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MEASURES DEFINED BY GAGES

Dedicated to the memory of Professor E. J. McShane

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ABSTRACT. Using ideas of McShane ([4, Example 3]), a detailed development of the Riemann integral in a locally compact Hausdorff space X was presented in [1]. There the Riemann integral is derived from a finitely additive volume v defined on a suitable semiring of subsets of X. Vis-à-vis the Riesz representation theorem ([8, Theorem 2.14]), the integral generates a Riesz measure ν in X, whose relationship to the volume v was carefully investigated in [1, Section 7].

In the present paper, we use the same setting as in [1] but produce the measure directly without introducing the Riemann integral. Specifically, we define an outer measure by means of *gages* and introduce a very intuitive concept of *gage measurability* that is different from the usual Carathéodory definition. We prove that if the outer measure is σ-finite, the resulting measure space is identical to that defined by means of the Carathéodory technique, and consequently to that of [1, Section 7]. If the outer measure is not σ-finite, we investigate the gage measurability of Carathéodory measurable sets that are σ-finite. Somewhat surprisingly, it turns out that this depends on the axioms of set theory.

1. **Preliminaries.** Throughout this paper, X is a locally compact Hausdorff space. If $A \subset X$, we denote by A^- and A° the closure and interior of A, respectively. If $\mathcal{E}$ and $\mathcal{F}$ are families of subsets of X, we say that $\mathcal{E}$ *refines* $\mathcal{F}$ whenever each $E \in \mathcal{E}$ is contained in some $F \in \mathcal{F}$.

We fix a family $\mathcal{S}$ of subsets of X that satisfies the following conditions.

1. If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$ and there are disjoint sets $C_1, \ldots, C_n$ in $\mathcal{S}$ such that $A - B = \bigcup_{i=1}^n C_i$.
2. If $A \in \mathcal{S}$, then A^- is compact.
3. For each $x \in X$ the collection $\mathcal{S}(x) = \{A \in \mathcal{S} : x \in A^\circ\}$ is a neighborhood base at x.

The following lemma, which was proved in [5, Section 1], summarizes some useful properties of the family $\mathcal{S}$.

LEMMA 1.1. *The following statements are true.*

1. *Each collection* $\{A_1, \ldots, A_m\} \subset \mathcal{S}$ *is refined by a disjoint collection* $\{B_1, \ldots, B_n\} \subset \mathcal{S}$ *with* $\bigcup_{j=1}^n B_j = \bigcup_{i=1}^m A_i$.
2. *For each* $A \in \mathcal{S}$ *and each collection* $\{A_1, \ldots, A_m\} \subset \mathcal{S}$ *there is a disjoint collection* $\{B_1, \ldots, B_n\} \subset \mathcal{S}$ *with* $\bigcup_{j=1}^n B_j = A - \bigcup_{i=1}^m A_i$.
3. *If* $A \in \mathcal{S}$, *then each open cover* $\mathcal{U}$ *of* A^- *is refined by a disjoint collection* $\{A_1, \ldots, A_m\} \subset \mathcal{S}$ *with* $A = \bigcup_{i=1}^m A_i$.

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A *partition* is a collection (possibly empty) $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are disjoint sets from $\mathcal{S}$ and $x_1, \dots, x_p$ are points of X. We say that P is *anchored* in a set $E \subset X$ if $\{x_1, \dots, x_p\} \subset E$. If $A \in \mathcal{S}$ and P is anchored in A^-, then P is called a partition *in* A or *of* A according to whether $\bigcup_{i=1}^p A_i \subset A$ or $\bigcup_{i=1}^p A_i = A$, respectively.

A *gage* in a set $E \subset X$ is a map γ that to each $x \in E$ assigns an open neighborhood $\gamma(x)$ of x in X. If γ is a gage in $E \subset X$, then a partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ anchored in E is called γ*-fine* whenever $A_i \subset \gamma(x_i)$ for $i = 1, \dots, p$.

The next simple lemma, proved in [1, Lemma 2.2], is of critical importance.

LEMMA 1.2. *If $A \in \mathcal{S}$, then a γ-fine partition of A exists for every gage γ in A^-.*

Throughout this paper, we assume that on $\mathcal{S}$ is defined a nonnegative real-valued function v, called *volume*, such that

$$v(A) = \sum_{i=1}^{n} v(A_i)$$

for each $A \in \mathcal{S}$ and each disjoint collection $\{A_1, \dots, A_n\} \subset \mathcal{S}$ for which $\bigcup_{i=1}^n A_i = A$.

EXAMPLE 1.3. A canonical example of the situation described above is obtained by letting

1. $X = \mathbf{R}$ where $\mathbf{R}$ is the set of all real numbers with its usual topology;
2. $\mathcal{S} = \{[a, b) : a, b \in \mathbf{R}, a \leq b\}$;
3. $v\big([a, b)\big) = \alpha(b) - \alpha(a)$ where $\alpha\colon \mathbf{R} \to \mathbf{R}$ is an increasing function.

If δ is a positive real-valued function defined on a set $E \subset \mathbf{R}$, then the map $\gamma\colon x \mapsto \big(x - \delta(x), x + \delta(x)\big)$ is a gage in E.

If f is a real-valued function defined on a set $E \subset X$, we let

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i)\, v(A_i)$$

for each partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ anchored in E.

DEFINITION 1.4. A real-valued function f defined on the closure of $A \in \mathcal{S}$ is called *integrable* in A if there is a real number I such that given $\varepsilon > 0$, we can find a gage γ in A^- such that $|\sigma(f, P) - I| < \varepsilon$ for each γ-fine partition P of A.

In view of Lemma 1.2, the number I of Definition 1.4 is uniquely determined by the function f. It is called the *integral* of f over A, denoted by $\int_A f$. For the basic properties of the integral we refer to [1, Sections 3–6].

2. **The outer measure.**

Let E be a subset of X. If γ is a gage in E, we let

$$v_\gamma(E) = \sup \sum_{i=1}^{p} v(A_i)$$

where the supremum is taken over all partitions $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E that are γ-fine. The number

$$v^*(E) = \inf_{\gamma} v_\gamma(E)$$

where the infimum is taken over all gages γ in E is called the *outer measure* of E. Our first task is to show that the map $v^* \colon E \mapsto v^*(E)$ is an outer measure in X in the usual sense.

PROPOSITION 2.1. *The following statements are true:*

1. $v^*(\emptyset) = 0$;

2. if $E \subset F \subset X$, *then* $v^*(E) \leq v^*(F)$;

3. if $\{E_n\}$ *is a sequence of subsets of* X, *then*

$$v^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} v^*(E_n);$$

4. if E *and* F *are subsets of* X *contained in disjoint open subsets of* X, *then*

$$v^*(E \cup F) = v^*(E) + v^*(F).$$

PROOF. The first statement is correct because only the empty partition $P = \emptyset$ is used in the definition of $v^*(\emptyset)$.

Let $E \subset F \subset X$ and let γ be a gage in F. The restriction of γ to E, still denoted by γ, is a gage in E and we have

$$v^*(E) \leq v_\gamma(E) \leq v_\gamma(F).$$

The second statement follows from the arbitrariness of γ.

In the third claim, assume first that the sets E_n are disjoint. If γ_n is a gage in E_n, define a gage γ in $E = \bigcup_{n=1}^{\infty} E_n$ by letting $\gamma(x) = \gamma_n(x)$ whenever $x \in E_n$. If $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a partition anchored in E that is γ-fine, then $\{(A_i, x_i) : x_i \in E_n\}$ is a partition anchored in E_n that is γ_n-fine, and consequently

$$\sum_{i=1}^{p} v(A_i) = \sum_{n=1}^{\infty} \sum_{x_i \in E_n} v(A_i) \leq \sum_{n=1}^{\infty} v_{\gamma_n}(E_n).$$

This and the arbitrariness of P implies

$$v^*(E) \leq v_\gamma(E) \leq \sum_{n=1}^{\infty} v_{\gamma_n}(E_n).$$

As γ_n is an arbitrary gage in E_n, the desired inequality follows. Now if E_n are any subsets of X, the previous result and the second statement yield

$$\begin{aligned} v^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) &= v^*\Big[\bigcup_{n=1}^{\infty}\Big(E_n - \bigcup_{k=1}^{n-1} E_k\Big)\Big] \\ &\leq \sum_{n=1}^{\infty} v^*\Big(E_n - \bigcup_{k=1}^{n-1} E_k\Big) \leq \sum_{n=1}^{\infty} v^*(E_n). \end{aligned}$$

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Finally, let E and F be subsets of X contained in disjoint open subsets of X, and let γ be a gage in $E \cup F$. We can find gages α and β in E and F, respectively, so that $\alpha(x) \subset \gamma(x), \beta(y) \subset \gamma(y)$, and $\alpha(x) \cap \beta(y) = \emptyset$ for each $x \in E$ and $y \in F$. Let $P = \{(E_1, x_1), \ldots, (E_p, x_p)\}$ and $Q = \{(F_1, y_1), \ldots, (F_q, y_q)\}$ be partitions anchored in E and F that are α- and β-fine, respectively. Then $P \cup Q$ is a partition anchored in $E \cup F$ that is γ-fine. Thus

$$\sum_{i=1}^{p} v(E_i) + \sum_{j=1}^{q} v(F_j) \leq v_\gamma(E \cup F),$$

and by the arbitrariness of P and Q,

$$v^*(E) + v^*(F) \leq v_\alpha(E) + v_\beta(F) \leq v_\gamma(E \cup F).$$

The arbitrariness of γ implies

$$v^*(E) + v^*(F) \leq v^*(E \cup F),$$

and applying the third statement completes the proof.

PROPOSITION 2.2. *If K is a compact subset of X, then*

$$v^*(K) = \inf \sum_{j=1}^{n} v(B_j)$$

where the infimum is taken over all disjoint collections $\{B_1, \ldots, B_n\} \subset \mathcal{S}$ for which $K \subset (\bigcup_{j=1}^n B_j)^\circ$.

PROOF. Denote by c the right side of the equation we want to establish.

If $v^*(K) < c$, then $v_\gamma(K) < c$ for a gage γ in K. Given $z \in K$, find a neighborhood $U_z \in \mathcal{S}$ of z in X with $U_z \subset \gamma(z)$. Since K is compact, there are $z_1, \ldots, z_n$ in K such that $\{U_{z_1}^\circ, \ldots, U_{z_n}^\circ\}$ covers K. According to Lemma 1.1, the collection $\{U_{z_1}, \ldots, U_{z_n}\}$ is refined by a disjoint collection $\{A_1, \ldots, A_p\} \subset \mathcal{S}$ for which $\bigcup_{i=1}^p A_i = \bigcup_{j=1}^n U_{z_j}$. For $i = 1, \ldots, p$, let $x_i = z_j$ where j is an integer with $1 \leq j \leq n$ and $A_i \subset U_{z_j}$. It is clear that $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a partition anchored in K that is γ-fine, and that

$$K \subset \bigcup_{j=1}^{n} U_{z_j}^\circ \subset \Big(\bigcup_{j=1}^{n} U_{z_j}\Big)^\circ = \Big(\bigcup_{i=1}^{p} A_i\Big)^\circ.$$

Thus $c \leq \sum_{i=1}^p v(A_i) \leq v_\gamma(K)$, a contradiction.

Conversely, if $c < v^*(K)$, we can find a disjoint collection $\{B_1, \ldots, B_n\} \subset \mathcal{S}$ so that $K \subset (\bigcup_{j=1}^n B_j)^\circ$ and $\sum_{j=1}^n v(B_j) < v^*(K)$. There is a gage γ in K with $\gamma(x) \subset \bigcup_{j=1}^n B_j$ for each $x \in K$. If $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a partition anchored in K that is γ-fine, then $\bigcup_{i=1}^p A_i \subset \bigcup_{j=1}^n B_j$. An easy application of Lemma 1.1 shows that $\sum_{i=1}^p v(A_i) \leq \sum_{j=1}^n v(B_j)$, and consequently

$$v^*(K) \leq v_\gamma(K) \leq \sum_{j=1}^{n} v(B_j).$$

This contradiction proves the proposition.

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PROPOSITION 2.3. *If G is an open subset of X, then*

$$v^*(G) = \sup_K v^*(K)$$

where the supremum is taken over all compact sets $K \subset G$.

PROOF. If c denotes the right side of the equation we want to establish, then $c \leq v^*(G)$ according to Proposition 2.1. There is a gage γ in G such that $\gamma(x)^- \subset G$ for each $x \in G$. Let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a partition anchored in G that is γ-fine. We select a gage β in the set $K = \bigcup_{i=1}^p A_i^-$, which is a compact subset of G. Using Lemma 1.2, find β-fine partitions $P_i = \{(A_1^i, x_1^i), \ldots, (A_{p_i}^i, x_{p_i}^i)\}$ of A_i, $i = 1, \ldots, p$, and observe that $P = \bigcup_{i=1}^p P_i$ is a partition anchored in K that is β-fine. Thus

$$\sum_{i=1}^p v(A_i) = \sum_{i=1}^p \sum_{j=1}^{p_i} v(A_j^i) \leq v_\beta(K)$$

and as β is arbitrary,

$$\sum_{i=1}^p v(A_i) \leq v^*(K) \leq c.$$

The arbitrariness of $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ implies that $v^*(G) \leq v_\gamma(G) \leq c$.

COROLLARY 2.4. *If A is the union of a disjoint collection $\{A_1, \ldots, A_k\} \subset \mathcal{S}$, then*

$$v^*(A^\circ) \leq \sum_{i=1}^k v(A_i) \leq v^*(A^-).$$

PROOF. If K is a compact subset of A°, then $v^*(K) \leq \sum_{i=1}^k v(A_i)$ by Proposition 2.2. The inequality $v^*(A^\circ) \leq \sum_{i=1}^k v(A_i)$ follows from Proposition 2.3.

Observe that the volume v has a unique additive extension w to the ring of sets generated by $\mathcal{S}$. If $\{B_1, \ldots, B_n\} \subset \mathcal{S}$ is a disjoint collection with $A^- \subset (\bigcup_{j=1}^n B_j)^\circ$, then

$$\sum_{i=1}^k v(A_i) = w(A) = \sum_{j=1}^n w(A \cap B_j) \leq \sum_{j=1}^n v(B_j),$$

and the corollary follows from Proposition 2.2.

EXAMPLE 2.5. In the context of Example 1.3, it is easy to show that

$$v^*\big([a, b)\big) = \alpha(b-) - \alpha(a-)$$

where $\alpha(c-) = \lim_{x \to c-} \alpha(x)$ for each $c \in \mathbf{R}$. Since $\alpha(c-) \leq \alpha(c)$, we see that for an $A \in \mathcal{S}$, there is *no* direct relationship between $v(A)$ and $v^*(A)$.

PROPOSITION 2.6. *If $E \subset X$, then*

$$v^*(E) = \inf_G v^*(G)$$

where the infimum is taken over all open sets $G \subset X$ containing E.

PROOF. If c denotes the right side of the equation we want to establish, then $v^*(E) \leq c$ according to Proposition 2.1, 2. Proceeding towards a contradiction, assume that $v^*(E) < c$ and select a gage η in E for which $v_\eta(E) < c$. If $G = \bigcup_{x \in E} \eta(x)^\circ$, then there is a gage γ in G such that $\gamma(x)^- \subset G$ for each $x \in G$ and $\gamma(x) \subset \eta(x)$ whenever $x \in E$. Let $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a partition anchored in G that is γ-fine, and fix an integer i with $1 \leq i \leq p$. Since $A_i^- \subset \gamma(x_i)^- \subset G$, we can find a disjoint collection $\{A_1^i, \ldots, A_{p_i}^i\} \subset \mathcal{S}$ which refines $\{\eta(x)^\circ : x \in E\}$ and such that $A_i = \bigcup_{j=1}^{p_i} A_j^i$ (Lemma 1.1, 3). For $j = 1, \ldots, p_i$, choose an $x_j^i \in E$ with $A_j^i \subset \eta(x_j^i)$ and observe that

$$\{(A_j^i, x_j^i) : j = 1, \ldots, p_i; i = 1, \ldots, p\}$$

is a partition anchored in E that is η-fine. Consequently

$$\sum_{i=1}^{p} v(A_i) = \sum_{i=1}^{p} \sum_{j=1}^{p_i} v(A_j^i) \leq v_\eta(E).$$

This and the arbitrariness of P imply

$$c \leq v^*(G) \leq v_\gamma(G) \leq v_\eta(E),$$

a contradiction.

3. **Gage measurability.** The following definition, which follows the spirit of Definition 1.4, closely reflects our intuition that a measurable set should not be too entangled with its complement.

DEFINITION 3.1. A set $E \subset X$ is called *gage measurable* if given $\varepsilon > 0$, there is a gage γ in X such that

$$\sum_{i=1}^{p} \sum_{j=1}^{q} v(A_i \cap B_j) < \varepsilon$$

for each γ-fine partitions $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ anchored in E and $X - E$, respectively.

The family of all gage measurable subsets of X, denoted by $\mathcal{S}^*$, is generally *incomparable* with the original semiring $\mathcal{S}$.

EXAMPLE 3.2. Let $X = \mathbf{R}$, let $\mathcal{S}$ be the ring of all bounded subsets of $\mathbf{R}$, and let $v: \mathcal{S} \longrightarrow [0, +\infty)$ be a *finitely additive* extension of the Lebesgue measure in $\mathbf{R}$ (see [7, Chapter 10, Problem 21]). Under these conditions, it is easy to verify that $\mathcal{S}^*$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbf{R}$.

PROPOSITION 3.3. *The following statements are true.*

1. *If $E \subset X$ is simultaneously closed and open or if $v^*(E) = 0$, then $E \in \mathcal{S}^*$.*
2. *The family $\mathcal{S}^*$ is an algebra in X.*
3. *If $E \in \mathcal{S}^*$ and $F \subset X - E$ is any set, then*

$$v^*(E \cup F) = v^*(E) + v^*(F).$$

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4. *If $H \subset X$ is the union of a disjoint sequence $\{H_n\}$ in $\mathcal{S}^*$, then*

$$v^*(H) = \sum_{n=1}^{\infty} v^*(H_n).$$

PROOF. The first statement is obvious. In view of Proposition 2.1, it implies that $\mathcal{S}^*$ contains the empty set. By symmetry, $\mathcal{S}^*$ is closed with respect to complementation. Thus to establish the second claim, it suffices to show that if two sets belong to $\mathcal{S}^*$, then so does their union.

Let $E, G \in \mathcal{S}^*$, $\varepsilon > 0$, and let α and β be, respectively, gages in X associated with E and G and ε according to Definition 3.1. Define a gage γ in X by setting $\gamma(x) = \alpha(x) \cap \beta(x)$ for each $x \in X$, and let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ be γ-fine partitions anchored in $E \cup G$ and $X - (E \cup G)$, respectively. Then

$$\sum_{i=1}^{p} \sum_{j=1}^{q} v(A_i \cap B_j) \leq \sum_{x_i \in E} \sum_{j=1}^{q} v(A_i \cap B_j) + \sum_{x_i \in G} \sum_{j=1}^{q} v(A_i \cap B_j) < 2\varepsilon$$

and we see that $E \cup G \in \mathcal{S}^*$.

If E is as above and $F \subset X - E$ is arbitrary, select a gage η in $E \cup F$ and define a gage δ on $E \cup F$ by setting $\delta(x) = \alpha(x) \cap \eta(x)$ for each $x \in E \cup F$. Let $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $Q = \{(B_1, y_1), \ldots, (B_q, y_q)\}$ be partitions anchored in E and F respectively. Employing Lemma 1.1, an easy induction on p produces a partition

$$R = \{(A_1, x_1), \ldots, (A_p, x_p), (D_1, z_1), \ldots, (D_r, z_r)\}$$

such that $\{z_1, \ldots, z_r\} \subset \{y_1, \ldots, y_q\}$, $D_k \subset B_j$ whenever $z_k = y_j$, and

$$\Big(\bigcup_{i=1}^{p} \bigcup_{j=1}^{q} (A_i \cap B_j)\Big) \cup \Big(\bigcup_{k=1}^{r} D_k\Big) = \bigcup_{j=1}^{q} B_j.$$

Thus if P and Q are δ-fine, then so is R and we see that

$$\begin{aligned} v_\eta(E \cup F) \geq v_\delta(E \cup F) &\geq \sum_{i=1}^{p} v(A_i) + \sum_{k=1}^{r} v(D_k) \\ &= \sum_{i=1}^{p} v(A_i) + \sum_{j=1}^{q} v(B_j) - \sum_{i=1}^{p} \sum_{j=1}^{q} v(A_i \cap B_j) \\ &> \sum_{i=1}^{p} v(A_i) + \sum_{j=1}^{q} v(B_j) - \varepsilon. \end{aligned}$$

As P and Q are arbitrary, we obtain

$$v_\eta(E \cup F) \geq v_\delta(E) + v_\delta(F) - \varepsilon \geq v^*(E) + v^*(F) - \varepsilon,$$

and since η and ε are arbitrary, this implies

$$v^*(E \cup F) \geq v^*(E) + v^*(F).$$

Now the third statement follows from Proposition 2.1.

Extending the third claim by induction and using Proposition 2.1 yields

$$\sum_{n=1}^{k} \nu^*(H_n) = \nu^*\Big(\bigcup_{n=1}^{k} H_n\Big) \leq \nu^*(H)$$

for $k = 1, 2, \ldots$. Thus $\sum_{n=1}^{\infty} \nu^*(H_n) \leq \nu^*(H)$ and another application of Proposition 2.1 completes the proof.

PROPOSITION 3.4. *If $\{E_n\}$ is a sequence in $\mathcal{S}^*$, then $E = \bigcup_{n=1}^{\infty} E_n$ belongs to $\mathcal{S}^*$ whenever $\nu^*(E) < +\infty$.*

PROOF. In view of Proposition 3.3, we may assume that the sets E_n are disjoint, and consequently that the series $\sum_{n=1}^{\infty} \nu^*(E_n)$ converges. Thus given $\varepsilon > 0$, there is a positive integer k such that $\sum_{n=k+1}^{\infty} \nu^*(E_n) < \varepsilon$. By Proposition 3.3, the set $F = \bigcup_{n=1}^{k} E_n$ belongs to $\mathcal{S}^*$ and $\nu^*(E - F) < \varepsilon$. Choose a gage γ in X associated with F and ε according to Definition 3.1 so that $\nu_\gamma(E - F) < \varepsilon$. If $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ are γ-fine partitions anchored in E and $X - E$, respectively, then

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=1}^{q} \nu(A_i \cap B_j) &\leq \sum_{x_i \in F}\sum_{j=1}^{q} \nu(A_i \cap B_j) + \sum_{x_i \in E-F}\sum_{j=1}^{q} \nu(A_i \cap B_j)\\ &< \varepsilon + \sum_{x_i \in E-F} \nu(A_i) \leq \varepsilon + \nu_\gamma(E - F) < 2\varepsilon,\end{aligned}$$

and the measurability of E is established.

The *characteristic function* of a set $E \subset X$ is a function χ_E on X such that $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in X - E$. The next proposition relates gage measurability to the integrability introduced in Definition 1.4.

PROPOSITION 3.5. *Let $A \in \mathcal{S}$ and $E \subset A^\circ$. Then E is gage measurable if and only if χ_E is integrable in A, in which case $\nu^*(E) = \int_A \chi_E$.*

PROOF. Assume that $E \in \mathcal{S}^*$, choose an $\varepsilon > 0$, and find a gage in X associated with E and $\varepsilon/2$ according to Definition 3.1. If $\{(A_1, x_1), \ldots, (A_n, x_n)\}$ and $\{(A_1, y_1), \ldots, (A_n, y_n)\}$ are γ-fine partitions in A, then

$$\{(A_i, x_i) : x_i \in E\} \text{ and } \{(A_i, y_i) : y_i \in E\}$$

are γ-fine partitions anchored in E, while

$$\{(A_j, x_j) : x_j \in X - E\} \text{ and } \{(A_j, y_j) : y_j \in X - E\}$$

are γ-fine partitions anchored in $X - E$. Thus

$$\begin{aligned}\sum_{i=1}^{n} |\chi_E(x_i) - \chi_E(y_i)| \nu(A_i) &= \sum\{\nu(A_i) : x_i \in E, y_i \in X - E\}\\ &+ \sum\{\nu(A_i) : y_i \in E, x_i \in X - E\} = \sum_{x_i \in E}\sum_{y_j \in X-E} \nu(A_i \cap A_j)\\ &+ \sum_{y_i \in E}\sum_{x_j \in X-E} \nu(A_i \cap A_j) < \varepsilon,\end{aligned}$$

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and χ_E is integrable in A by [1, Proposition 3.8].

Conversely, assume that χ_E is integrable in A, choose an $\varepsilon > 0$, and use [1, Proposition 3.8] to find a gage δ in A^- so that

$$\sum_{i=1}^{n} |\chi_E(x_i) - \chi_E(y_i)| \nu(A_i) < \varepsilon$$

for all partitions $\{(A_1, x_1), \ldots, (A_n, x_n)\}$ and $\{(A_1, y_1), \ldots, (A_n, y_n)\}$ in A that are δ-fine. Let η be a gage in X such that $\eta(x) \subset \delta(x)$ if $x \in A^-$, $\eta(x) \subset X - A$ if $x \in X - A^-$, and $\eta(x) \subset A$ if $x \in E$. Select η-fine partitions $P = \{(B_1, t_1), \ldots, (B_p, t_p)\}$ and $Q = \{(C_1, z_1), \ldots, (C_q, z_q)\}$ anchored in E and $X - E$, respectively. Since $B_i \cap C_j = \emptyset$ whenever $z_j \notin A^-$, we may assume that Q is anchored in $A^- - E$. By the choice of η, the families $\{(B_i \cap C_j, t_i)\}$ and $\{(B_i \cap C_j, z_j)\}$, where $i = 1, \ldots, p$ and $j = 1, \ldots, q$, are δ-fine partitions in A. Hence

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \nu(B_i \cap C_j) = \sum_{i=1}^{p} \sum_{j=1}^{q} |\chi_E(t_i) - \chi_E(z_j)| \nu(B_i \cap C_j) < \varepsilon$$

and we see that $E \in \mathcal{S}^*$.

To establish the equation $\nu^*(E) = \int_A \chi_E$, let γ be a gage in E such that $\nu_\gamma(E) < \nu^*(E) + \varepsilon$. The gage γ can be extended to a gage in A^-, still denoted by γ, so that $|\sigma(\chi_E, P) - \int_A \chi_E| < \varepsilon$ for each γ-fine partition P of A. If $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a γ-fine partition of A, then

$$\int_A \chi_E - \varepsilon < \sum_{x_i \in E} \nu(A_i) < \int_A \chi_E + \varepsilon$$

and consequently

$$\int_A \chi_E - \varepsilon < \nu_\gamma(E) \leq \int_A \chi_E + \varepsilon.$$

We conclude that $|\nu^*(E) - \int_A \chi_E| < 2\varepsilon$ and the proposition follows from the arbitrariness of ε.

COROLLARY 3.6. *Each compact subset of X is gage measurable.*

PROOF. Let K be a compact subset of X and assume first that $K \subset A^\circ$ for an $A \subset \mathcal{S}$. Since χ_K is upper semicontinuous, $K \in \mathcal{S}^*$ by Proposition 3.5 and [1, Corollary 5.7]. If K is arbitrary and $x \in K$, choose neighborhoods $U_x, V_x \in \mathcal{S}$ of x in X so that $U_x^- \subset V_x^\circ$. Then K is covered by a collection $\{U_{x_1}, \ldots, U_{x_n}\}$, and each $K \cap U_{x_i}^-$ belongs to $\mathcal{S}^*$ by the first part of the proof. An application of Proposition 3.3 completes the argument.

REMARK 3.7. It is easy to prove Corollary 3.6 directly without referring to the integral (*cf.* Remark 4.3).

THEOREM 3.8. *If $G \subset X$ is open and $\nu^*(G) < +\infty$, then $G \in \mathcal{S}^*$.*

PROOF. It follows from Proposition 2.3 that there is a σ-compact set $K \subset G$ with $\nu^*(K) = \nu^*(G)$. Since $K \in \mathcal{S}^*$ by Corollary 3.6 and Proposition 3.4, an application of Proposition 3.3 shows that $G \in \mathcal{S}^*$.

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COROLLARY 3.9. *Let $E \in \mathcal{S}^*$ and $v^*(E) < +\infty$. Then $E \cap G \in \mathcal{S}^*$ for each open set $G \subset X$.*

PROOF. By Proposition 2.6, there is an open set $U \subset X$ such that $E \subset U$ and $v^*(U) < +\infty$. If $G \subset X$ is open, then $E \cap G = E \cap (G \cap U)$ and the corollary follows from Theorem 3.8 and Propositions 3.3.

The next example shows that the family $\mathcal{S}^*$ *need not* be closed with respect to countable unions.

EXAMPLE 3.10. Let Y be an uncountable discrete space and let $Z = \{0\} \cup \{2^{-n} : n = 1, 2, \ldots\}$ be topologized as a subspace of $\mathbf{R}$. By w we denote a weighted counting measure in $X = Y \times Z$ such that $w\big(\{(y, z)\}\big) = z$ for each $(y, z) \in X$. Let $\mathcal{S}$ be the ring generated by the sets

$$\{(y, 2^{-n})\} \text{ and } \{(y, 0)\} \cup \{(y, 2^{-k}) : k = n, n+1, \ldots\}$$

where $n = 1, 2, \ldots$, and let v be the restriction of w to $\mathcal{S}$.

Under this setting, it follows from Proposition 3.3 that for each integer $n \geq 1$, the set $E_n = Y \times \{2^{-n}\}$ belongs to $\mathcal{S}^*$. Yet, it is easy to show that the union $\bigcup_{n=1}^{\infty} E_n$ is not gage measurable.

4. Measurable sets. The *Borel σ-algebra* in X is the σ-algebra in X generated by all open subsets of X; its members are called *Borel sets*. Let $\mathcal{N}$ be a σ-algebra in X containing all Borel sets, and let ν be a measure on $\mathcal{N}$ such that $\nu(K) < +\infty$ for each compact set $K \subset X$. We recall a few standard definitions.

A set $E \in \mathcal{N}$ is called

1. *ν-σ-finite* if $E = \bigcup_{n=1}^{\infty} E_n$ where $E_n \in \mathcal{N}$ and $\nu(E_n) < +\infty$ for $n = 1, 2, \ldots$;
2. *ν-outer regular* if $\nu(E) = \inf_G \nu(G)$ where the infimum is taken over all open sets $G \subset X$ containing E;
3. *ν-Radon* if $\nu(E) = \sup_K \nu(K)$ where the supremum is taken over all compact subsets of E.

We say that the measure ν is

1. *σ-finite* if X is ν-σ-finite;
2. *Radon* if each $E \in \mathcal{N}$ is ν-Radon;
3. *regular* (or *Riesz*) if each open set $G \subset X$ is ν-Radon and each $E \in \mathcal{N}$ is ν-outer regular;
4. *complete* if $\mathcal{N}$ contains all subsets of each set $E \in \mathcal{N}$ with $\nu(E) = 0$;
5. *saturated* if $\mathcal{N}$ contains all sets $E \subset X$ such that $E \cap F \in \mathcal{N}$ for every $F \in \mathcal{N}$ with $\nu(F) < +\infty$;
6. *diffused* if $\nu(\{x\}) = 0$ for each $x \in X$.

Throughout this section, $\mathcal{M}$ denotes the σ-algebra of all subsets of X that are v^*-measurable in the Carathéodory sense, and μ denotes the measure on $\mathcal{M}$ that is the restriction of the outer measure v^*. In view of Propositions 2.1, 2.3, and 2.6, standard arguments reveal that μ is a complete saturated and regular measure (see [6, Exercises (13-7) through (13-10)]).

REMARK 4.1. It follows from Proposition 2.2, [1, Proposition 7.1], and [6, Corollary (9.10)] that the measure space $(X, \mathcal{M}, \mu)$ coincides with the measure space $(X, \mathcal{N}, \nu)$ of [1, Section 7].

The primary goal of this section is to clarify the relationship between the families $\mathcal{S}^*$ and $\mathcal{M}$. The following lemma is our main tool.

LEMMA 4.2. *A set $E \subset X$ is gage measurable if and only if for each $\varepsilon > 0$ there is an open set $G \subset X$ and a closed set $F \subset X$ such that*

$$F \subset E \subset G \text{ and } \mu(G - F) < \varepsilon.$$

PROOF. Let $E \in \mathcal{S}^*$ and $\varepsilon > 0$. Select a gage γ in X associated with E and ε according to Definition 3.1, and let

$$G = \bigcup_{x \in E} \gamma(x) \text{ and } F = X - \bigcup_{x \in X-E} \gamma(x).$$

If $K \subset G - F$ is a compact set, then it follows from Lemma 1.1 that there are γ-fine partitions $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ anchored in E and $X - E$, respectively, and such that

$$K \subset \Big(\bigcup_{i=1}^{p} A_i\Big)^\circ \cap \Big(\bigcup_{j=1}^{q} B_j\Big)^\circ \subset \Big(\bigcup_{i=1}^{p}\bigcup_{j=1}^{q}(A_i \cap B_j)\Big)^\circ.$$

Now by Proposition 2.2,

$$\mu(K) \leq \sum_{i=1}^{p}\sum_{j=1}^{q} v(A_i \cap B_j) < \varepsilon.$$

Consequently $\mu(G - F) < \varepsilon$, since μ is regular and $G - F$ is open.

Conversely, let G and F satisfy the conditions of the lemma for a given $\varepsilon > 0$. Choose a gage γ in X so that $\gamma(x)^- \subset G$ for every $x \in E$ and $\gamma(x) \subset X - F$ for every $x \in X - E$. If $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ are γ-fine partitions anchored in E and $X - E$, respectively, then $K = \bigcup_{i=1}^{p}\bigcup_{j=1}^{q}(A_i \cap B_j)^-$ is a subset of $G - F$. Thus by Corollary 2.4,

$$\sum_{i=1}^{p}\sum_{j=1}^{q} v(A_i \cap B_j) \leq v^*(K) \leq \mu(G - F) < \varepsilon$$

and the gage measurability of E is established.

REMARK 4.3. An immediate consequence of Lemma 4.2 and Proposition 2.6 is that each compact subset of X is gage measurable (*cf.* Remark 3.7).

THEOREM 4.4. *If $E \in \mathcal{S}^*$, then $E \in \mathcal{M}$ and E is μ-Radon.*

PROOF. Given $E \in \mathcal{S}^*$, use Lemma 4.2 to find open sets $G_n \subset X$ and closed sets $F_n \subset X$ such that

$$F_n \subset E \subset G_n \text{ and } \mu(G_n - F_n) < \frac{1}{n}$$

for $n = 1, 2, \ldots$. The sets $G = \bigcap_{n=1}^{\infty} G_n$ and $F = \bigcup_{n=1}^{\infty} F_n$ belong to $\mathcal{M}$, $F \subset E \subset G$, and $\mu(G - F) = 0$. Since μ is complete, $E \in \mathcal{M}$.

If $c < \mu(E)$, then $c + 1/n < \mu(E) \leq \mu(G_n)$ for a positive integer n, and we can find a compact set $K \subset G_n$ so that $c + 1/n < \mu(K)$. Now $K \cap F_n$ is a compact subset of E and

$$\mu(K \cap F_n) = \mu(K) - \mu(K - F_n) > c + \frac{1}{n} - \mu(G_n - F_n) > c.$$

It follows that E is μ-Radon and the theorem is proved.

PROPOSITION 4.5. *If $E \in \mathcal{M}$ and $\mu(E) < +\infty$, then $E \in \mathcal{S}^*$.*

PROOF. By [6, Lemma (9.2)], the set E is μ-Radon. Since it is also μ-outer regular, we can readily verify that it satisfies the condition of Lemma 4.2.

COROLLARY 4.6. *A set $E \subset X$ belongs to $\mathcal{M}$ if and only if $E \cap F \in \mathcal{S}^*$ for each $F \in \mathcal{S}^*$ with $\mu(F) < +\infty$.*

PROOF. Since μ is saturated and $\mathcal{S}^* \subset \mathcal{M}$, the corollary is a direct consequence of Proposition 4.5.

THEOREM 4.7. *If μ is σ-finite, then $\mathcal{S}^* = \mathcal{M}$.*

PROOF. It follows from Propositions 2.6, 2.3, and 2.1 that $X = N \cup Y$ where $\mu(N) = 0$ and Y is σ-compact. As Y is paracompact, it can be covered by a sequence $\{U_n\}$ of open subsets of X such that each U_n^- is compact and each $x \in Y$ has a neighborhood that meets only finitely many $U_n \cap Y$. If $E \in \mathcal{M}$, then all sets $E_n = E \cap U_n \cap Y$ belong to $\mathcal{S}^*$ according to Proposition 4.5. Choose an $\varepsilon > 0$, and using Lemma 4.2, find open sets $G_n \subset X$ and closed sets $F_n \subset X$ such that

$$F_n \subset E_n \subset G_n \text{ and } \mu(G_n - F_n) < \varepsilon 2^{-n}$$

for $n = 1, 2, \ldots$. Since $\{U_n \cap Y\}$ is an open locally finite cover of Y, it is easy to verify that $\bigcup_{n=1}^{\infty} F_n$ is a relatively closed subset of Y. Hence there is a closed set $F \subset X$ with $F \cap Y = \bigcup_{n=1}^{\infty} F_n$. Select an open set $G_0 \subset X$ so that $N \subset G_0$ and $\mu(G_0) < \varepsilon$. If $G = \bigcup_{n=0}^{\infty} G_n$, then $F \subset E \cup N \subset G$ and

$$\mu(G - F) \leq \mu(G_0) + \sum_{n=1}^{\infty} \mu(G_n - F_n) < 2\varepsilon.$$

Now it follows from Lemma 4.2 and Proposition 3.1 that $E \in \mathcal{S}^*$.

THEOREM 4.8. *If $\mathcal{S}^* = \mathcal{M}$, then μ is σ-finite whenever it is diffused.*

PROOF. If $\mathcal{S}^* = \mathcal{M}$, then the complete saturated and regular measure μ is also Radon by Theorem 4.4. It follows from [3, Section 2, (C)] that there is a disjoint family $\mathcal{D}$ of nonempty compact subsets of X having the following properties:

1. If $G \subset X$ is open, then $\mu(D \cap G) > 0$ for each $D \in \mathcal{D}$ with $D \cap G \neq \emptyset$.
2. If $E \subset X$ and $D \cap E \in \mathcal{M}$ for each $D \in \mathcal{D}$, then $E \in \mathcal{M}$.

3. If $E \in \mathcal{M}$, then $\mu(E) = \sum_{D \in \mathcal{D}} \mu(D \cap E)$.
In each $D \in \mathcal{D}$ select a point x_D. By 2, the set $E = \{x_D : D \in \mathcal{D}\}$ belongs to $\mathcal{M}$ and, assuming that μ is diffused, $\mu(E) = 0$ by 3. Since E is μ-outer regular, there is an open set $G \subset X$ such that $E \subset G$ and $\mu(G) < +\infty$. According to 1, we have $\mu(D \cap G) > 0$ for each $D \in \mathcal{D}$. In view of 3, this implies that $\mathcal{D}$ is countable and consequently that μ is σ-finite.

EXAMPLE 4.9. Let X be an uncountable discrete space, let $\mathcal{S}$ be the family of all finite subsets of X, and let ν be the counting measure in X restricted to $\mathcal{S}$. Then μ is not σ-finite, and yet by Proposition 3.3, the family $\mathcal{S}^*$ contains all subsets of X; in particular, $\mathcal{S}^* = \mathcal{M}$. Thus Theorem 4.8 is false when μ is not diffused.

If μ is not σ-finite, then $\mathcal{M}$ contains a proper σ-ideal Σ consisting of all μ-σ-finite elements of $\mathcal{M}$. The natural question whether Σ is a subfamily of $\mathcal{S}^*$ has interesting answers.

LEMMA 4.10. *A set $E \in \Sigma$ belongs to $\mathcal{S}^*$ if and only if for each $\varepsilon > 0$ there is a closed set $F \subset X$ such that $F \subset E$ and $\mu(E - F) < \varepsilon$.*

PROOF. There are sets $E_n \in \mathcal{M}$ such that $E = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < +\infty$ for $n = 1, 2, \ldots$. Find open sets $G_n \subset X$ so that $\mu(G_n - E_n) < \varepsilon 2^{-n}$, and let $G = \bigcup_{n=1}^{\infty} G_n$. Clearly, G is an open subset of X containing E, and if the condition of the lemma is satisfied, then

$$\mu(G - F) = \mu(G - E) + \mu(E - F) < 2\varepsilon.$$

Thus $E \in \mathcal{S}^*$ by Lemma 4.2. The converse is an obvious consequence of Lemma 4.2.

A family $\mathcal{E}$ of subsets of X is called, respectively, *point-finite* or *point-countable* if the set $\{E \in \mathcal{E} : x \in E\}$ is finite or countable for each $x \in X$. We say that X is, respectively, *metacompact* or *metalindelöf* if each open cover of X has a point-finite or point-countable open refinement.

The *continuum hypothesis* and *Martin's axiom* are abbreviated as CH and MA, respectively.

THEOREM 4.11. *The inclusion $\Sigma \subset \mathcal{S}^*$ is implied by either of the conditions:*
1. *X is metacompact;*
2. *X is metalindelöf and* MA + $\neg$ CH *holds.*

PROOF. Let $E \in \Sigma$ and $Y = E^-$. For a Borel set $B \subset Y$, set $\lambda(B) = \mu(B \cap E)$, and observe that by [6, Corollary (9.3)], λ is a σ-finite Radon measure on the Borel σ-algebra in Y. As Y is a closed subset of X, it follows from [2, Corollary 12.5 and Theorem 12.11] that either condition of the theorem implies the regularity of λ. By Lemma 4.10, however, the λ-outer regularity of $Y - E$ is equivalent to $E \in \mathcal{S}^*$.

It follows from [2, Example 12.7] that there is a nonmetalindelöf space X in which Σ is not a subfamily of $\mathcal{S}^*$. Moreover, [2, Example 12.12] shows that under CH, there is a metalindelöf space X in which Σ is not a subfamily of $\mathcal{S}^*$. Thus whether the inclusion $\Sigma \subset \mathcal{S}^*$ holds in all metalindelöf spaces *cannot* be decided within the usual universe of the Zermelo-Fraenkel set theory including the axiom of choice.

REFERENCES

1. S. I. Ahmed and W. F. Pfeffer, *A Riemann integral in a locally compact Hausdorff space*, J. Australian Math. Soc. **41**(1986),115–137.

2. R. J. Gardner and W. F. Pfeffer, *Borel measures*. In: Handbook of Set-theoretic Topology, (ed. K. Kunen and J. E. Vaughan), North-Holland, New York, 1984, 961–1043.

3. R. J. Gardner and W. F. Pfeffer, *Decomposability of Radon measures*, Trans. American Math. Soc. **283**(1984), 283–293.

4. E. J. McShane, *A Riemann type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Mem. American Math. Soc. **88**(1969).

5. W. F. Pfeffer, *On the lower derivative of a set function*, Canadian J. Math. **20**(1968), 1489–1498.

6. ———, *Integrals and Measures*, Marcel Dekker, New York, 1977.

7. H. L. Royden, *Real Analysis*, Macmillan, New York, 1968.

8. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987.

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