).$$

We have only to show it in the case $k = 0$. The proof of Lemma 4.7 is omitted since we can show it in the same way as in the case that $b = 0$ ([11] p. 169).

Now we prove Proposition 4.6.

Proof of Proposition 4.6. We recall (2.19): $\Sigma(H_N) = \Lambda(H_{N-1}) + \Lambda(T_N^2)$. Here T_N is T_j with $j = N$. From (2.11) and Proposition 4.3 it follows that

$$\Sigma(H_N) = \Lambda(H_{N-1}) + \Sigma(T_N^2) = \Lambda(H_{N-1}) + t^{-2}. \tag{4.14}$$

We consider the following two cases separated.

(I) The case that $\Lambda(H_{N-1}) < \Sigma(H_{N-1})$.

Let $x = (x', x^N) \in \mathbf{R}^{3N}$. In this case there is a normalized eigenfunction $\eta(x') \in D(H_{N-1})$ satisfying

$$H_{N-1}\eta = \Lambda(H_{N-1})\eta \text{ in } L^2(\mathbf{R}^{3N-3}). \tag{4.15}$$

(II) The case that $\Lambda(H_{N-1}) = \Sigma(H_{N-1})$.

In this case there exists a sequence of orthonormal function $\{\eta_l(x')\}_l \subset D(H_{N-1})$ such that

$$(H_{N-1} - \Lambda(H_{N-1}))\eta_l \to 0 \text{ strongly in } L^2(\mathbf{R}^{3N-3}) \text{ as } l \to \infty. \tag{4.16}$$

We only consider the case (II), since the case (I) can be treated similarly. For any $\lambda \geq 0$, let $\mu = t^{-2} + \lambda$. By the proof of Proposition 4.3, there exists a sequence of functions $\{\phi_m(x^N)\}_m \subset C^2(\mathbf{R}^3) \cap D(T_N^2)$ such that

$$\begin{cases} \|\phi_m\|_{L^2(\mathbf{R}^3)} = 1 \ (m \in \mathbf{N}), \ (\phi_j, \phi_k)_{L^2(\mathbf{R}^3)} = 0 \ (j \neq k), \\ \text{supp}\, \phi_m \subset \{x^N; |x^N| \geq m\} \ (m \in \mathbf{N}) \text{ and} \\ (T_N^2 - \mu)\phi_m \to 0 \text{ strongly in } L^2(\mathbf{R}^3) \text{ as } m \to \infty. \end{cases} \tag{4.17}$$

Let $\psi_m^l(x) = \eta_l(x')\phi_m(x^N) \in D(H_N)$ $(m, l \in \mathbf{N})$ and remark that $\|\psi_m^l\|_{L^2} = 1$ $(m, l \in \mathbf{N})$ and

$$(\psi_m^l, \psi_{m'}^{l'})_{L^2} = 0 \text{ if } m \neq m' \text{ or } l \neq l'.$$

Then, by using the equality

$$H_N = H_{N-1} + T_N^2 - \frac{Z}{|x^N|} + \sum_{j=1}^{N-1} \frac{1}{|x^j - x^N|},$$

we have

$$\begin{aligned}(4.18) \quad H_N\psi_m^l = (\Lambda(H_{N-1}) + \mu)\psi_m^l + \{(H_{N-1} - \Lambda(H_{N-1}))\eta_l(x')\}\phi_m(x^N) \\ + \{(T_N^2 - \mu)\phi_m(x^N)\}\eta_l(x') - \frac{Z}{|x^N|}\psi_m^l + \sum_{j=1}^{N-1} \frac{1}{|x^j - x^N|}\psi_m^l.\end{aligned}$$

By (4.16) and (4.17), as is easily seen, the second, third and fourth terms in the right-hand side of (4.18) strongly converge to zero in $L^2(\mathbf{R}^{3N})$ as l and m tend to infinity. We estimate the last term. By (4.17), we have

$$\int_{|x^j|<|x^N|/2} \frac{1}{|x^j - x^N|^2} |\psi_m^l|^2 dx \leq \int_{|x^j|<|x^N|/2} 4m^{-2} |\psi_m^l|^2 dx \leq 4m^{-2} \to 0 \text{ as } m \to \infty.$$

On the other hand, from Lemma 4.7 it follows that

$$\int_{\mathbf{R}^3} \frac{1}{|x^j - x^N|^2} |\phi_m|^2 dx^N \leq 4(T_N^2\phi_m, \phi_m)_{L^2(\mathbf{R}^3)} \leq 4\mu + 1 < +\infty$$

for large m. Using this fact,

$$\begin{aligned}\int_{|x^j|\geq|x^N|/2} \frac{1}{|x^j - x^N|^2} |\psi_m^l|^2 dx &\leq \int_{|x^j|\geq m/2} \frac{1}{|x^j - x^N|^2} |\psi_m^l|^2 dx \\ &\leq (4\mu + 1) \int_{|x'|\geq m/2} |\eta_l(x')|^2 dx' \to 0 \text{ as } m \to \infty.\end{aligned}$$

Hence we have

$$\frac{1}{|x^j - x^N|}\psi_m^l \to 0 \text{ strongly in } L^2(\boldsymbol{R}^{3N}) \text{ as } m \to \infty \text{ (for any fixed } l).$$

Summing up and using (4.18), we can find $m = m(l)$ satisfying

$$(H_N - (\Lambda(H_{N-1}) + \mu))\psi_{m(l)}^l \to 0 \text{ strongly in } L^2(\mathbf{R}^{3N}) \text{ as } l \to \infty,$$

which implies $\Lambda(H_{N-1}) + \mu \in \sigma_e(H_N)$, so we obtain (4.13). □

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5. Proof of theorems

In this section we prove Theorems 1.1 and 1.2. We want to use Theorem 3.2 in order to derive the finiteness of discrete spectrum. Before going into the proof, we prepare the following proposition, from which it follows that, by virtue of Proposition 2.10, $M \equiv M(N) \neq S^{3N-1}$ holds. We denote $H_{N,Z}(b)$ by $H_N(b)$ for short.

PROPOSITION 5.1. *Let $b_t(y)$ be defined by* (4.3). *Let ε be arbitrary positive number and $Z \geq \varepsilon$. Then there exists $t(\varepsilon) > 1$, which depends only on ε, such that*

$$\Lambda(H_1(b_{t(\varepsilon)})) < \Sigma(H_1(b_{t(\varepsilon)})), \tag{5.1}$$

that is, $\sigma_d(H_1(b_{t(\varepsilon)})) \neq \emptyset$.

Proof of Proposition 5.1. We can pick up two real-valued functions $\phi_0(\vec{\rho}) \in C_0^\infty(\rho \leq 1)$ and $\eta_0(y_3) \in C_0^\infty(|y_3| \leq 1)$ normalized as $\|\phi_0\|_{L^2(\mathbf{R}^2)} = \|\eta_0\|_{L^2(\mathbf{R}^1)} = 1$, where $y = (y_1, y_2, y_3) \in \mathbf{R}^3$, $\vec{\rho} = (y_1, y_2)$ and $\rho = |\vec{\rho}|$. Letting

$$\phi_t(\vec{\rho}) = t^{-1}\phi_0(\vec{\rho}/t) \text{ and } \eta_t(y_3) = t^{-1/2}\eta_0(y_3/t)$$

for $t > 1$, we define

$$\psi_t(y) = \phi_t(\vec{\rho})\eta_t(y_3) \in C_0^\infty(\mathbf{R}^3) \quad (t > 1).$$

Then ψ_t is real-valued, $\|\psi_t\|_{L^2(\mathbf{R}^3)} = 1$ and

$$\operatorname{supp}\psi_t \subset \{(\vec{\rho}, y_3) \in \mathbf{R}^3; \rho \leq t, |y_3| \leq t\} \subset \{y \in \mathbf{R}^3; |y| \leq \sqrt{2}t\}. \tag{5.2}$$

Since ψ_t is real-valued, we have

$$(T_t^2\psi_t, \psi_t)_{L^2} = \|\nabla\psi_t\|_{L^2}^2 + \|b_t\psi_t\|_{L^2}^2,$$

where $T_t = -i\nabla_y - b_t(y)$. By a change of variables and (5.2) it is easy to see that

$$\|\nabla\psi_t\|_{L^2}^2 = t^{-2}(\|\nabla_2\phi_0\|_{L^2(\mathbf{R}^2)}^2 + \|\eta_0'\|_{L^2(\mathbf{R}^1)}^2) = d_1 t^{-2},$$

$$\|b_t\psi_t\|_{L^2}^2 = t^{-4}\int_{\operatorname{supp}\psi_t} |f(\rho/t)|^2\rho^2|\psi_t|^2 dx \leq d_2 t^{-2}$$

for some positive constants d_1 and d_2 which are independent of t. Hence, by using (5.2) again, we have

$$(H_1(b_t)\phi_t, \phi_t)_{L^2} = (T_t^2\phi_t, \phi_t)_{L^2} - \int_{\mathrm{supp}\,\phi_t} \frac{Z}{|y|} |\phi_t|^2 dy$$
$$\leq (d_1 + d_2)t^{-2} - \frac{\varepsilon}{\sqrt{2}t}.$$

Hence, for a sufficiently large t,

$$\Lambda(H_1(b_t)) \leq (H_1(b_t)\phi_t, \phi_t)_{L^2} < 0 < t^{-2} = \Sigma(H_1(b_t)),$$

where we have used Proposition 4.3. Note that this t depends only on ε. □

Now we go into the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $N \geq 2$. At first, for any $\varepsilon > 0$, we pick up $t(\varepsilon)$ (being fixed) in Proposition 5.1. Also let $Z \geq \varepsilon$. Now we show that the number of the discrete spectrum of $H_{N,Z}(b_{t(\varepsilon)})$ is finite, that is, we have only to put $b_\varepsilon = b_{t(\varepsilon)}$. For the sake of convenience, we denote $H_{N,Z}(b_{t(\varepsilon)})$ by H_N and put $B = t(\varepsilon)^{-2}$. From Propositions 5.1 and 2.10, it follows that $M \neq S^{3N-1}$, in particular,

$$M \subset \bigcup_{i_1,\dots,i_{N-1}} M_{i_1,\dots,i_{N-1}} \equiv M'. \tag{5.3}$$

Let $\delta_0 = 16^{-2^{N-2}}$ for the present. It is easy to show that

$$\begin{aligned}(M')_{\delta_0} &\subset \bigcup_{i_1,\dots,i_{N-1}} (M_{i_1,\dots,i_{N-1}})_{\delta_0} \\ &\subset \bigcup_{i_1,\dots,i_{N-1}} [\{(M_{i_1,\dots,i_{N-1}})_{\delta_0} \setminus \overline{(\bigcup_{i_1,\dots,i_{N-2}} M_{i_1,\dots,i_{N-2}})_{\delta_0^{1/2}/2}}\}] \cup (\bigcup_{i_1,\dots,i_{N-2}} M_{i_1,\dots,i_{N-2}})_{\delta_0^{1/2}} \\ &\subset \dots\dots,\end{aligned}$$

hence the set $(M')_{\delta_0}$ is covered as follows:

$$(M')_{\delta_0} \subset (\bigcup_i M_i)_{\delta_1} \bigcup_{k=2}^{N-1} \bigcup_{i_1,\dots,i_k} A_{i_1,\dots,i_k}(\delta_0) \equiv \hat{M}(\delta_0), \tag{5.4}$$

where

$$\begin{cases} A_{i_1,\dots,i_k}(\delta_0) = (M_{i_1,\dots,i_k})_{\delta_k} \setminus \overline{(\bigcup_{i_1,\dots,i_{k-1}} M_{i_1,\dots,i_{k-1}})_{\delta_{k-1}/2}} \ (k = 2,\dots, N-1), \\ \delta_k = \delta_0^{(1/2)^{N-k-1}} \quad (k = 1,\dots, N-1). \end{cases} \tag{5.5}$$

Here we exchange δ_0 for such a small number that $\hat{M}(\delta_0) \neq S^{3N-1}$. We want to show the following statement: there exist positive numbers $R_0(> 1)$ and α such that

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$$(5.6)\quad \begin{cases} ((H_N - \alpha\,|\,x\,|^{-a})\phi,\ \phi)_{L^2} \geq \Sigma(H_N)\ \|\,\phi\,\|_{L^2}^2 \\ \text{for } \phi \in C_0^\infty(\Gamma(A_{i_1,\ldots,i_k}(\delta_0),\ R_0))\ldots\ldots \text{ the case (I)} \\ \qquad (\{i_1,\ldots,\ i_k\} \subset \{1,\ldots,\ N\},\ k = 2,\ldots,\ N-1) \\ \text{and for } \phi \in C_0^\infty(\Gamma((\cup_i M_i)_{\delta_1},\ R_0))\ldots\ldots \text{ the case (II)}, \end{cases}$$

where α is in (4.1). For a while we admit the above statement and continue the proof. We remark that (5.6) holds with R_0 replaced by R such that $R \geq R_0$. Since $\hat{M}(\delta_0)$ is an open covering of $\overline{M'_{\delta_0/2}}$ which is compact in S^{3N-1}, there exists a partition of unity

$$\{J_1,\ J_{i_1,\ldots i_k};\ \{i_1,\ldots,\ i_k\} \subset \{1,\ldots,\ N\},\ k = 2,\ldots,\ N-1\} \equiv \{J_\beta\}_\beta$$

such that

$$(5.7)\quad \begin{cases} \sum_{k=2}^{N-1} \sum_{i_1,\ldots,i_k} J_{i_1,\ldots,i_k}^2(\omega) + J_1^2(\omega) = 1 \text{ on } \overline{M'_{\delta_0/2}}, \\ 0 \leq J_{i_1,\ldots,i_k} \leq 1,\ 0 \leq J_1 \leq 1, \\ \operatorname{supp} J_1 \subset (\cup_i M_i)_{\delta_1},\ \operatorname{supp} J_{i_1,\ldots,i_k} \subset A_{i_1,\ldots,i_k}(\delta_0). \end{cases}$$

We define $J_\beta(x) = J_\beta(x/|\,x\,|)$ $(|\,x\,| \geq 1)$, then it is easy to see that

$$(5.8)\qquad |\,(\nabla J_\beta)(x)\,|^2 \leq C(\delta_0)\,|\,x\,|^{-2}\quad (|\,x\,| \geq 1)$$

for some positive constant $C(\delta_0)$ depending on δ_0. Then , for $R \geq R_0$, we have

$$(5.9)\qquad K(M_{\delta_0/2},\ R,\ H_N - \alpha\,|\,x\,|^{-2}\chi_{B(1)^c})$$

$$\geq \inf\{((H_N - \alpha\,|\,x\,|^{-2})\phi,\ \phi)_{L^2};\ \phi \in C_0^\infty(\Gamma(M'_{\delta_0/2},\ R)),\ \|\,\phi\,\|_{L^2} = 1\}$$

$$= \inf\{\sum_\beta \{((H_N - \alpha\,|\,x\,|^{-2})J_\beta\phi,\ J_\beta\phi)_{L^2} - (|\,\nabla J_\beta\,|^2\phi,\ \phi)_{L^2}\};$$

$$\phi \in C_0^\infty(\Gamma(M'_{\delta_0/2},\ R)),\ \|\,\phi\,\|_{L^2} = 1\}$$

$$\geq \inf\{\sum_\beta ((H_N - \alpha|x|^{-a})J_\beta\phi, J_\beta\phi)_{L^2} + (\{-\ \#\{\beta\}C(\delta_0)\,|x|^{-2} + \alpha(|\,x\,|^{-a} - |\,x\,|^{-2})\}\phi,\ \phi)_{L^2};$$

$$\phi \in C_0^\infty(\Gamma(M'_{\delta_0/2},\ R)),\ \|\,\phi\,\|_{L^2} = 1\}$$

$$\geq \inf\{\sum_\beta \{((H_N - \alpha\,|\,x\,|^{-a})J_\beta\phi,\ J_\beta\phi)_{L^2}\};\ \phi \in C_0^\infty(\Gamma(M'_{\delta_0/2},\ R)),\ \|\,\phi\,\|_{L^2} = 1\}$$

$$\geq \Sigma(H_N)$$

when R is sufficiently large. Here we have used Lemma 2.2, (5.6), (5.8) and the fact $a < 1$. It follows from Lemma 2.3 and (2.7) that

$$K(M_{\delta_0/2},\ R,\ H_N - \alpha\,|\,x\,|^{-2}\chi_{B(1)^c}) \leq \Sigma(H_N).$$

Hence by Remark 3.4 the hypothesis of Theorem 3.2 is satisfied. Therefore putting $b_\varepsilon = b_{t(\varepsilon)}$, by Theorem 3.2 and Proposition 4.6, we have

$$\# \sigma_d(H_N) < +\infty.$$

This b_ε is independent of N and Z.

It remains to show (5.6). First we show it for the case (II). Since $\{(M_i)_{\delta_1}\}_i$ is disjoint, we have $(\cup_i M_i)_{\delta_1} = \cup_i (M_i)_{\delta_1}$ and

$$\Gamma((\bigcup_i M_i)_{\delta_1}, R) = \bigcup_i \Gamma((M_i)_{\delta_1}, R) \quad (R \geq 1),$$

both of which are disjoint unions. So, for any $\phi \in C_0^\infty(\Gamma((\cup_i M_i)_{\delta_1}, R))$, we can represent

$$\phi = \sum_{i=1}^{N} \phi_i, \ \phi_i \in C_0^\infty(\Gamma((M_i)_{\delta_1}, R)) \quad (i = 1, \ldots, N). \tag{5.10}$$

Let us consider only the case $i = N$, since the other cases are similar. If $x = (x', x^N) \in \Gamma((M_N)_{\delta_1}, R)$ $(R \geq 1)$, then it follows from [8] (Appendix C) that

$$|x'| \leq \delta_1 |x|, \ |x^j| \leq \delta_1 |x| \ (1 \leq j \leq N-1), \ |x^N| \geq \sqrt{1-\delta_1^2}\, |x|.$$

Then

$$\begin{aligned}(H_N\phi, \phi)_{L^2} &= (H_{N-1}\phi, \phi)_{L^2} + (T_N^2\phi, \phi)_{L^2} + \left(\left(-\frac{Z}{|x^N|} + \sum_{j=1}^{N-1}\frac{1}{|x^j - x^N|}\right)\phi, \phi\right)_{L^2} \\ &\geq \Lambda(H_{N-1})\|\phi\|_{L^2}^2 + (T_N^2\phi, \phi)_{L^2} - \left(\frac{Z}{\sqrt{1-\delta_1^2}}|x|^{-1}\phi, \phi\right)_{L^2}\end{aligned}$$

for $\phi \in C_0^\infty(\Gamma((M_N)_{\delta_1}, R))$. Here $T_N = T_j$ with $j = N$. Recall $B = \Lambda(T_N^2) = \Sigma(T_N^2) = t(\varepsilon)^{-2}$. By Lemma 4.2

$$\begin{aligned}(T_N^2\phi, \phi)_{L^2} &\geq B\|\phi\|_{L^2}^2 + (\min\{d_1, d_2|x^N|^{-a}\}\phi, \phi)_{L^2} \\ &\geq B\|\phi\|_{L^2}^2 + (d_2|x|^{-a}\phi, \phi)_{L^2}\end{aligned}$$

for $\phi \in C_0^\infty(\Gamma((M_N)_{\delta_1}, R))$ when R is large. From the above two inequalities

$$\begin{aligned}(H_N\phi, \phi)_{L^2} &\geq (\Lambda(H_{N-1}) + B)\|\phi\|_{L^2}^2 + (d_2/2|x|^{-a}\phi, \phi)_{L^2} \\ &\geq \Sigma(H_N)\|\phi\|_{L^2}^2 + (d_2/2|x|^{-a}\phi, \phi)_{L^2},\end{aligned}$$

for $\phi \in C_0^\infty(\Gamma((M_N)_{\delta_1}, R))$ when R is sufficiently large, where we have used (2.19) and $a < 1$. Hence there exists $R_1 > 0$ such that

$$((H_N - d_2/2|x|^{-a})\phi, \phi)_{L^2} \geq \Sigma(H_N)\|\phi\|_{L^2}^2 \text{ for } \phi \in C_0^\infty(\Gamma((M_N)_{\delta_1}, R)). \tag{5.11}$$

In the same way (5.11) holds for $\phi \in C_0^\infty(\Gamma((M_i)_{\delta_1}, R_1))$ $(i = 1, \ldots, N)$, hence, by (5.10), also holds for $\phi \in C_0^\infty(\Gamma((\cup_i M_i)_{\delta_1}, R_1))$.

Next we show (5.6) for the case (I). We consider only the case $(i_1, \ldots, i_k) = (1, \ldots, k)$ $(2 \leq k \leq N-1)$, since the other cases are similiar. If $x = (x^1, \ldots, x^k, x') \in \Gamma(A_{12\ldots k}(\delta_0), R)$ $(R \geq 1)$, then it follows that

$$|x^j| \geq \sqrt{\delta_k}\,|x|/4 \geq \sqrt{\delta_{N-1}}\,|x|/4 \quad (j = 1, \ldots, k), \tag{5.12}$$

which we admit for a while. Then, by using the equality again that

$$H_N = H_{N-1} + T_1^2 - \frac{Z}{|x^1|} + \sum_{j=2}^{N} \frac{1}{|x^1 - x^j|},$$

we can follow the similar way to the case (II). Hence there exists $R_2 > 0$ such that

$$((H_N - d_2/2\,|x|^{-a})\phi, \phi)_{L^2} \geq \Sigma(H_N)\,\|\phi\|_{L^2}^2 \text{ for } \phi \in C_0^\infty(\Gamma(A_{12\ldots k}(\delta_0), R_2)). \tag{5.13}$$

At the end we show (5.12). Let $x = (x^1, \ldots, x^k, x') \in \Gamma(A_{12\ldots k}(\delta_0), R)$ and let $\gamma(x) = \sum_{j=1}^{k-1} |x^j|^2$. Since $\mathrm{dist}(x/|x|, M_{12\ldots k}) \leq \delta_k$, it is easy to see that $|x'| \leq \delta_k |x|$. If $\gamma(x) = 0$, then $x = (0, \ldots, 0, x^k, x')$ and

$$\mathrm{dist}(x/|x|, \bigcup_{i_1,\ldots,i_{k-1}} M_{i_1,\ldots,i_{k-1}}) \leq |x/|x| - \xi| < 2\delta_k < \delta_{k-1}/2,$$

where $\xi = |x^k|^{-1}(0, \ldots, 0, x^k, 0, \ldots, 0)$. By the definition of $\Gamma(A_{12\ldots k}(\delta_0), R_0)$, this is contradiction. Hence $\gamma(x) \neq 0$. Now let $\omega = \gamma(x)^{-1/2}(x^1, \ldots, x^{k-1}, 0, \ldots, 0) \in M_{12\ldots k-1}$. Since $\mathrm{dist}(x/|x|, M_{1\ldots k-1}) \geq \delta_{k-1}/2 = \sqrt{\delta_k}/2$, we see that the inequality

$$|x/|x| - \omega| \geq \sqrt{\delta_k}/2 \tag{5.14}$$

holds. By a simple manipulation

$$\left|\frac{x^j}{|x|} - \frac{x^j}{\gamma(x)^{1/2}}\right| = \frac{|x| - \gamma(x)^{1/2}}{|x|\,\gamma(x)^{1/2}}\,|x^j| = \frac{|x'|^2 + |x^k|^2}{|x|\,\gamma(x)^{1/2}(|x| + \gamma(x)^{1/2})}\,|x^j|$$
$$(j = 1, \ldots, k-1).$$

Since $(|x'|^2 + |x^k|^2)^{1/2} \leq |x| \leq |x| + \gamma(x)^{1/2}$, we have

$$\left|\frac{x^j}{|x|} - \frac{x^j}{\gamma(x)^{1/2}}\right| \leq \frac{\sqrt{|x'|^2 + |x^k|^2}}{|x|\,\gamma(x)^{1/2}}\,|x^j| \quad (j = 1, \ldots, k-1).$$

Hence

$$(5.15)\qquad |x/|x| - \omega|^2 \le \sum_{j=1}^{k-1} \left| \frac{x^j}{|x|} - \frac{x^j}{\gamma(x)^{1/2}} \right|^2 + \frac{|x^k|^2}{|x|^2} + \frac{|x'|^2}{|x|^2}$$
$$\le 2\,\frac{|x'|^2 + |x^k|^2}{|x|^2} \le 2\Big(\frac{|x^k|^2}{|x|^2} + \delta_k^2\Big).$$

By the definition of δ_k, it follows $1 - 8\delta_k \ge 1/2$. Hence we have by (5.14) and (5.15)

$$\frac{|x^k|^2}{|x|^2} \ge \delta_k/8 - \delta_k^2 \ge \delta_k/16,$$

which implies

$$|x^k| \ge \sqrt{\delta_k}\,|x|/4.$$

Replacing $\gamma(x)$ by $\sum_{j=1}^{k} |x^j|^2 - |x^i|^2$ and following the same way, we have (5.12). Thus the proof of Theorem 1.1 is complete. □

Now we start the proof of Theorem 1.2. First we consider the other vector potential. Recall (4.1) and let $g(s) = f(s) - 1/2$. We can follow the same argument as in §4 by replacing $f(s)$ by $g(s)$. Then letting

$$(5.16)\qquad \begin{cases} g_t(s) = t^{-2} g(s/t) \quad (t > 1), \\ \tilde{b}_t(y) = f_t(\rho)(-y_2, y_1, 0) \\ (y = (y_1, y_2, y_3) \in \mathbf{R}^3,\ \vec{\rho} = (y_1, y_2),\ \rho = |\vec{\rho}|), \end{cases}$$

we have the following results:

$$(5.17)\qquad \begin{cases} T(\tilde{b}_t)^2 \ge \min\{c_1(t),\ c_2(t)\rho^{-a}\} \text{ in the form sense,} \\ \sigma_e(\tilde{H}_t) = [0, \infty) \text{ where } \tilde{H}_t = T(\tilde{b}_t)^2 - Z/|y| \text{ in } L^2(\mathbf{R}^3), \\ \sigma_e(H_{N,Z}(\tilde{b}_t)) = [\Sigma(H_{N,Z}(\tilde{b}_t)),\ \infty). \end{cases}$$

The other lemmas and propositions in §4 and §5 also hold in the same form. Furthermore, by using these facts, we can follow the same argument as in the proof of Theorem 1.1. Hence we have

PROPOSITION 5.2. *For any $\varepsilon > 0$, there exists a vector potential $\tilde{b}_\varepsilon$ of the above form* (5.16), *which is independent of N and Z, such that the number of the discrete spectrum of $H_{N,Z}(\tilde{b}_\varepsilon)$ is finite for $N \ge 2$ and $Z \ge \varepsilon$.*

Next we consider the case $N=1$.

PROPOSITION 5.3. *For any* $\varepsilon>0$, *we pick up the vector potential* $\tilde{b}_{\varepsilon}$, *of the form* (5.16) *in Proposition* 5.2. *Then*

$$\#\,\sigma_d(H_{1,Z}(\tilde{b}_{\varepsilon}))<+\infty \text{ for any } Z>0.$$

Proof of Proposition 5.3. By (5.17), it follows that

$$(T_1(\tilde{b}_{\varepsilon})^2\phi,\ \phi)_{L^2}\geq\int_{\mathbf{R}^3}\min\{c_1(\varepsilon),\ c_2(\varepsilon)\,|\,x\,|^{-a}\}\,|\,\phi\,|^2dx \text{ for } \phi\in C_0^{\infty}(\mathbf{R}^3),$$

where $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are some positive constants. For the sake of convenience, we denote $H_{1,Z}(\tilde{b}_{\varepsilon})$, $T_1(\tilde{b}_{\varepsilon})$, $c_1(\varepsilon)$, $c_2(\varepsilon)$ by H_1, T_1, c_1, c_2, respectively. Letting

$$W(x)=\min\{c_1,\ c_2\,|\,x\,|^{-a}\}/2-\frac{Z}{|\,x\,|},$$

we have

$$H_1\geq T_1^2/2+W(x) \text{ in the form sense.}$$

Since $a<1$, there exists a positive number R_Z such that $W(x)\geq 0$ for $|\,x\,|\geq R_Z$. If we put $\tilde{W}(x)=\min\{2W(x),\ 0\}$, then $\tilde{W}$ has a compact support and $2W\geq\tilde{W}$. Hence

$$H_1\geq(T_1^2+\tilde{W})/2\equiv H_1'/2 \text{ in the form sense.}$$

From the estimate of the number of negative eigenvalues in [3] (Theorem 2.15):

$$\#\,\{\sigma_d(H_1')\cap(-\infty,\ 0)\}\leq(\text{constant})\cdot\int_{\mathbf{R}^3}|\,\tilde{W}_-(x)\,|^{3/2}dx,$$

it follows that $\#\,\sigma_d(H_1')<+\infty$. Hence, by noting that $\Sigma(H_1)=\Sigma(H_1')=0$, we obtain

$$\#\,\sigma_d(H_1)<+\infty. \qquad \square$$

Combining Propositions 5.2 with 5.3, we have Theorem 1.2, which includes the case $N=1$.

Remark 5.4. In particular, by Proposition 5.3 and the proof of Proposition 5.2 it follows that

$$1 \leq \# \sigma_d(H_{1,Z}(\tilde{b_\varepsilon})) < +\infty \textit{ for } Z \geq \varepsilon.$$

Note Added in Proof. Recently the related work has done by Steven D. Underwood (University of Alabama at Birmingham). He also studied the Agmon function for magnetic vector potentials to get the other result which is extension of Evans-Lewis-Saitō's results to the magnetic case. I am thankful to Professor Yoshimi Saitō for giving me this information.

Also I should have referred to the other works on the essential spectrum of many body Schrödinger operators with magnetic fields. For example, B. Helffer, A. Mohamed; Ann. Institut Fourier, 38 (1988), 95-113. I am grateful to Professor B. Helffer for telling me many references.

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