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# SIMULTANEOUS MONOTONE $L_p$ APPROXIMATION, $p \longrightarrow \infty$

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ABSTRACT. Suppose that $f, g \in L_\infty[0, 1]$ have discontinuities of the first kind only. Using the measure, $\max\{\|f-h\|_p, \|g-h\|_p\}$, of simultaneous $L_p$ approximation, we show that the best simultaneous approximations to $f$ and $g$ by nondecreasing functions converge uniformly as $p \longrightarrow \infty$. Part of the proof involves a discussion of discrete simultaneous approximation in a general context. Assuming only that $f$ and $g$ are approximately continuous, we show that their simultaneous best monotone $L_p$ approximation is continuous.

**Introduction.** If $K \subset L_p = L_p(\Omega, \Sigma, \mu), f, g, h \in L_p$ and $A \subset \Omega$, let $d_p^A(h,f) = \left(\int_A |h-f|^p\right)^{1/p}$, $d_p^A(K,f) = \inf_{h\in K} d_p^A(h,f)$, $d_p^A(h;f,g) = \max\{d_p^A(h,f), d_p^A(h,g)\}$ and $d_p^A(K;f,g) = \inf_{h\in K} d_p^A(h;f,g)$. If $A = \Omega$, we will abbreviate $d_p^A$ by $d_p$ in each of the above definitions and for $d_p(h,f)$ we will use the familiar notation $\|h-f\|_p$. We say that $h^* \in K$ is a *best simultaneous $L_p$ approximation to $f$ and $g$ from $K$* ($p$-b.s.a.) if

$$d_p(h^*;f,g) = d_p(K;f,g). \tag{1}$$

There are other ways of defining a simultaneous norm [12, 14], but the *max norm* we are using, being central to the theory of Chebyshev centers, appears to have the longest history. It is the norm in which the simultaneous approximation is most naturally generalized from two, to a compact set, of approximatees. See [2] for a survey of max-norm simultaneous approximation.

In the present paper, we are primarily interested in the continuous case, where $\Omega = [0, 1] \subset \mathbb{R}$, $\mu$ is Lebesgue measure and $\Sigma$ consists of all Lebesgue measurable sets. In a preliminary step however, we will consider the discrete case, where $\Omega = \{1, 2, \ldots, n\}$ and $\mu$ is the counting measure. In the continuous case $K$ will consist of all *nondecreasing* functions, i.e., $f \in K$ if and only if $f(t_1) \le f(t_2)$ whenever $t_1 < t_2$. In the discrete case $K$ will be any closed convex basically tubular set. If $K$ is *any* closed convex subset of $L_p$ and $1 < p < \infty$, then there is exactly one $p$-b.s.a. to $f$ and $g$ [6], which we denote by $S_p(f, g)$. We call $S_p(f,f)$ the best $L_p$ approximation to $f$ from $K$ ($p$-b.a.).

Suppose that $f, g \in L_\infty$. Then $f, g \in L_p$ for all $p$ in $[1, \infty]$. If $S_p(f, g)$ converges almost everywhere as $p \longrightarrow \infty$, we say that the (simultaneous) *Pólya algorithm* converges. In the present paper we show that the Pólya algorithm converges in both contexts mentioned

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above. We solve the discrete problem using the ideas of Descloux [5]. To solve the continuous problem, we assume $f$ and $g$ are *quasi-continuous*, i.e., $f$ and $g$ have discontinuities of the first kind only. This enables uniform approximation of $f$ and $g$ by step functions, and, thereby, reduction to the discrete case. We also show that, under mild restrictions on $f$ and $g$, $S_p(f,g)$ is continuous, with the result that $\lim_{p\to\infty} S_p(f,g)$ is continuous. These results generalize those published in [4]. We conclude with a discussion of an alternative method of proving convergence, which generalizes that of Legg and Townsend [11].

One of the reasons for our interest in monotone simultaneous approximation is that it is related to the statistical problem of isotonic regression. In [13] the method proposed to simultaneously approximate $f_1,\ldots,f_n$ is to approximate the single function $\bar{f} = n^{-1}\sum f_i$. We find this method theoretically suboptimal and despair over the amount of data that is lost in the calculation of $\bar{f}$. The other major theme in this paper, the Pólya algorithm, is of theoretical interest in that its focus is the compatability of $L_p$, $p < \infty$, with $L_\infty$. When $p = \infty$, the set, $K_\infty$ of best simultaneous approximations to $f$ and $g$ from $K$ may contain more than one element. The Pólya algorithm induces a single-valued *selection* from $K_\infty$ [8].

1. **Simultaneous Approximation in $\mathbb{R}^n$.** In this section we show that the discrete simultaneous Pólya algorithm converges for a large class of closed convex approximating sets: the *basically tubular* sets. Let $\Omega = \{1,\ldots,n\}$, let $\Sigma$ consist of all subsets of $\Omega$ and let $\mu$ be the counting measure. In this context, it is common to denote $L_p$ by $\ell_p$ and to describe a function $\mathbf{x} : \Omega \longrightarrow \mathbb{R}$ by equating $\mathbf{x}$ and the vector of its values, i.e., $\mathbf{x} := (x_1,\ldots,x_n)$. We begin by assuming only that $K$ is a closed convex subset of $\mathbb{R}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be fixed elements of $\ell_p$. For $1 < p < \infty$ we denote by $\mathbf{s}^p$, $\mathbf{x}^p$ and $\mathbf{y}^p$ the $p$-b.s.a. to $\mathbf{x}$ and $\mathbf{y}$ from $K$, the $p$-b.a. to $\mathbf{x}$ from $K$ and the $p$-b.a. to $\mathbf{y}$ from $K$, respectively.

LEMMA 2. *As $p \longrightarrow \infty$, $\|\mathbf{x}-\mathbf{x}^p\|_p$ converges to $d_\infty(K,\mathbf{x})$ and $d_p(\mathbf{s}^p;\mathbf{x},\mathbf{y})$ to $d_\infty(K;\mathbf{x},\mathbf{y})$.*

PROOF. If there exist $p < \infty$ and $\mathbf{z} \in K$ such that $\|\mathbf{x}-\mathbf{x}^p\|_p > \|\mathbf{x}-\mathbf{z}\|_\infty + 2\epsilon$ and $\big|\|\mathbf{x}-\mathbf{z}\|_p - \|\mathbf{x}-\mathbf{z}\|_\infty\big| < \epsilon$, then $\|\mathbf{x}-\mathbf{z}\|_p + \epsilon < \|\mathbf{x}-\mathbf{z}\|_\infty + 2\epsilon < \|\mathbf{x}-\mathbf{x}^p\|_p$, which contradicts the definition of $\mathbf{x}^p$. The proof of the second claim is similar. ∎

A corollary of Lemma 2 is that for sufficiently large $p$, both $\{\|\mathbf{x}^p\|_\infty : p \geq 1\}$ and $\{\|\mathbf{s}^p\|_\infty : p \geq 1\}$ are uniformly bounded sets. Also it is clear that any convergent sequence $\{\mathbf{s}^{p_\nu} : p_\nu \longrightarrow \infty\}$ converges to an $\infty$-b.s.a. of $\mathbf{x}$ and $\mathbf{y}$.

We now describe the context in which we call $K$ a *tubular* set. If $\mathbf{x},\mathbf{v} \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, let $L(\mathbf{x},\mathbf{v})$ be the straight line in $\mathbb{R}^n$ which contains $\mathbf{x}$ and is parallel to the line containing $\mathbf{0}$ and $\mathbf{v}$, and let $N(A,\delta) = \{\mathbf{z} \in \mathbb{R}^n : d_\infty(A,\mathbf{z}) < \delta\}$. A subset $A$ of $\mathbb{R}^n$ is said to be $\mathbf{v}$*-tubular at* $\mathbf{x}$ if for any $\epsilon > 0$ there exists $\delta = \delta(\mathbf{x},\epsilon) > 0$ such that $d_\infty\big(L(\mathbf{y},\mathbf{v}) \cap A, \mathbf{x}\big) < \epsilon$ whenever $\mathbf{y} \in A$ and $d_\infty\big(L(\mathbf{x},\mathbf{v}),\mathbf{y}\big) < \delta$. (This is equivalent to the definition of $\mathbf{v}$*-cylindrical* in [9].) The set $A$ is said to be $\mathbf{v}$*-tubular* if it is $\mathbf{v}$-tubular at every $\mathbf{x}$ in the closure of $A$. A subset $A$ of $\mathbb{R}^n$ is said to be *basically tubular* if $A$ is $\mathbf{e}_i$-tubular for each vector $\mathbf{e}_1,\ldots,\mathbf{e}_n$ in the standard basis of $\mathbb{R}^n$. Included in the class of

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basically tubular sets are all smooth rotund convex bodies and all polyhedral convex sets in $\mathbb{R}^n$. An example of the latter is the set of all nondecreasing vectors, i.e., all vectors $\mathbf{x}$ such that $x_i \leq x_j$ whenever $i < j$. See [9] for a discussion of tubular sets and a proof of the following lemma. We suppose for the remainder of this section that $K$ is $\|\cdot\|_\infty$-closed, convex and basically tubular.

LEMMA 3. *Suppose $B \subset \mathbb{R}^m$ is basically tubular. If $\mathbf{u} \in B$, if $1 \leq k \leq m$ and if $\epsilon > 0$, then there is a $\delta = \delta(\mathbf{u}, \epsilon) > 0$ such that, if $\mathbf{v} \in K$ and $|u_i - v_i| < \delta$, $1 \leq i \leq k$, then there is a vector $\mathbf{w} \in K$ such that $w_i = v_i$, $1 \leq i \leq k$, and $\|\mathbf{u} - \mathbf{w}\|_\infty < \epsilon$.*

To streamline our discussion, we will embed $\mathbb{R}^n$ in $\mathbb{R}^{2n}$. Let $\mathbf{z} = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ and let $\tilde{K} = \{\tilde{\mathbf{v}} = (v_1, v_1, v_2, v_2, \ldots, v_n, v_n) : \mathbf{v} \in K\}$. Then $\tilde{K}$ is a basically tubular subset of $\mathbb{R}^{2n}$ [7]. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, let $|||(\mathbf{a}, \mathbf{b})|||_p = \max\{\|\mathbf{a}\|_p, \|\mathbf{b}\|_p\}$. Then $|||\cdot|||_p$ is a norm on $\mathbb{R}^{2n}$ and $\mathbf{s}^p$ is a $|||\cdot|||_p$-best approximation to $\mathbf{z}$ from $\tilde{K}$. Thus, the set $K_\infty := \{\tilde{\mathbf{v}} \in \tilde{K} : d_\infty(\mathbf{v}; \mathbf{x}, \mathbf{y}) = d_\infty(K; \mathbf{x}, \mathbf{y})\}$ is not empty. We construct the *strict* approximation to $\mathbf{z}$ as follows: if $\tilde{\mathbf{v}} \in K_\infty$, let $\phi(\tilde{\mathbf{v}})$ be the vector whose components are given by $|\tilde{v}_i - z_i|$, arranged in non-increasing order. The strict approximation to $\mathbf{z}$ is the unique $\tilde{\mathbf{s}} \in K_\infty$ with $\phi(\tilde{\mathbf{s}})$ minimal in the lexicographic ordering on $\phi(K_\infty)$.

THEOREM 4. *With $K$, $\mathbf{x}$ and $\mathbf{y}$ as defined above, the net $\{\mathbf{s}^p\}$ converges as $p \longrightarrow \infty$.*

PROOF. We will show by induction that $\lim \tilde{\mathbf{s}}^p = \tilde{\mathbf{s}}$. Let $I = \{1, \ldots, 2n\}$, let $r_1, r_2, \ldots$ be the distinct values, in decreasing order, of $\{|\tilde{s}_i - z_i| : i \in I\}$, let $C_0$ be the empy set, and for each $k \in \mathbb{N}$, let $C_k = \{i \in I : |\tilde{s}_i - z_i| = r_k\}$ and $J_k = I - C_k$. By default, $\tilde{s}_i^p \longrightarrow \tilde{s}_i$ for each $i \in C_0$.

Suppose that $\lim_{p\to\infty} \tilde{\mathbf{s}}_i^p = \tilde{\mathbf{s}}_i$ for each $i$ in $C_k$, and that $\{\tilde{\mathbf{s}}^{p_\nu}\}$ is a sequence in $\{\tilde{\mathbf{s}}^p : 1 < p < \infty\}$ such that $\lim_\nu \tilde{\mathbf{s}}^{p_\nu} = \tilde{\mathbf{v}}$. Suppose, for contradiction, that there is a $\lambda$ in $C_{k+1}$ such that $\tilde{v}_\lambda \neq \tilde{s}_\lambda$. Since $\tilde{\mathbf{v}} \in K_\infty$, the minimality of $\phi(\tilde{\mathbf{s}})$ implies that there is a $\mu$ in $J := J_k$ such that

$$|\tilde{v}_\mu - z_\mu| = r > r_{k+1}, \tag{5}$$

where $r = \max\{|\tilde{v}_i - z_i| : i \in J\}$. By Lemma 3, we may choose a sequence $\{\mathbf{w}^\nu\}_\nu \subset K$ such that, for every $\nu \in \mathbb{N}$,

$$\tilde{w}_i^\nu = \tilde{s}_i^{p_\nu}, \quad i \in C_k, \quad \nu \in \mathbb{N} \tag{6}$$

and $\lim_{\nu\to\infty} \|\mathbf{w}^\nu - \mathbf{s}\|_\infty = 0$. Thus, as $\nu \longrightarrow \infty$, $d_{p_\nu}^J(\mathbf{w}^\nu; \mathbf{x}, \mathbf{y}) \longrightarrow d_\infty^J(\mathbf{s}; \mathbf{x}, \mathbf{y})$ and $d_{p_\nu}^J(\mathbf{s}^{p_\nu}; \mathbf{x}, \mathbf{y}) \longrightarrow d_\infty^J(\mathbf{v}; \mathbf{x}, \mathbf{y})$. It then follows from (5) and (6) that there exists a $\sigma$ such that

$$d_{p_\sigma}(\mathbf{w}^\sigma; \mathbf{x}, \mathbf{y}) < d_{p_\sigma}(\mathbf{s}^{p_\sigma}; \mathbf{x}, \mathbf{y}),$$

a contradiction. ∎

The above proof requires only notational changes to apply to the case where the $\ell_p$ norms are weighted, i.e., $\|\mathbf{x}\|_p = (\sum w_i |x_i|^p)^{1/p}$, $1 \leq p < \infty$, and $\|\mathbf{x}\|_\infty = \sup\{w_i |x_i|\}$, where $w_i \geq 0$, $1 \leq i \leq n$.

2. **Simultaneous Monotone Approximation on [0,1].** Let $K$ be the set of all nondecreasing real valued functions on $\Omega := [0,1]$ and let $S_p\colon L_p \times L_p \longrightarrow L_p$ be defined as in the introduction. The next lemma shows that $S_p$ is a monotone operator.

LEMMA 7. *Suppose that $f_i, g_i \in L_p$, $i = 1,2$, $1 < p < \infty$. If $f_1 \leq f_2$ and $g_1 \leq g_2$, then $S_p(f_1, g_1) \leq S_p(f_2, g_2)$.*

PROOF. Let $h_i = S_p(f_i, g_i)$, $i = 1,2$, $T_1 = h_1 \wedge h_2 = \min(h_1, h_2)$ and $T_2 = h_1 \vee h_2 = \max(h_1, h_2)$; let $a_i = |f_i - h_i|$, $b_i = |g_i - h_i|$, $c_i = |f_i - T_i|$ and $d_i = |g_i - T_i|$, $i = 1,2$. By Lemma 2 in [10],

$$a_2^p + a_1^p \geq c_2^p + c_1^p \text{ and } b_2^p + b_1^p \geq d_2^p + d_1^p,$$

so

$$a_2^p \vee b_2^p \geq c_2^p \vee d_2^p \text{ or } a_1^p \vee b_1^p \geq c_1^p \vee d_1^p.$$

If the first case holds, then

$$\max(\|f_2 - h_2\|_p, \|g_2 - h_2\|_p) \geq \max(\|f_2 - T_2\|_p, \|g_2 - T_2\|_p).$$

Since $S_p(f_2, g_2)$ is uniquely defined, $h_2 = T_2 \geq h_1$. By similar reasoning, if the second case holds, then $h_1 = T_1 \leq h_2$. This completes the proof of Lemma 7. ∎

Let $f, g \in L_\infty$ be fixed and let $h_p = S_p(f,g)$, $1 < p < \infty$.

LEMMA 8. *For $1 < p < \infty$, if $f, g \in L_p$ and $c \in \mathbb{R}$, then $S_p(f+c, g+c) = h_p + c$.*

PROOF. It easily follows from the definition of $h_p$ that $d_p(h_p + c; f + c, g + c) \leq d_p(h + c; f + c, g + c)$, for all $h \in K$. Since $K = \{h + c : h \in K\}$, the proof is complete. ∎

LEMMA 9. *For $1 < p < \infty$, if $f, g \in L_p$, $I$ is an open interval and both $f$ and $g$ are constant on $I$, then $S_p(f,g)$ is constant on $I$.*

PROOF. Let $h = S_p(f,g)$ and let $h'|_I = -h + g + f$ and $h'|_{\Omega\setminus I} = h$. Note that $h'|_I$ is nonincreasing. For notational convenience, we let $\|k - l\| = d_p^I(k,l)$. Then $\|f - h\| = \|g - h'\|$ and $\|g - h\| = \|f - h'\|$. If $h'' = (h + h')/2$ and $d = (\|f - h\| + \|g - h\|)/2$, then both $\|f - h''\| \leq d$ and $\|g - h''\| \leq d$. But this implies that

$$\|f - h''\| \vee \|g - h''\| \leq \|f - h\| \vee \|g - h\|.$$

Since $h'' = (g+f)/2$ is constant on $I$ and $h = S_p(f,g)$, it must be that $h'' = h$, so $h' = h$. Thus $h|_I$ is both nondecreasing and nonincreasing, hence constant. ∎

Let $f$ and $g$ be quasi-continuous functions on [0,1]. Since step functions are uniformly dense in the space of quasi-continuous functions (see [15]), for any $n \in \mathbb{N}$ there are step functions

$$f_n = a_1\chi_{[0,t_1]} + \sum_{i=2}^{k_n} a_i\chi_{(t_{i-1},t_i]} \text{ and } g_n = b_1\chi_{[0,t_1]} + \sum_{i=2}^{k_n} b_i\chi_{(t_{i-1},t_i]}, \tag{10}$$

(where $\chi_A$ is the indicator function of $A$, i.e., $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$) such that $\|f - f_n\|_\infty < n^{-1}$ and $\|g - g_n\|_\infty < n^{-1}$, where $\{0 = t_0 < t_1 < \ldots < t_n =$

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1} is the common refinement of the partitions of [0,1] associated with the canonical representations of $f_n$ and $g_n$. Let $h_{n,p} = S_p(f_n, g_n)$. By the last lemma, $h_{n,p}$ must have the form

$$h_{n,p} = c_1^p \chi_{[0,t_1]} + \sum_{i=2}^{k_n} c_i^p \chi_{(t_{i-1},t_i]}. \tag{11}$$

Thus, we are in the context of weighted discrete simultaneous approximation (where $f_n = \{a_i\}_{i=1}^{k_n}$, $g_n = \{b_i\}_{i=1}^{k_n}$, $h_{n,p} = \{c_i^p\}_{i=1}^{k_n}$ and $w_i = t_i - t_{i-1}$ so, by Theorem 4, there are numbers $c_i^\infty$, $1 \le i \le k_n$, such that

$$\lim_{p\to\infty} h_{n,p} = h_{n,\infty} = c_1^\infty \chi_{[0,t_1]} + \sum_{i=2}^{k_n} c_i^\infty \chi_{(t_{i-1},t_i]}. \tag{12}$$

LEMMA 13. *In the above context, for every $\epsilon > 0$, there exists an $N = N(f, g, \epsilon)$ such that for all $n \ge N$ and $p \in (1, \infty)$, $\|h_{n,p} - h_p\|_\infty < \epsilon$.*

PROOF. Let $\epsilon > 0$ be given. Then there is an integer $N$ such that $\|f - f_n\|_\infty < \epsilon$ and $\|g - g_n\|_\infty < \epsilon$ for all $n \ge N$. Thus, except on a set of measure zero, $n \ge N$ implies that

$$f_n < f + \epsilon, \quad g_n < g + \epsilon, \quad f < f_n + \epsilon, \text{ and } g < g_n + \epsilon. \tag{14}$$

The Lemma follows from (7), (8) and (14). ∎

We are now ready to state our principal result.

THEOREM 15. *Let $f$ and $g$ be as given in (13). Then the net $\{h_p\}$ converges uniformly as $p \to \infty$.*

PROOF. Let $\epsilon > 0$ be given. Then there exists $N$ such that $\|f_n - f_m\|_\infty < \epsilon$, and $\|g_n - g_m\|_\infty < \epsilon$ for all $n, m \ge N$. An argument similar to that in the last proof shows that there exists an $N = N(f, g, \epsilon)$ such that for every $n, m \ge N$ and $p \in (1, \infty)$, $h_{n,p} < h_{m,p} + \epsilon$ and $h_{m,p} < h_{n,p} + \epsilon$. Letting $p \to \infty$, we obtain

$$\|h_{n,\infty} - h_{m,\infty}\|_\infty < \epsilon, \quad n, m \ge N. \tag{16}$$

Hence $\{h_{n,\infty} : n = 1, 2, \ldots\}$ converges uniformly to, say, $h_\infty$. Since the values of $N$ in (13) and (16) are independent of $p$, (12), (13) and (16) and the triangle inequality imply that $h_p$ converges uniformly to $h_\infty$ as $p \to \infty$. ∎

The example in §4 of [3] may easily be altered to show that mere *approximate* continuity of $f$ and $g$ does not guarantee the convergence of the Pólya algorithm. However it does guarantee that $h_p$ is continuous, $1 < p < \infty$, as is shown in the following theorem. If $f$ and $g$ are continuous, then both are quasi-continuous and approximately continuous, so, by (15) and (17), $h_\infty$ is continuous on (0,1).

THEOREM 17. *If $f$ and $g$ are approximately continuous and $p \in (1, \infty)$, then $h_p$ is continuous on (0,1).*

PROOF. Suppose for contradiction that $h_p$ has a jump discontinuity at $a \in (0, 1)$. We may assume without loss of generality that $g(a) \le f(a)$.

We may approximate the above functions by step functions. Indeed, let $\sigma = g(a)$, $\tau = f(a)$, $\lambda = h_p(a^-) = \lim_{t\uparrow a} h_p(t)$ and $\mu = h_p(a^+)$ and suppose that $\alpha > 0$. By Lemma 9, there exists an $\eta \in [\lambda, \mu]$ and $\epsilon = \epsilon(\alpha) > 0$ such that $\max\{\alpha(|\tau-\mu|^p + |\tau-\lambda|^p), \alpha(|\lambda-\sigma|^p + |\mu-\sigma|^p)\} = \max\{2\alpha|\tau-\eta|^p, 2\alpha|\eta-\sigma|^p\} + \epsilon$. If $\alpha$ is replaced by a multiple of $\alpha$ in the last equality, then $\epsilon$ is replaced by the same multiple of $\epsilon$. Thus there exists a $K > 0$ such that $\epsilon(\alpha) = K\alpha$. Hence

$$\max\{|\tau-\mu|^p + |\tau-\lambda|^p, |\lambda-\sigma|^p + |\mu-\sigma|^p\} = \max\{2|\tau-\eta|^p, 2|\eta-\sigma|^p\} + K.$$

Let $h_p^r(t) = h_p(t)$ if $t > a$ and $h_p^r(t) = \mu$ if $t \le a$, and define $h_p^\ell$ similarly, with reversed inequalities. Then each of $h_p^r$ and $h_p^\ell$ is continuous at $a$ so, by Theorem 5.4 in [1], each of $|h_p^j - k|^p, j = r, l,\ k = f, g$, is approximately continuous at $a$. By Theorem 8.2 in [1],

$$\lim_{\delta\to 0} \delta^{-1} \int_a^{a+\delta} |h_p^r - k|^p = |h_p^r(a) - k(a)|^p, \quad k = f, g,$$

and similar statements hold for $h_p^\ell$, with integration from $a-\delta$ to $a$. Since $K > 0$ there exists a $\delta > 0$ such that $M(h_p) > M(\eta)$, where

$$M(z) = \max\left\{\delta^{-1}\int_{a-\delta}^{a+\delta} |z-f|^p, \delta^{-1}\int_{a-\delta}^{a+\delta} |z-g|^p\right\}.$$

If $h_p^*$ is defined by

$$h_p^* = \begin{cases} \eta, & t \in [a-\delta, a+\delta), \\ h_p(t), & \text{otherwise}, \end{cases}$$

then $h_p^*$ is a better simultaneous $L_p$ approximation to $f$ and $g$ than is $h_p$, a contradiction. ■

3. **The Legg-Townsend Method.** In [11], Legg and Townsend showed that if $f$ has one-sided limits at a few select points, then $\lim_{p\to\infty} f_p$ can actually be constructed. Their method appears to require only trivial modifications to apply in the context of simultaneous approximation. We now provide an outline of this method, along with some of the necessary changes:

1. Divide $[0, 1]$ into a finite number of sub-intervals, starting with the interval $[x_{11}, y_{11}]$ where $b_{11}$ is the essential maximum, over all pairs $(x, y)$ such that $x \le y$, of

$$\{(f(x) - g(y))^+, (g(x) - f(y))^+, (f(x) - f(y))^+, (g(x) - g(y))^+\}.$$

On this interval, let $h_\infty(x) = (1/2)[(f, g)(x_{11}^1) + (f, g)(y_{11}^1)] = A_{11}$, where $(f, g)$ depends on the outcome of $b_{11}$, and $x_{11}^1$ , $y_{11}^1$ satisfy $x_{11} \le x_{11}^1 \le y_{11}^1 \le y_{11}$ and $(f, g)(x_{11}^1) - (f, g)(y_{11}^1) = b_{11}$.

2. Continue partitioning $[0, 1] - [x_{11}, y_{11}]$ in a similar manner, and defining $h_\infty$ on every subinterval, bearing in mind the monotonicity of $h_\infty$.

3. Generalize Theorem 3.1 in [11]. If $f$ and $g$ are simple Lebesgue measurable functions and if $\lim_{x\to x_{ij}^+} f(x)$, $\lim_{x\to x_{ij}^+} g(x)$, $\lim_{x\to x_{ij}^-} f(x)$ and $\lim_{x\to x_{ij}^-} g(x)$ exist for each $x_{ij}$, $y_{ij}$ as computed from the construction steps, then $h_p$ converges uniformly to $h_\infty$ on $[0, 1]$.

An outline of the proof follows. We will consider the case where $b = b_{11}$, the essential maximum, over all pairs $(x, y)$ such that $x \leq y$, of $\left(g(x) - f(y)\right)^+$, is positive, $x_{11}^{11} < y_{11}^{11}$ and

$$x_{11}^{11} = \sup\{x_{11} \leq x \leq y_{11} : \mu[x_{11} < t < x : g(t) \neq g(x_{11}^1)] = 0\},$$

and

$$y_{11}^{11} = \inf\{x_{11} \leq y \leq y_{11} : \mu[y < t < y_{11} : f(t) \neq f(y_{11}^1)] = 0\}.$$

All other cases are similar. Let $\epsilon > 0$ be given. Choose $p_0$ large enough so that if $p \geq p_0$, then $(b/2 + \epsilon)^p(y_{11}^{11} - y_{11}) > (b/2)^p$. If $h_p(x) \geq h_\infty(x) + \epsilon$ for some $x \in (x_{11}, y_{11}^{11}]$. Then by monotonicity, $h_p(x) \geq h_\infty(x) + \epsilon$ for all $x \in (y_{11}^{11}, y_{11}]$. Also we have $|f(x) - h_p(x)| \geq b/2 + \epsilon$ a.e. on $(y_{11}^{11}, y_{11}]$. From the way $h_\infty$ was constructed, we conclude that $|g(x) - h_\infty(x)| \leq b/2$ a.e. on $[0, 1]$. Hence

$$\begin{aligned} \|g - h_\infty\|_p^p &\leq (b/2)^p < (b/2 + \epsilon)^p(y_{11} - y_{11}^{11}) \\ &\leq \int_{x_{11}}^{y_{11}} |f(x) - h_p(x)|^p dx \leq \|f - h_p\|_p^p. \end{aligned}$$

Similarly $\|f - h_\infty\|_p^p \leq \|f - h_p\|_p^p$. Combining the last two inequalities, we conclude that $h_\infty$ is a better best $L_p$-simultaneous approximation to $f$ and $g$ than is $h_p$, a contradiction.

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