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SEIBERG-WITTEN INVARIANTS OF GENERALISED RATIONAL BLOW-DOWNS

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One of the main problems in Seiberg-Witten theory is to find (SW)-basic classes and their invariants for a given smooth 4-manifold. The rational blow-down procedure introduced by Fintushel and Stern is one way to compute these invariants for some smooth 4-manifolds. In this paper, we extend their results to the general case. That is, we find (SW)-basic classes and Seiberg-Witten invariants for generalised rational blow-down 4-manifolds by using index computations.

1. INTRODUCTION

As gauge theory (Donaldson theory and Seiberg-Witten theory) is developed, the fundamental problem in this area is to find its invariants for a given smooth 4-manifold.

In 1993, Fintushel and Stern introduced a surgical procedure, called rational blow-down, to compute the Donaldson series for simply connected regular elliptic surfaces with multiple fibres of relatively prime orders. 'Rational blow-down' means that if a smooth 4-manifold X contains a certain configuration C_p of transversally intersecting 2-spheres whose boundary is $L(p^2, 1-p)$, then one can construct a new smooth 4-manifold X_p from X by replacing C_p with a rational ball B_p.

In fact, Casson and Harer [**2**] showed that for any pair of relatively prime integers p and q, $L(p^2, 1-pq)$ bounds a rational ball $B_{p,q}$. Hence one can extend this rational blow-down procedure to the general case, that is, whenever a smooth 4-manifold X contains a certain configuration $C_{p,q}$ of transversally intersecting 2-spheres whose boundary is $L(p^2, 1-pq)$, one can always construct a new smooth 4-manifold $X_{p,q}$ by replacing $C_{p,q}$ with a rational ball $B_{p,q}$.

For the $q = 1$ case, Fintushel and Stern initially computed the Donaldson series of $X_p = X_{p,1}$ from the Donaldson series of X, and later they computed the Seiberg-Witten invariants of X_p [**5**]. In Section 3 of this paper we extend these results to the general case. Explicitly, we prove the following theorem by using index computations:

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THEOREM 1.1. *Suppose X is a smooth 4-manifold which contains a configuration $C_{p,q}$. If L is a characteristic line bundle on X such that $SW_X(L) \neq 0$, $\left(L|_{C_{p,q}}\right)^2 = -b_2(C_{p,q})$ and $c_1\left(L|_{L(p^2,1-pq)}\right) = mp \in \mathbf{Z}_{p^2} \cong H^2\left(L(p^2, 1-pq); \mathbf{Z}\right)$ with $m \equiv (p-1) \pmod 2$, then L induces a characteristic line bundle $\overline{L}$ on $X_{p,q}$ such that $SW_{X_{p,q}}(\overline{L}) = SW_X(L)$.*

Furthermore, we prove the following theorem:

THEOREM 1.2. *If a simply connected smooth 4-manifold X contains a configuration $C_{p,q}$ satisfying condition $(*)$ below, then the SW-invariants of $X_{p,q}$ are completely determined by those of X. That is, for any characteristic line bundle $\overline{L}$ on $X_{p,q}$ with $SW_{X_{p,q}}(\overline{L}) \neq 0$, there exists a characteristic line bundle L on X such that $SW_X(L) = SW_{X_{p,q}}(\overline{L})$.*

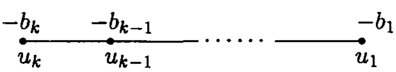
The condition $(*)$ in the theorem above is the following:

$$(*) \quad \left\{\partial\left(\sum_{i=1}^{k} \varepsilon_i e_i|_{B_{p,q}}\right) : \varepsilon_i = \pm 1, \forall i\right\}$$
$$= \left\{mp : -(p-1) \leqslant m \leqslant (p-1) \text{and } m \equiv (p-1) \pmod 2\right\}$$

All known configurations $C_{p,q}$ satisfy this condition.

2. THE TOPOLOGY OF RATIONAL BLOW-DOWNS

In this section we describe topological aspects and several examples of rational blow-down 4-manifolds. For any relatively prime integers p and q with $1 \leqslant q < p$, we define a configuration $C_{p,q}$ as a smooth 4-manifold obtained by plumbing disk bundles over the 2-sphere instructed by the following linear diagram



where $p^2/(pq-1) = [b_k, b_{k-1}, \ldots, b_1]$ is a unique continued linear fraction with all $b_i \geqslant 2$, and each vertex u_i represents a disk bundle over the 2-sphere whose Euler number is $-b_i$. Then the configuration $C_{p,q}$ has the following properties:

1. It is a simply connected smooth 4-manifold whose boundary is the lens space $L(p^2, 1-pq)$.
2. $H_2(C_{p,q}; \mathbf{Z}) \cong \bigoplus_{i=1}^{k} \mathbf{Z}$ has generators $\{u_i : 1 \leqslant i \leqslant k\}$ which can be represented by embedded 2-spheres, that is, each u_i is represented by the zero-section S_i^2 of the disk bundle u_i over S^2. (We use u_i for both a generator and the corresponding disk bundle.)

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3. The plumbing matrix for $C_{p,q}$ with respect to the basis $\{u_i : 1 \leqslant i \leqslant k\}$ is given by the symmetric $k \times k$ matrix

$$P = \begin{pmatrix} -b_1 & 1 & 0 & & & \\ 1 & -b_2 & 1 & & 0 & \\ 0 & 1 & -b_3 & & & \\ & & & \ddots & & \\ & 0 & & & -b_{k-1} & 1 \\ & & & & 1 & -b_k \end{pmatrix}$$

so that $C_{p,q}$ is negative definite.

4. The intersection form on $H^2(C_{p,q}; \mathbf{Z})$ with respect to the dual basis $\{\gamma_i : 1 \leqslant i \leqslant k\}$ (that is, $\langle \gamma_i , u_j \rangle = \delta_{ij}$) is given by

$$Q := (\gamma_i \cdot \gamma_j) = P^{-1}.$$

PROOF: Note that the intersection form Q on $H^2(C_{p,q}; \mathbf{Z})$ is defined by

$$\gamma_i \cdot \gamma_j := \frac{1}{p^2} \langle \gamma_i , PD\gamma'_j \rangle$$

where $\gamma'_j \in H^2(C_{p,q}, \partial C_{p,q}; \mathbf{Z})$ is determined by $\jmath^*(\gamma'_j) = p^2 \cdot \gamma_j$ in the sequence

$$0 \longrightarrow H^2(C_{p,q}, \partial C_{p,q}; \mathbf{Z}) \xrightarrow{\jmath^*} H^2(C_{p,q}; \mathbf{Z}) \xrightarrow{\partial} H^2(\partial C_{p,q}; \mathbf{Z}) \longrightarrow 0.$$

Since $\jmath^* = P$, we have

$$\begin{aligned} \gamma_i \cdot \gamma_j &:= \frac{1}{p^2} \langle \gamma_i , PD\gamma'_j \rangle = \frac{1}{p^2} \langle \gamma_i , P^{-1}(p^2 \cdot PD\gamma_j) \rangle = \langle \gamma_i , P^{-1}(PD\gamma_j) \rangle \\ &= \left(P^{-1}\right)_{ij}. \end{aligned}$$

∎

LEMMA 2.1. *The inclusion induced homomorphism* $\partial : H^2(C_{p,q}; \mathbf{Z}) \longrightarrow H^2(\partial C_{p,q}; \mathbf{Z}) \cong \mathbf{Z}_{p^2}$ *is given by* $\partial(\gamma_i) = n_i$, *where* n_i *is a number satisfying*

$$\begin{pmatrix} * \\ n_i \end{pmatrix} := \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ b_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} -1 & 0 \\ b_{i-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

PROOF: By Poincaré duality, it suffices to show $\partial : H_2(C_{p,q}, \partial C_{p,q}; \mathbf{Z}) \to H_1(\partial C_{p,q}; \mathbf{Z})$ is given by $\partial(PD\gamma_i) = n_i$. For each i, choose a fibre D_i^2 of a disk bundle u_i over S^2

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so that $D_i^2 \cdot S_j^2 = \delta_{ij}$. Then D_i^2 is a representative for $PD(\gamma_i) \in H_2(C_{p,q}, \partial C_{p,q}; \mathbf{Z})$. Since

$$\begin{aligned}\partial C_{p,q} &= D^+ \times S_k^1 \cup_{A_k} \partial D^- \times S_k^1 \cup_B \partial D^+ \times S_{k-1}^1 \cup_{A_{k-1}} \cdots \cup_{A_1} D^- \times S_1^1 \\ &= D^+ \times S_k^1 \cup_A D^- \times S_1^1\end{aligned}$$

where $S_i^1 := \partial D_i^2$ and $A := A_k B A_{k-1} \cdots A_1$ with $A_i := \begin{pmatrix} -1 & 0 \\ b_i & 1 \end{pmatrix}$, and $B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have

$$\begin{aligned}\partial(PD\gamma_i) &= \partial(D_i^2) \\ &= \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} -1 & 0 \\ b_{i-1} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ n_i \end{pmatrix}\end{aligned}$$

which is homologous to $\begin{pmatrix} 0 \\ n_i \end{pmatrix}$ in $H_1(\partial C_{p,q}; \mathbf{Z})$. Hence, by choosing $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a generator of $H_1(\partial C_{p,q}; \mathbf{Z})$, we have $\partial(PD\gamma_i) = n_i$. ∎

LEMMA 2.2. *The lens space $L(p^2, 1-pq) = \partial C_{p,q}$ bounds a rational ball $B_{p,q}$ with $\pi_1(B_{p,q}) = \mathbf{Z}_p$, and the inclusion induced homomorphism*

$$\iota^* : H^2(B_{p,q}; \mathbf{Z}) \cong \mathbf{Z}_p \longrightarrow H^2(L(p^2, 1-pq); \mathbf{Z}) \cong \mathbf{Z}_{p^2}$$

can be given by $n \longmapsto np$.

PROOF: The first part was proved by Casson and Harer [**2**]. For the second part, since the Mayer-Vietoris sequence for $X \equiv C_{p,q} \cup_L \overline{B_{p,q}}$ which is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$

$$0 \longrightarrow H_2(C_{p,q}; \mathbf{Z}) \oplus H_2(B_{p,q}; \mathbf{Z}) \longrightarrow H_2\left(\sharp k\overline{\mathbf{CP}}^2; \mathbf{Z}\right) \longrightarrow \cdots$$

implies $H_2(B_{p,q}; \mathbf{Z})$ is torsion free, by Poincaré duality, $H^2(B_{p,q}, \partial B_{p,q}; \mathbf{Z}) \cong H_2(B_{p,q}) = 0$. On the other hand, since the exact sequence for $(B_{p,q}, \partial B_{p,q})$ also implies that

$$\iota^* : H^2(B_{p,q}; \mathbf{Z}) \cong \mathbf{Z}_p \longrightarrow H^2(\partial B_{p,q}; \mathbf{Z}) \cong \mathbf{Z}_{p^2}$$

is injective, $\iota^*(1) = lp$ for some l with $\gcd(l, p) = 1$. Hence, by re-choosing a generator of $H^2(\partial B_{p,q}; \mathbf{Z}) \cong \mathbf{Z}_{p^2}$, we may assume that $\iota^*(1) = p$, so that $\iota^*(n) = np$. ∎

LEMMA 2.3. *$B_{p,q}$ is spin if p is odd, and $B_{p,q}$ is not spin if p is even.*

PROOF: If p is odd, then $H_1(B_{p,q}) \cong \mathbf{Z}_p$ implies $H^2(B_{p,q}; \mathbf{Z}_2) \cong Ext(H_1(B_{p,q}); \mathbf{Z}_2) = 0$. Assume p is even and $B_{p,q}$ is spin. Then the index of the Dirac operator on $B_{p,q}$ should be an integer. But the index computation on $B_{p,q}$ (Proposition 3.3 and its remark) shows that it is not an integer—a contradiction! ∎

Now we define the rational blow-down procedure: Suppose X is a smooth 4-manifold which contains a configuration $C_{p,q}$ for some relatively prime integers p and q. We construct a new smooth 4-manifold $X_{p,q}$, called the **rational blow-down of** X, by replacing $C_{p,q}$ with the rational ball $B_{p,q}$ (Figure 1). We call this procedure a '**(generalised) rational blow-down**'. Note that this procedure is well defined, that is, $X_{p,q}$ is uniquely constructed (up to diffeomorphism) from X because each diffeomorphism of $\partial B_{p,q} = L(p^2, 1-pq)$ extends over the rational ball $B_{p,q}$ by the same argument as in [**5**, Corollary 2.2].

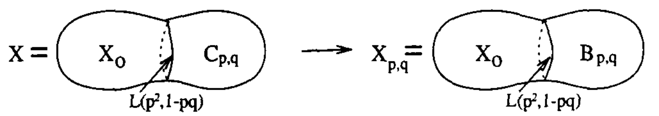

Figure 1

Lemma 2.4. *$b^+(X_{p,q}) = b^+(X)$ and $c_1^2(X_{p,q}) = c_1^2(X) + k$, where $k = b_2(C_{p,q})$.*

Proof: Since $C_{p,q}$ is negative definite, $b^+(X_{p,q}) = b^+(X)$ and

$$\begin{aligned} c_1^2(X_{p,q}) &= 3\sigma(X_{p,q}) + 2e(X_{p,q}) \\ &= 3(\sigma(X) + k) + 2(e(X) - k) \\ &= c_1^2(X) + k. \end{aligned}$$

where $\sigma(X)$ is the signature of X and $e(X)$ is the Euler characteristic of X. ∎

Here are several configurations $C_{p,q}$ that will be used later.

Case $q = 1$. This case is studied in [**5**], whose configuration $C_{p,1}$ is

$$\underset{u_{p-1}}{\overset{-(p+2)}{\bullet}} \!-\!\!-\!\! \underset{u_{p-2}}{\overset{-2}{\bullet}} \!-\!\!-\!\! \cdots\cdots \!-\!\!-\!\! \underset{u_1}{\overset{-2}{\bullet}}$$

Fintushel and Stern used this configuration to show that the rational blow-down of $E(n)\sharp(p-1)\overline{\mathbf{CP}}^2$ is diffeomorphic to $E(n;p)$, p-log transform on $E(n)$, and to compute the Donaldson and Seiberg-Witten invariants of simply connected elliptic surfaces with multiple fibres. Here $E(n)$ is a simply connected elliptic surface with no multiple fibres and holomorphic Euler characteristic n, and 'p-log transform on $E(n)$' is the result of removing a tubular neighbourhood of a torus fibre in $E(n)$, say $T^2 \times D^2$, and regluing it by a diffeomorphism

$$\varphi : T^2 \times \partial D^2 \longrightarrow T^2 \times \partial D^2$$

such that the absolute value of the degree of the map

$$\mathrm{proj}_{\partial D^2} \circ \varphi : pt \times \partial D^2 \longrightarrow \partial D^2$$

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is p. Note that 'p-log transform on $E(n)$' is well defined, that is, $E(n;p)$ is uniquely determined up to diffeomorphism by the fact that if $\mathrm{proj}_{\partial D^2} \circ \varphi$ and $\mathrm{proj}_{\partial D^2} \circ \varphi'$ have the same degree up to sign, then the resulting two manifolds are diffeomorphic [**6,** Proposition 2.1].

CASE $p = kq - 1$ $(k, q \geqslant 2)$. We assume $q \geqslant 3$ (the $q = 2$ case is also obtained in a similar way). The configuration $C_{p,q}$ is given by

which can be embedded in $\sharp(k + q - 2)\overline{\mathbf{CP}}^2$ by choosing

$$u_i := \begin{cases} e_{k+q-2-i} - e_{k+q-1-i} & i = 1, \dots, k-2 \\ e_{q-2} - e_{q-1} - e_q & i = k-1 \\ e_{k+q-3-i} - e_{k+q-2-i} & i = k, \dots, k+q-4 \\ -2e_1 - e_2 - \cdots - e_{q-1} & i = k+q-3 \\ e_{q-1} - e_q - \cdots - e_{k+q-2} & i = k+q-2 \end{cases}$$

where each e_i $(1 \leqslant i \leqslant k + q - 2)$ is the exceptional divisor in $\sharp(k + q - 2)\overline{\mathbf{CP}}^2$. Furthermore, by using Lemma 2.1, we get its boundary values

$$\text{(1)} \qquad \partial\gamma_i = \begin{cases} i & i = 1, \dots, k-1 \\ (i + 2 - k)k - i & i = k, \dots, k+q-3 \\ pq - 1 & i = k+q-2 \end{cases}$$

which imply that $C_{kq-1,q}$ satisfies the condition $(*)$ mentioned in the introduction.

THEOREM 2.1. *For any integers k and q $(k, q \geqslant 2)$, there is an embedding $C_{kq-1,q} \subset E(n)\sharp(k + q - 2)\overline{\mathbf{CP}}^2$ such that the rational blow-down is diffeomorphic to $E(n; kq - 1)$.*

PROOF: Consider the homology class f of the fibre in $E(n)$ which can be represented by an immersed 2-sphere with one positive double point and self-intersection 0 (a nodal fibre). Blow up this double point so that $f - 2e_1$ (e_1 is the exceptional divisor) is represented by an embedded sphere. Since e_1 intersects $f - 2e_1$ at two positive points, blow up one of these points again. By continuing in this way, we get a configuration $C_{kq-1,q}$ in $E(n)\sharp(k + q - 2)\overline{\mathbf{CP}}^2$. We draw the case $q \geqslant 3$ (Figure 2) (the $q = 2$ case is similar). The claim that the rational blow-down of $E(n)\sharp(k + q - 2)\overline{\mathbf{CP}}^2$ is diffeomorphic to $E(n; kq - 1)$ can be proved by Kirby calculus on the neighbourhood of a cusp fibre as in [**5,** Theorem 3.1]. ∎

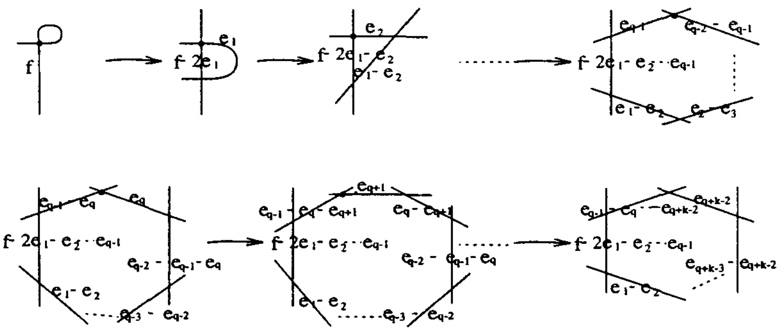

Figure 2

Here are a few remarks on this theorem:

1. The theorem implies that there are many ways to obtain $E(n;p)$, p-log transform on $E(n)$, from $E(n)$ via a rational blow-down procedure; so one can choose an 'economical' way to get $E(n;p)$. For example, $E(n,11)$ is diffeomorphic to the rational blow-down of $C_{11,1} \subset E(n)\sharp 10\overline{\mathbf{CP}}^2$, of $C_{11,2} \subset E(n)\sharp 6\overline{\mathbf{CP}}^2$, and of $C_{11,3} \subset E(n)\sharp 5\overline{\mathbf{CP}}^2$.

2. One expects that for any relative prime integers p and q, there is an embedding $C_{p,q}$ in $E(n)\sharp k\overline{\mathbf{CP}}^2$, for some $k \in \mathbf{Z}$, such that the rational blow-down is diffeomorphic to $E(n;p)$.

3. The key ingredient in the proof of the theorem is to find such a configuration $C_{kq-1,q}$. We chose u_i exactly the same as u_i embedded in $\sharp(k+q-2)\overline{\mathbf{CP}}^2$ except

$$u_{k+q-3} = f - 2e_1 - e_2 \cdots - e_{q-1} \quad (u_{k-1} = f - 2e_1 - e_2, \text{ if } q = 2)$$

4. One can extend the 'logarithmic transform' procedure to any 4-manifold which contains a cusp neighbourhood. A *cusp* in a 4-manifold means a PL embedded 2-sphere of self-intersection 0 with a single non-locally flat point whose neighbourhood is the cone on the right-hand trefoil knot, and we define a *cusp neighbourhood* in a 4-manifold to be a manifold N obtained by performing 0-framed surgery on the trefoil knot in the boundary of the 4-ball. Note that since the trefoil knot is a fibred knot with a genus 1 fibre, N is fibred by tori with one singular fibre which is a cusp. Hence one can perform 'p-log transform' on a regular torus fibre in N exactly the same way as in $E(n)$, so that the theorem above is also true for any smooth 4-manifold containing a cusp neighbourhood.

3. Seiberg-Witten Theory of Rational Blow-downs of 4-Manifolds

In this section we compute the Seiberg-Witten invariants of rational blow-downs of 4-manifolds. We start by recalling the basics of Seiberg-Witten invariants introduced by Seiberg and Witten (see [**8, 10**]).

Let X be an oriented, closed Riemannian 4-manifold, and let L be a characteristic line bundle on X, that is, $c_1(L)$ is an integral lift of $w_2(X)$. This determines a Spin^c-structure on X. We denote the associated $U(2)$-bundles by $W^\pm := S^\pm \otimes L^{1/2}$, where $S^\pm$ is a (locally defined) spinor bundle on X. (One may choose a Spin^c-structure first, and associated $U(2)$-bundles $W^\pm$ on X. Then $L := \det(W^+) \cong \det(W^-)$ is the associated characteristic line bundle on X.) For simplicity we assume that $H^2(X;\mathbf{Z})$ has no 2-torsion so that the set Spin$^c(X)$ of Spin^c-structures on X is identified with the set of characteristic line bundles on X.

Note that the Clifford multiplication $c : T^*X \to \operatorname{Hom}(W^+, W^-)$ leads to an isomorphism

$$\rho : \Lambda^+ \otimes \mathbf{C} \longrightarrow sl(W^+)$$

taking Λ^+ to $su(W^+)$, and the Levi-Civita connection on TX together with a unitary connection A on L induces a connection $\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$. This connection, followed by Clifford multiplication, induces a Spin^c-Dirac operator $D_A : \Gamma(W^+) \to \Gamma(W^-)$. The Seiberg-Witten equations **[10]** are the following pair of equations for a unitary connection A of L and a section Ψ of $\Gamma(W^+)$:

$$\text{(2)} \qquad \begin{cases} D_A\Psi &= 0 \\ \rho(F_A^+) &= i(\Psi \otimes \Psi^*)_0 \end{cases}$$

where F_A^+ is the self-dual part of the curvature of A and $(\Psi \otimes \Psi^*)_0$ is the trace-free part of $(\Psi \otimes \Psi^*)$ which is interpreted as an endomorphism of W^+.

The gauge group $\mathcal{G} := Aut(L) \cong Map(X, S^1)$ acts on the space $\mathcal{A}_X(L) \times \Gamma(W^+)$ by

$$g \cdot (A, \Psi) = \big(g \cdot A \cdot g^{-1}, g \cdot \Psi\big).$$

In particular, if $b_1(X) = 0$, then the gauge group $\mathcal{G}$ is homotopy equivalent to S^1 so that the quotient

$$\mathcal{B}_X^*(L) := \mathcal{A}_X(L) \times \big(\Gamma(W^+) - 0\big)/S^1$$

is homotopy equivalent to $\mathbf{CP}^\infty$. Since the set of solutions is invariant under the action, it induces an orbit space, called the (*Seiberg-Witten*) *moduli space*, denoted by $M_X(L)$, whose formal dimension is

$$\dim M_X(L) = \frac{1}{4}\Big(c_1(L)^2 - 3\sigma(X) - 2e(X)\Big)$$

where $\sigma(X)$ is the signature of X and $e(X)$ is the Euler characteristic of X.

DEFINITION: A solution (A, Ψ) of the Seiberg-Witten equation (2) is called *irreducible* (*reducible*) if $\Psi \not\equiv 0$ ($\Psi \equiv 0$).

Note that if $b^+(X) > 0$ and $M_X(L) \neq \emptyset$, then for a generic metric on X the moduli space $M_X(L)$ contains no reducible solutions, so that it is a compact, smooth manifold of the given dimension. Furthermore the moduli space $M_X(L)$ is orientable and its orientation is determined by a choice of orientation on $\det\big(H^0(X;\mathbf{R})\oplus H^1(X;\mathbf{R}) \oplus H^2_+(X;\mathbf{R})\big)$.

DEFINITION: The *Seiberg-Witten invariant* for X with $b_1(X) = 0$ is the function $SW_X : \mathrm{Spin}^c(X) \to \mathbf{Z}$ defined by

$$SW_X(L) = \begin{cases} 0 & \text{if } \dim M_X(L) < 0 \text{ or odd} \\ \sum\limits_{(A,\Psi)\in M_X(L)} sign(A,\Psi) & \text{if } \dim M_X(L) = 0 \\ \langle \beta^{d_L}, [M_X(L)] \rangle & \text{if } \dim M_X(L) := 2d_L > 0 \quad \text{and even} \end{cases}$$

where $sign(A,\Psi)$ is ± 1 whose sign is determined by an orientation on $M_X(L)$, and β is a generator of $H^2(\mathcal{B}^*_X(L);\mathbf{Z}) \cong H^2(\mathbf{CP}^\infty;\mathbf{Z})$. For convenience, we denote the Seiberg-Witten invariant for X by $SW_X = \sum\limits_L SW_X(L)\cdot e^L$.

Note that if $b^+(X) > 1$, the Seiberg-Witten invariant $SW_X = \sum SW_X(L)\cdot e^L$ is a diffeomorphism invariant, that is, SW_X does not depend on the choice of generic metric on X and generic perturbation of the Seiberg-Witten equation. Furthermore, only finitely many Spin^c-structures on X have a non-zero Seiberg-Witten invariant.

DEFINITION: Let X be an oriented, smooth 4-manifold with $b_1 = 0$ and $b^+ > 1$. We say a cohomology class $c_1(L) \in H^2(X;\mathbf{Z})$ is a *Seiberg-Witten basic class* (for brevity, *SW-basic class*) for X if $SW_X(L) \neq 0$.

DEFINITION: An oriented, smooth 4-manifold X is called a *Seiberg-Witten simple type* (for brevity, *SW-simple type*) if $SW_X(L) = 0$, for all L satisfying $\dim M_X(L) > 0$.

Next we describe a (Seiberg-Witten) gluing theory for computing Seiberg-Witten invariants of a smooth 4-manifold $X = X_+ \cup_Y X_-$ which is separated into two pieces X_+, X_- by an embedded 3-manifold Y. Let (X_R, g_R) be the Riemannian manifold obtained from X by cutting along Y and inserting a cylinder $[-R,R]\times Y$ on which g_R is a product metric. As in Donaldson theory, if the moduli space $M_{X_R}(L)$ is non-empty for all sufficiently large R, then by stretching the neck along Y in X (that is, $R \to \infty$) each solution $(A,\Psi) \in M_X(L)$ is split into three relative solutions

$$((A_+,\Psi_+),(A_0,\Psi_0),(A_-,\Psi_-)) \in M_{X_+}(L|_{X_+}) \times M_{R\times Y}(L|_{R\times Y}) \times M_{X_-}(L|_{X_-}),$$

and conversely any such three relative solutions $(A_+,\Psi_+),(A_0,\Psi_0)$ and (A_-,Ψ_-) induce a global solution $(A_+,\Psi_+)\sharp_{g_1}(A_0,\Psi_0)\sharp_{g_2}(A_-,\Psi_-) \in M_X(L)$, where g_1 and g_2 are gluing parameters. (In general, there is an obstruction to construct a global solution from relative solutions [**3**].) In particular, if the embedded 3-manifold Y in X has a positive scalar curvature metric (for example, $Y = S^3, L(p^2, 1-pq)$), then any such solution $(A_0, \Psi_0) \in M_{R\times Y}(L|_{R\times Y})$ is reducible. That is,

$$\begin{aligned} M_{R\times Y}(L|_{R\times Y}) &= \{(A_0, 0) : A_0 \text{ is an ASD U(1) - connection on } Y\} \\ &\cong H^1(Y;\mathbf{R})/H^1(Y;\mathbf{Z}). \end{aligned}$$

For example, if $Y = S^3$ or $L(p^2, 1-pq)$, then $M_{R\times Y}(L|_{R\times Y})$ is a single reducible solution. Furthermore, since L is a $U(1)$-bundle, the gluing parameters are S^1. In summary, we have

Proposition 3.1. *If a smooth 4-manifold X is split into two pieces X_+ and X_- by an embedded 3-manifold $Y = S^3$ or $L(p^2, 1-pq)$, then each solution $(A, \Psi) \in M_X(L)$ can be obtained from two relative solutions $((A_+, \Psi_+), (A_-, \Psi_-)) \in M_{X_+}(L|_{X_+}) \times M_{X_-}(L|_{X_-})$ and*

$$\dim M_X(L) = \dim M_{X_+}(L|_{X_+}) + \dim M_{X_-}(L|_{X_-}) + 1$$

where $M_{X_i}(L|_{X_i})$ is the set of solutions (modulo the gauge group) which converge asymptotically to a reducible solution in $M_Y(L|_Y)$.

Note that if $\dim M_{X_-}(L|_{X_-}) < 0$, then $M_{X_-}(L|_{X_-})$ consists of reducible solutions. The technical part in the rest of this section is to show that $\dim M_{B_{p,q}}(L|_{B_{p,q}}) = -1$ and $\dim M_{C_{p,q}}(L|_{C_{p,q}}) \leqslant -1$, so that both $M_{B_{p,q}}(L|_{B_{p,q}})$ and $M_{C_{p,q}}(L|_{C_{p,q}})$ consist of a single reducible solution. Before doing this, as a warm-up, we can get a well-known blow-up formula [**4**] for Seiberg-Witten invariants by using index computations.

Proposition 3.2. *If X is a SW-simple type 4-manifold, then the blow-up $\widetilde{X} \equiv X\sharp\overline{\mathbf{CP}^2}$ is also of SW-simple type, and the Seiberg-Witten invariants of $\widetilde{X} \equiv X\sharp\overline{\mathbf{CP}}^2$ are*

$$SW_{\widetilde{X}} = SW_X \cdot \left(e^E + e^{-E}\right)$$

where E is the exceptional divisor of $\overline{\mathbf{CP}}^2$.

Proof: Note that a characteristic line bundle on $\widetilde{X} \equiv X\sharp\overline{\mathbf{CP}}^2$ is of the form $L + (2k+1)E$, where L is a characteristic line bundle on X and $k \in \mathbf{Z}$. (We identify the exceptional divisor E with its corresponding line bundle on $\overline{\mathbf{CP}}^2$.) Suppose $\widetilde{L} := L + (2k+1)E$ is a characteristic line bundle on $\widetilde{X}$ such that $SW_{\widetilde{X}}\left(\widetilde{L}\right) \neq 0$. Then, when splitting apart $\widetilde{X}$ along S^3, Proposition 3.1 implies that any solution in $M_{\widetilde{X}}\left(\widetilde{L}\right)$ can be obtained from two relative solutions which are identified with two (absolute) solutions in $M_X(L) \times M_{\overline{\mathbf{CP}}^2}((2k+1)E)$. (Since stretching the neck along S^3 corresponds to choosing a sequence of metric so that the neck is pinched down to a point, the last statement follows from a simple removable singularities argument.) But since

$$\begin{aligned}\dim M_{\overline{\mathbf{CP}}^2}((2k+1)E) &= 2\cdot \operatorname{ind} D_A|_{\overline{\mathbf{CP}}^2} + \operatorname{ind}\left(d^+ + d^*\right)|_{\overline{\mathbf{CP}}^2} \\ &= 2\cdot\left(e^{((2k+1)E)/2}\cdot \widehat{A}\left(\overline{\mathbf{CP}}^2\right)\right)\cdot[\overline{\mathbf{CP}}^2] + \left(h^1 - h^0 - h^+\right)\left(\overline{\mathbf{CP}}^2\right) \\ &= 2\cdot \int_{\overline{\mathbf{CP}}^2}\left(\frac{((2k+1)E)^2}{8} - \frac{p_1}{24}\right) - 1 \\ &= 2\cdot\frac{-\left(4k^2+4k\right)}{8} - 1 \\ &\leqslant -1.\end{aligned}$$

(In case $Y = S^3$, $\operatorname{ind} D_A$ has no boundary terms.) Thus $M_{\overline{\mathbf{CP}}^2}((2k+1)E)$ consists of a single reducible solution, and $M_{\widetilde{X}}\left(\widetilde{L}\right)$ can be identified with $M_X(L)$. Furthermore, since

$$\begin{aligned}\dim M_{\widetilde{X}}\left(\widetilde{L}\right) &= \frac{1}{4}\left\{(c_1(L) + (2k+1)E)^2 - \left(3\sigma\left(\widetilde{X}\right) + 2e\left(\widetilde{X}\right)\right)\right\} \\ &= \frac{1}{4}\{c_1(L)^2 - (3\sigma(X) + 2e(X))\} - \left(k^2 + k\right) \\ &= \dim M_X(L) - \left(k^2+k\right),\end{aligned}$$

the SW-simple type condition on X and $SW_{\widetilde{X}}\left(\widetilde{L}\right) \neq 0$ imply that $\dim M_{\widetilde{X}}\left(\widetilde{L}\right) = 0$ and $k = 0$ or -1. Hence $\widetilde{X}$ is also of SW-simple type and $SW_X(L) = SW_{\widetilde{X}}(L+E) = SW_{\widetilde{X}}(L-E)$. ∎

In order to compute $\operatorname{ind} D_A$ on $B_{p,q}$ and $C_{p,q}$, we need the following two elementary trigonometric computations.

LEMMA 3.1. *For relatively prime integers p and q, and $z = e^{(2\pi i)/p^2}$*

$$\sum_{k=1}^{p^2-1}\frac{z^{tpk}}{\left(z^k-1\right)\left(z^{(pq-1)k}-1\right)} = \sum_{k=1}^{p^2-1}\frac{1}{\left(z^k-1\right)\left(z^{(pq-1)k}-1\right)}, \quad \text{for all } t\in\mathbf{Z}.$$

PROOF: There exist integers r and s satisfying $rp + sq = 1$; so $z^{tpk} = z^{stpqk}$. Thus it suffices to show

$$\sum_{k=1}^{p^2-1}\frac{z^{tpqk}-1}{\left(z^k-1\right)\left(z^{(pq-1)k}-1\right)} = 0, \quad \text{for all } t\in\mathbf{Z}.$$

Given $t \in \mathbf{Z}$ and setting $w = z^{pq-1}$,

$$\begin{aligned}
&\sum_{k=1}^{p^2-1} \frac{z^{(t+1)pqk} - z^{tpqk}}{(z^k-1)\left(z^{(pq-1)k}-1\right)} \\
&= \sum_{k=1}^{p^2-1} \frac{z^{tpqk}\{(z^k-1)(w^k-1)\} + z^{tpqk}\{(w^k-1)+(z^k-1)\}}{(z^k-1)(w^k-1)} \\
&= \sum_{k=1}^{p^2-1}\left\{z^{tpqk} + \frac{2}{(z^k-1)}\right\} + \sum_{k=1}^{p^2-1}\left\{\frac{(z^{tpqk}-1)}{(z^k-1)} + \frac{(w^{-(pq+1)tpqk}-1)}{(w^k-1)}\right\} \\
&= \sum_{k=1}^{p^2-1}\left\{z^{tpqk} + \frac{2}{(z^k-1)}\right\} + \sum_{k=1}^{p^2-1}\left\{\frac{(z^{tpqk}-1)}{(z^k-1)} - \frac{(w^{tpqk}-1)}{w^{tpqk}(w^k-1)}\right\} \\
&= \sum_{k=1}^{p^2-1}\left\{z^{tpqk} + \frac{2}{(z^k-1)}\right\} + \sum_{l=0}^{tpq-1}\sum_{k=1}^{p^2-1}\left\{z^{lk} - (w^{-1})^{(tpq-l)k}\right\} \\
&= \sum_{k=1}^{p^2-1}\left\{z^{tpqk} + \frac{2}{(z^k-1)}\right\} + \sum_{l=0}^{tpq-1}\sum_{k=1}^{p^2-1} z^{lk} - \sum_{l=1}^{tpq}\sum_{k=1}^{p^2-1} (w^{-1})^{lk} \\
&= \sum_{k=1}^{p^2-1}\left\{z^{tpqk} + \frac{2}{(z^k-1)}\right\} + \sum_{l=0}^{tpq-1}\sum_{k=1}^{p^2-1} z^{lk} - \sum_{l=1}^{tpq}\sum_{k=1}^{p^2-1} z^{lk} \\
&= \sum_{k=1}^{p^2-1} \frac{2}{(z^k-1)} + (p^2-1) \\
&= 0.
\end{aligned}$$

Hence the lemma follows by induction on t. ∎

LEMMA 3.2. *For relatively prime integers p and q, and $z = e^{(2\pi i)/p^2}$*

$$s(1-pq,p^2) = \sum_{k=1}^{p^2-1} \cot\left(\frac{\pi k}{p^2}\right)\cdot\cot\left(\frac{\pi k(1-pq)}{p^2}\right) = \frac{2}{3}(1-p^2),$$

equivalently, $$\sum_{k=1}^{p^2-1} \frac{1}{(z^k-1)\left(z^{(pq-1)k}-1\right)} = \frac{1}{12}(p^2-1)$$

Note that this lemma can also be proved by using a different method [**7**].

PROOF: An easy computation shows that

$$s(1-pq,p^2) = (1-p^2) + \sum_{k=1}^{p^2-1} \frac{4}{(z^k-1)\left(z^{(pq-1)k}-1\right)}$$

Note that for $0 \leqslant t \leqslant p-1$ and $w = z^p$,

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{w^{tk}-1}{(w^k-1)(w^{-k}-1)} &= \sum_{l=0}^{t-1}\sum_{k=1}^{p-1} \frac{w^{lk}}{(w^{-k}-1)} \\
&= \sum_{k=1}^{p-1} \frac{-t}{(w^k-1)} - \sum_{l=1}^{t}\sum_{k=1}^{p-1} \frac{(w^{lk}-1)}{(w^k-1)} \\
&= \frac{t(p-1)}{2} - \sum_{l=1}^{t}((p-1)-(l-1)) \\
&= \frac{t^2-tp}{2}.
\end{aligned}$$

(The third equality follows from the fact that $\sum_{k=1}^{p-1} w^{lk} = -1$, for $1 \leqslant l \leqslant p-1$.) Hence by using the equality $\sum_{t=0}^{p-1} w^{tk} = 0$ for $1 \leqslant k \leqslant p-1$,

$$\begin{aligned}
0 &= \sum_{t=1}^{p-1}\sum_{k=1}^{p-1} \frac{w^{tk}}{(w^k-1)(w^{-k}-1)} + \sum_{k=1}^{p-1} \frac{1}{(w^k-1)(w^{-k}-1)} \\
&= \sum_{t=1}^{p-1} \frac{(t^2-tp)}{2} + \sum_{k=1}^{p-1} \frac{p}{(w^k-1)(w^{-k}-1)},
\end{aligned}$$

so that

$$\frac{p}{12}(p^2-1) = \sum_{k=1}^{p-1} \frac{p}{(w^k-1)(w^{-k}-1)}.$$

Finally by using the fact that $\sum_{l=0}^{p-1} z^{lpqk} = 0$ if $k \neq tp$ and $\sum_{l=0}^{p-1} z^{lpqk} = p$ if $k = tp$, and by Lemma 3.1, we have

$$\begin{aligned}
\sum_{k=1}^{p^2-1} \frac{p}{(z^k-1)\left(z^{(pq-1)k}-1\right)} &= \sum_{l=0}^{p-1}\sum_{k=1}^{p^2-1} \frac{z^{lpk}}{(z^k-1)\left(z^{(pq-1)k}-1\right)} \\
&= \sum_{t=1}^{p-1} \frac{p}{(z^{tp}-1)\left(z^{(pq-1)tp}-1\right)} \\
&= \sum_{t=1}^{p-1} \frac{p}{(w^t-1)(w^{-t}-1)} \\
&= \frac{p}{12}(p^2-1).
\end{aligned}$$

□

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Proposition 3.3. *For any characteristic line bundle L_B on $B_{p,q}$ with a cylindrical end*

$$B^+_{p,q} = B_{p,q} \cup L(p^2, 1-pq) \times [1, \infty)$$

$\dim M_{B^+_{p,q}}(L_B) = -1$; so the moduli space $M_{B^+_{p,q}}(L_B)$ consists of a single reducible solution.

Proof: It suffices to show that $\operatorname{ind}\left(D_A|_{B^+_{p,q}}\right) = 0$ because

$$\begin{aligned}\dim M_{B^+_{p,q}}(L_B) &= 2\cdot \operatorname{ind}\left(D_A|_{B^+_{p,q}}\right) + \operatorname{ind}\left(d^+ + d^*\right)|_{B^+_{p,q}} \\ &= 2\cdot \operatorname{ind}\left(D_A|_{B^+_{p,q}}\right) + \left(b^1 - b^0 - b^+\right)(B^+_{p,q}) \\ &= 2\cdot \operatorname{ind}\left(D_A|_{B^+_{p,q}}\right) \ -1\end{aligned}$$

where A is a $U(1)$-connection on $L_B \to B^+_{p,q}$. Now compute

$$\begin{aligned}\operatorname{ind}\left(D_A|_{B^+_{p,q}}\right) &= \left(e^{(c_1(L_B))/2}\cdot \widehat{A}(B^+_{p,q})\right)\cdot [B^+_{p,q}] \\ &= \int_{B^+_{p,q}} \left(\frac{c_1(L_B)^2}{8} - \frac{p_1}{24}\right) - \left(\frac{h+\eta(0)}{2}\right).\end{aligned}$$

Since L_B is a flat connection on $B^+_{p,q}$ the first term $\left(c_1(L_B)^2\right)/8 = 0$, and the second term can be computed by using [**1**, Proposition 2.12]

$$0 = \sigma(B^+_{p,q}) = \int_{B^+_{p,q}} \left(\frac{p_1}{3}\right) + \frac{1}{p^2}\sum_{k=1}^{p^2-1} \cot\left(\frac{\pi k}{p^2}\right)\cdot \cot\left(\frac{\pi k(1-pq)}{p^2}\right).$$

Hence, by Lemma 3.2,

$$\int_{B^+_{p,q}} \left(\frac{p_1}{24}\right) = \frac{-1}{8p^2}\cdot s(1-pq, p^2) = \frac{1}{12p^2}(p^2-1).$$

The boundary term, $(h+\eta(0))/2$, can also be computed by using the Atiyah-Singer fixed point theorem [**9**, Section 19] for a Spin^c-Dirac operator D_A on $D^4/\mathbf{Z}_{p^2} \cong$ cone on $L(p^2, 1-pq)$:

$$\begin{aligned}\frac{h+\eta(0)}{2} &= \frac{-1}{p^2}\sum_{g\in \mathbf{Z}_{p^2}-\{0\}} \operatorname{Spin}\left(g, D^4\right) \\ &= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{\left(e^{\pi ki/p^2} - e^{-\pi ki/p^2}\right)\left(e^{(1-pq)\pi ki/p^2} - e^{-(1-pq)\pi ki/p^2}\right)\cdot e^{mp\cdot \pi ki/p^2}}{\left(1-e^{\pi ki/p^2}\right)\left(1-e^{-\pi ki/p^2}\right)\left(1-e^{(1-pq)\pi ki/p^2}\right)\left(1-e^{-(1-pq)\pi ki/p^2}\right)} \\ &= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{e^{mp\cdot \pi ki/p^2}}{\left(e^{\pi ki/p^2} - e^{-\pi ki/p^2}\right)\left(e^{(1-pq)\pi ki/p^2} - e^{-(1-pq)\pi ki/p^2}\right)}\end{aligned}$$

where $c_1\left(L_B|_{L(p^2,1-pq)}\right) = mp \in H^2(L(p^2,1-pq);\mathbf{Z}) \cong \mathbf{Z}_{p^2}$ (Lemma 2.2). Since L_B is a characteristic line bundle, we can always choose an integer m so that $m+q$ is even. (If p and $m+q$ are odd, choose $m+p+q \equiv m+q \pmod p$. If p is even, then m and q are odd.) By setting $z := e^{2\pi i/p^2}$ and $t := (m+q)/2 \in \mathbf{Z}$, we have

$$\begin{aligned}
\frac{h+\eta(0)}{2} &= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{e^{\pi(m+q)ki/p}}{\left(e^{2\pi ki/p^2}-1\right)\left(e^{2\pi(pq-1)ki/p^2}-1\right)} \\
&= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{z^{tpk}}{(z^k-1)\left(z^{(pq-1)k}-1\right)} \\
&= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{1}{(z^k-1)\left(z^{(pq-1)k}-1\right)} && \text{(by Lemma 3.1)} \\
&= \frac{1}{12p^2}\left(1-p^2\right) && \text{(by Lemma 3.2).}
\end{aligned}$$

Combining these computations we get $\operatorname{ind}\left(D_A|_{B^+_{p,q}}\right) = 0$. □

REMARK. In the proof of Proposition 3.3 above, if both p and m are even (in particular if $m=0$), a similar computation shows that $\operatorname{ind} D_A$ on $B_{p,q}$ is not an integer. This contradiction means that $B_{p,q}$ is not spin for p even (see Lemma 2.3).

COROLLARY 3.1. *For any characteristic line bundle L_C on $C^+_{p,q} = C_{p,q} \cup L(p^2,1-pq)\times[1,\infty)$, $\dim M_{C^+_{p,q}}(L_C)$ is odd and $\leqslant -1$; so the moduli space $M_{C^+_{p,q}}(L_C)$ consists of a single reducible solution.*

PROOF: Since $\operatorname{ind}\left(d^+ + d^*|_{C^+_{p,q}}\right) = (b^1 - b^0 - b^+)(C^+_{p,q}) = -1$, in the same way as the proof above, it suffices to show that $\operatorname{ind}\left(D_A|_{C^+_{p,q}}\right) \leqslant 0$. Since $X = C^+_{p,q} \cup_L \overline{B^+_{p,q}}$ is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$ with $k = b_2(C_{p,q})$, for any characteristic line bundle L on X, $c_1(L)^2 \leqslant -k$ and

$$\operatorname{ind}\left(D_A|_{C^+_{p,q}}\right) + \operatorname{ind}\left(D_A|_{\overline{B^+_{p,q}}}\right) = \operatorname{ind}(D_A|X) = \int_X \frac{\left(c_1(L)^2+k\right)}{8} \leqslant 0.$$

Hence $\operatorname{ind}(D_A|_{C^+_{p,q}}) \leqslant -\operatorname{ind}(D_A|_{B^+_{p,q}}) = 0$. □

LEMMA 3.3. *Let X be a smooth 4-manifold containing a configuration $C_{p,q}$, that is, $X = X_0 \cup_{L(p^2,1-pq)} C_{p,q}$, and let $X_{p,q}$ be its rational blow-down. Then a line bundle L on $X_{p,q}$ is characteristic if and only if both $L|_{X_0}$ on X_0 and $L|_{B_{p,q}}$ on $B_{p,q}$ are characteristic.*

PROOF: Since $H^1(B_{p,q};\mathbf{Z}_2) \to H^1(L(p^2,1-pq);\mathbf{Z}_2)$ is surjective, $i^* \oplus j^*$: $H^2(X_{p,q};\mathbf{Z}_2) \to H^2(X_0;\mathbf{Z}_2) \oplus H^2(B_{p,q};\mathbf{Z}_2)$ is injective. Hence the proof follows from the following commutative diagram

$$\begin{array}{ccccc} & & 0 \longrightarrow & H^2(X_{p,q};\mathbf{Z}) & \longrightarrow & H^2(X_0;\mathbf{Z}) \oplus H^2(B_{p,q};\mathbf{Z}) \\ & & & \downarrow & & \downarrow \\ H^1(L(p^2,1-pq);\mathbf{Z}_2) & \longrightarrow & & H^2(X_{p,q};\mathbf{Z}_2) & \xrightarrow{i^*\oplus j^*} & H^2(X_0;\mathbf{Z}_2) \oplus H^2(B_{p,q};\mathbf{Z}_2) \end{array}$$

∎

THEOREM 3.1. *Suppose X is a smooth 4-manifold which contains a configuration $C_{p,q}$. If L is a characteristic line bundle on X such that $SW_X(L) \neq 0$, $(L|_{C_{p,q}})^2 = -b_2(C_{p,q})$ and $c_1\left(L|_{L(p^2,1-pq)}\right) = mp \in \mathbf{Z}_{p^2} \cong H^2(L(p^2,1-pq);\mathbf{Z})$ with $m \equiv (p-1) \pmod 2$, then L induces a characteristic line bundle $\overline{L}$ on $X_{p,q}$ such that $SW_{X_{p,q}}(\overline{L}) = SW_X(L)$.*

PROOF: The condition $c_1\left(L|_{L(p^2,1-pq)}\right) = mp$ with $m \equiv (p-1) \pmod 2$ and Lemma 2.2 imply that the characteristic line bundle $L|_{X_0}$ on X_0 extends uniquely to a characteristic line bundle $\overline{L}$ on $X_{p,q}$. Then the rest of proof is the same argument as the proof of [**5**, Theorem 8.2]. That is, first we study the solutions of Seiberg-Witten equations on X for L by pulling apart $X = X_0 \cup_{L(p^2,1-pq)} C_{p,q}$ along $L(p^2,1-pq)$. Then Proposition 3.1 and Corollary 3.1 imply that each solution in $M_X(L)$ can be obtained by gluing a solution $(A_{X_0},\Psi_{X_0}) \in M_{X_0}(L|_{X_0})$ with a unique reducible solution $(A_{C_{p,q}},0) = M_{C_{p,q}}(L|_{C_{p,q}})$. But, not every solution in $M_{X_0}(L|_{X_0})$ produces a global solution in $M_X(L)$. Explicitly, using Corollary 3.1, the inequality

$$\begin{aligned} 2d_L = \dim M_X(L) &= \dim M_{X_0}(L|_{X_0}) + \dim M_{C_{p,q}}(L|_{C_{p,q}}) + 1 \\ &\leqslant \dim M_{X_0}(L|_{X_0}) = 2d_{L|_{X_0}} \end{aligned}$$

implies that there is an obstruction bundle ξ of rank $d_{L|_{X_0}} - d_L$ associated to the basepoint fibration over $M_{X_0}(L|_{X_0})$ such that the zero set of a generic section of ξ is homologous to $M_X(L)$ in $\mathcal{B}^*_X(L)$ [**3**, Theorem 4.53], or [**4**, Section 4]. Hence

$$SW_X(L) = \langle \beta^{d_L}, [M_X(L)] \rangle = \langle \beta^{d_L}, \beta^{d_{L|_{X_0}}-d_L} \cap [M_{X_0}(L|_{X_0})] \rangle = \langle \beta^{d_{L|_{X_0}}}, [M_{X_0}(L|_{X_0})] \rangle$$

where β is a generator of $H^2(\mathcal{B}^*_X(L);\mathbf{Z})$. Similarly, since $\dim M_{B_{p,q}}(\overline{L}|_{B_{p,q}}) = -1$ by Proposition 3.3, the same argument as above shows

$$SW_{X_{p,q}}(\overline{L}) = \langle \beta^{d_{L|_{X_0}}}, [M_{X_0}(L|_{X_0})] \rangle$$

so that $SW_{X_{p,q}}(\overline{L}) = SW_X(L)$. ∎

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Corollary 3.2. *If two characteristic line bundles L and L' on X satisfying the hypothesis in Theorem 3.1 induce the same characteristic line bundle $\overline{L}$ on $X_{p,q}$, then $SW_X(L) = SW_X(L')$.*

Freedman's classification of simply connected topological 4-manifolds implies that $X \equiv C_{p,q} \cup_L \overline{B_{p,q}}$ is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$ with $k = b_2(C_{p,q})$. Each generator e_i of $H^2(X;\mathbf{Z})$ when restricted to $B_{p,q}$ has the boundary value $\partial(e_i|_{B_{p,q}}) = mp \in H^2(L(p^2, 1-pq);\mathbf{Z})$ for some m. We impose the following condition $(*)$ on $C_{p,q}$:

$$(*) \quad \left\{\partial\Big(\sum_{i=1}^{k} \varepsilon_i e_i|_{B_{p,q}}\Big) : \varepsilon_i = \pm 1, \forall i\right\} = \{mp : -(p-1) \leqslant m \leqslant (p-1) \text{ and } m \equiv (p-1) \pmod 2\}.$$

All known configurations $C_{p,q}$ satisfy the condition $(*)$ above. (One expects that all relatively prime integers (p,q) satisfy the condition $(*)$.) Under this assumption, we prove

Lemma 3.4. *Suppose X is a simply connected smooth 4-manifold which contains a configuration $C_{p,q}$ satisfying the condition $(*)$, and let $X_{p,q}$ be its rational blow-down. If $\overline{L}$ is a characteristic line bundle on $X_{p,q}$, there exists a characteristic line bundle L on X such that $L|_{X_0} = \overline{L}|_{X_0}$ and $c_1(L|_{C_{p,q}})^2 = -k$, where $k = b_2(C_{p,q})$.*

Proof: The condition $(*)$ on $C_{p,q}$ implies that there exists $\varepsilon_i = \pm 1$, for $1 \leqslant i \leqslant k$, such that $\partial\Big(\sum_{i=1}^{k} \varepsilon_i e_i|_{B_{p,q}}\Big) = mp = \partial c_1(\overline{L}|_{B_{p,q}})$. Since the corresponding line bundle, denoted by the same notation $\sum_{i=1}^{k} \varepsilon_i e_i$, is characteristic on $C_{p,q} \cup_L \overline{B_{p,q}}$ which is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$, its restriction $\sum_{i=1}^{k} \varepsilon_i e_i|_{C_{p,q}}$ is also characteristic on $C_{p,q}$ and $\Big(\sum_{i=1}^{k} \varepsilon_i e_i|_{C_{p,q}}\Big)^2 = \Big(\sum_{i=1}^{k} \varepsilon_i e_i\Big)^2 - \Big(\sum_{i=1}^{k} \varepsilon_i e_i|_{\overline{B_{p,q}}}\Big)^2 = \Big(\sum_{i=1}^{k} \varepsilon_i e_i\Big)^2 = -k$. Now define a line bundle L on X by

$$L = \begin{cases} \overline{L}|_{X_0} & \text{on} \quad X_0 \\ \sum_{i=1}^{k} \varepsilon_i e_i|_{C_{p,q}} & \text{on} \quad C_{p,q} \end{cases}$$

Then L has the desired properties except (possibly) characteristic, that is, if p is odd, then L is automatically a characteristic line bundle on X, so we are done. If p is even, we can change L (see below) so that L is characteristic on X satisfying the same properties.

Suppose p is even.

$$\begin{array}{ccccc} & 0 & \longrightarrow & H^2(X;\mathbf{Z}) & \longrightarrow & H^2(X_0;\mathbf{Z})\oplus H^2(C_{p,q};\mathbf{Z}) \\ & & & \downarrow h_* & & \downarrow \\ H^1(L(p^2,1-pq);\mathbf{Z}_2) & \xrightarrow{\delta} & & H^2(X;\mathbf{Z}_2) & \xrightarrow{i^*\oplus j^*} & H^2(X_0;\mathbf{Z}_2)\oplus H^2(C_{p,q};\mathbf{Z}_2) \end{array}$$

Since X is simply connected, $H_1(X_0;\mathbf{Z}) \cong \mathbf{Z}_t$ for some t dividing p^2. If t is even, then $i^*\oplus j^* : H^2(X;\mathbf{Z}_2) \to H^2(X_0;\mathbf{Z}_2)\oplus H^2(C_{p,q};\mathbf{Z}_2)$ is injective so that L is characteristic. If t is odd, then $i^*\oplus j^*$ is not injective, and in this case $h_*(c_1(L)) = w_2(X)$ or $w_2(X)+\delta(1)$.

Since $C_{p,q}$ satisfies the condition $(*)$, there exists $\delta_i = \pm 1$ satisfying $\sum_{i=1}^{k} \delta_i e_i|_{C_{p,q}} = (p-m)p$. Then setting $\gamma_i \equiv (\varepsilon_i + \delta_i)/2$ we have

$$(1)\quad \partial\Big(\sum_{i=1}^{k} \gamma_i e_i|_{C_{p,q}}\Big) = (p/2)p \neq 0,$$

$$(2)\quad \partial\Big(\sum_{i=1}^{k} (\varepsilon_i - 2\gamma_i) e_i|_{C_{p,q}}\Big) = \partial\Big(\sum_{i=1}^{k} \varepsilon_i e_i|_{C_{p,q}}\Big) = mp,$$

$$(3)\quad \sum_{i=1}^{k} (\varepsilon_i - 2\gamma_i) e_i|_{C_{p,q}} = \sum_{i=1}^{k} \varepsilon_i' e_i|_{C_{p,q}}, \quad \text{for some } \varepsilon_i' = \pm 1.$$

Hence there exists a bundle L' on X such that $L'|_{X_0} = L|_{X_0}$ and $L'|_{C_{p,q}} = \sum_{i=1}^{k} (\varepsilon_i - 2\gamma_i) e_i|_{C_{p,q}}$. Then we claim either L or L' is characteristic: Suppose neither L nor L' is characteristic, that is, $h_*(c_1(L)) = h_*(c_1(L')) = w_2(X)+\delta(1)$. Then $h_*(L-L') = 0$, so that there exists an element $\alpha \in H^2(X;\mathbf{Z})$ satisfying $2\alpha = L - L'$. Since both $H^2(X_0;\mathbf{Z})$ and $H^2(C_{p,q};\mathbf{Z})$ are 2-torsion free,

$$2(\alpha|_{X_0}, \alpha|_{C_{p,q}}) = (i^*\oplus j^*)(2\alpha) = (i^*\oplus j^*)(L-L') = 2\Big(0, \sum_{i=1}^{k} \gamma_i e_i|_{C_{p,q}}\Big)$$

implies $\alpha|_{X_0} = 0$ and $\alpha|_{C_{p,q}} = \sum_{i=1}^{k} \gamma_i e_i|_{C_{p,q}}$ which contradicts $\partial\big(\sum_{i=1}^{k} \gamma_i e_i|_{C_{p,q}}\big) = (p/2)p \neq 0$. ∎

Finally, by using the same argument as in the proof of Theorem 3.1 with the characteristic line bundle L on X constructed in the Lemma 3.4 above, we get our main theorem.

THEOREM 3.2. *If a simply connected smooth 4-manifold X contains a configuration $C_{p,q}$ satisfying the condition $(*)$, then the Seiberg-Witten invariants of $X_{p,q}$*

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are completely determined by those of X. That is, for any characteristic line bundle $\overline{L}$ on $X_{p,q}$ with $SW_{X_{p,q}}(\overline{L}) \neq 0$, there exists a characteristic line bundle L on X such that $SW_X(L) = SW_{X_{p,q}}(\overline{L})$. Furthermore, if X is of SW- simple type, then $X_{p,q}$ is also of SW-simple type.

4. Examples

In this section we apply the result of the previous section to several examples of rational blow-downs. We compute the Seiberg-Witten invariants of a manifold constructed from $E(n)$ via blowing up and rationally blowing down.

Example 1. Consider a 4-manifold $X \equiv E(3)\sharp 2\overline{\mathbf{CP}}^2$ constructed by the following blowing up process (Figure 3):

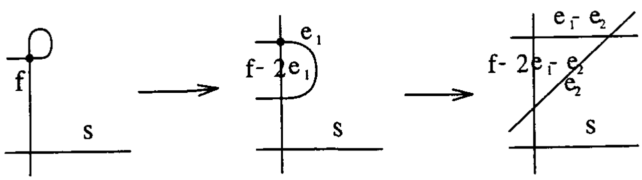
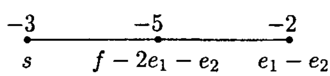


Figure 3

Then we get a configuration $C_{5,2} \subset X$



where s is a section in $E(3)$ and e_i $(i = 1, 2)$ is the exceptional divisor in $\overline{\mathbf{CP}}^2$. Since SW-basic classes in $E(3)$ are $\pm f$, up to sign the SW-basic classes of X are of the form

$$L = f + \varepsilon_1 e_1 + \varepsilon_2 e_2 \qquad (\varepsilon_i = \pm 1).$$

By using boundary values (see equation (1)), compute $L|_{C_{5,2}}$ and $\partial(L|_{C_{5,2}})$

$$\begin{aligned} L|_{C_{5,2}} &= (L \cdot u_1)\gamma_1 + (L \cdot u_2)\gamma_2 + (L \cdot u_3)\gamma_3 \\ &= (\varepsilon_2 - \varepsilon_1)\gamma_1 + (2\varepsilon_1 + \varepsilon_2)\gamma_2 + \gamma_3, \\ \partial(L|_{C_{5,2}}) &= (\varepsilon_2 - \varepsilon_1) + 2(2\varepsilon_1 + \varepsilon_2) + 9 \\ &= 3(\varepsilon_1 + \varepsilon_2) + 9. \end{aligned}$$

Then $\partial(L|_{C_{5,2}})$ is a multiple of $p = 5$ if and only if $\varepsilon_1 = \varepsilon_2 = 1$. Hence by Theorem 3.1, only $L = f + e_1 + e_2$ descends to a SW-basic class $\overline{L}$ of $X_{5,2}$, and by Theorem 3.2, $\overline{L}$

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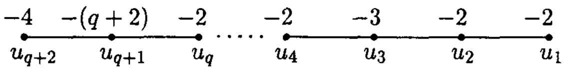
is the only SW-basic class of $X_{5,2}$. Since $c_1(\overline{L})^2 = c_1(L)^2 - c_1(L|_{C_{5,2}})^2 = -2+3 = 1$, $X_{5,2}$ is a SW-simple type 4-manifold with $c_1^2 = 1$ which has one basic class $\overline{L} = \overline{f+e_1+e_2}$ (up to sign) and its Seiberg-Witten invariant is $SW_{X_{5,2}}(\overline{L}) = SW_X(L) = 1$.

Next, let us consider a configuration $C_{4q-1,q}$



whose boundary values (see equation (1)) are given by

$$\partial\gamma_i = \begin{cases} i & i = 1,2 \\ 4i-9 & i = 3,\dots,q+1 \\ (4q-1)q-1 & i = q+2\,. \end{cases}$$

Then we have

Proposition 4.1. *Suppose X is a simply connected smooth 4-manifold containing a configuration $C_{p,q}$ $(p = 4q-1)$. If each u_i satisfies $|L\cdot u_i| + u_i^2 \leqslant -2$, for each basic class L in X, then the Seiberg-Witten invariants of $X_{p,q}$ are given by*

$$SW_{X_{p,q}}(\overline{L}) = \begin{cases} SW_X(L) & \text{if } L\cdot u_3 = \varepsilon,\ \ L\cdot u_{q+1} = \varepsilon q \ \text{ and } \ L\cdot u_{q+2} = 2\varepsilon \quad (\varepsilon = \pm 1) \\ 0 & \text{otherwise}\,. \end{cases}$$

Remark. The hypothesis, $|L\cdot u_i| + u_i^2 \leqslant -2$, in Proposition 4.1 above comes from the adjunction inequality in [4]. Our assumption is that the u_i are generic in the sense that they do not fall into the special case of [**4**, Theorem 1.3].

Proof: The condition $|L\cdot u_i| + u_i^2 \leqslant -2$ implies $L\cdot u_i = 0$ $(i = 1,2,4,\dots,q)$, so that

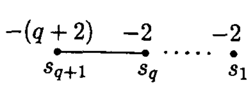
$$\begin{aligned} L|_{C_{p,q}} &= (L\cdot u_3)\gamma_3 + (L\cdot u_{q+1})\gamma_{q+1} + (L\cdot u_{q+2})\gamma_{q+2} \\ \partial(L|_{C_{p,q}}) &= 3(L\cdot u_3) + (4q-5)(L\cdot u_{q+1}) + (pq-1)(L\cdot u_{q+2}) \\ &\equiv 3(L\cdot u_3) - 4(L\cdot u_{q+1}) - (L\cdot u_{q+2}) \pmod p. \end{aligned}$$

Since $L|_{C_{p,q}}$ is characteristic, the condition $\partial(L|_{C_{p,q}}) \equiv 0 \pmod p$ in Theorem 3.1 implies that only basic class $\overline{L}$ in $X_{p,q}$ comes from L of X satisfying

$$L\cdot u_3 = \varepsilon,\ L\cdot u_{q+1} = \varepsilon q \text{ and } L\cdot u_{q+2} = 2\varepsilon \quad (\varepsilon = \pm 1).$$

The rest of the proof follows from Theorem 3.2. ∎

Example 2. Let $X \equiv E(q+2)\sharp 2\overline{\mathbf{CP}}^2$ be a manifold constructed as follows: Consider the following configuration in $E(q+2)$



where $f \cdot s_{q+1} = 1$ and $f \cdot s_i = 0$, for $i = 1, \cdots, q$. (One can choose such a configuration lying in the canonical resolution Q of a singularity of $z_1^2 + z_2^{2q+3} + z_3^{4q+5} = 0$ in $\mathbf{C}^3$. Note that an elliptic surface $E(q+2)$, as a genus $q+1$ Lefschetz fibration, can be constructed as follows:

$$E(q+2) \cong Q \cup_{\Sigma(2,2q+3,4q+5)} C(2,2q+3) \sharp_{\Sigma} C(2,2q+3) \cup_{\Sigma(2,2q+3,4q+5)} Q$$

where $C(2,2q+3)$ is a blow-up of the manifold obtained from $+1$ surgery on the $(2,2q+3)$ torus knot and Σ is an embedded surface of genus $q+1$ and self-intersection 0 in $C(2,2q+3)$.) By blowing up the double point of a nodal fibre f in $E(q+2)$ and a regular point in s_3, we have a configuration $C_{4q-1,q} \subset X$ such that

$$u_{q+2} = f - 2e_1,\ u_3 = s_3 - e_2 \text{ and } u_i = s_i,\ i \neq 3, q+2.$$

Since the SW-basic classes of X have the form

$$L = kf + \varepsilon_1 e_1 + \varepsilon_2 e_2 \ (|k| \leqslant q,\ k \equiv q \pmod 2 \text{ and } \varepsilon_i = \pm 1)$$

this example satisfies the hypothesis of the Proposition 4.1 above. It follows that $X_{p,q}$ has one basic class $\overline{L} = \overline{qf + e_1 + e_2}$ (up to sign) with $c_1(\overline{L})^2 = q$. Hence $X_{p,q}$ is a SW-simple type irreducible smooth 4-manifold lying in $c_1^2 = \chi - 2$ which has one basic class and cannot admit a complex structure.

EXAMPLE 3. (p-log transform) As we see in [**5**] (or Theorem 2.1), $E(n;p)$ is obtained by blowing up and rational blow-down from $E(n)$, so that the Seiberg-Witten invariants of $E(n;p)$ can be computed explicitly as the same way as in Example 1:

THEOREM 4.1. ([**5**].) *The Seiberg-Witten invariants of $E(n;p)$ are*

$$SW_{E(n;p)} = SW_{E(n)} \cdot \left(e^{(p-1)f_p} + e^{(p-3)f_p} + \cdots + e^{-(p-1)f_p}\right)$$

where f_p is a multiple fibre obtained by p-log transform on $E(n)$.

Furthermore, by extending the notion of 'p-log transform' to any smooth 4-manifold containing a cusp neighbourhood, we extend this result.

COROLLARY 4.1. *Let $X(p)$ be the result of p-log transform in the neighbourhood of a cusp, say f, in a SW-simple type irreducible 4-manifold X. Then the Seiberg-Witten invariants of $X(p)$ are*

$$SW_{X(p)} = SW_X \cdot \left(e^{(p-1)f_p} + e^{(p-3)f_p} + \cdots + e^{-(p-1)f_p}\right)$$

where f_p is a multiple fibre in $X(p)$ obtained by p-log transform on X.

PROOF: It suffices to show that $f \cdot L = 0$ for each basic class L of X. Since genus$(f) = 1$ and $f^2 = 0$, this is implied by the adjunction inequality

$$f^2 + |f \cdot L| \leqslant 2 \cdot \text{ genus } (f) - 2. \qquad \square$$

We close this paper by suggesting that Corollary 4.1 allows us to answer partially the uniqueness problems of irreducible 4-manifolds.

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