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OPTIMAL DESIGN OF DYNAMIC DEFAULT RISK MEASURES

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Abstract

We consider the question of an optimal transaction between two investors to minimize their risks. We define a dynamic entropic risk measure using backward stochastic differential equations related to a continuous-time single jump process. The inf-convolution of dynamic entropic risk measures is a key transformation in solving the optimization problem.

Keywords: Inf-convolution; dynamic entropic risk measure; single jump process; backward stochastic differential equation

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1. Introduction

In this paper we consider the optimal structure of a contract depending on a nontradable risk related to a nonfinancial risk, such as natural catastrophe. Barrieu and El Karoui [1], [2] discussed a related problem in a continuous diffusion setting. In an earlier paper [13] we constructed backward stochastic differential equations associated with a single jump process. This process might relate to a natural disaster or default. Our results could describe how risk should be optimally allocated between an insurer and the insured. In Section 2 we first review risk measures used in mathematical finance, including static and dynamic risk measures. We next recall results relating to backward stochastic differential equations associated with a finite-horizon, continuous-time, single jump process developed in [13]. Then we introduce the dynamic entropic risk measure based on the solution of a backward stochastic differential equation and generate new dynamic risk measures as the inf-convolution of dynamic entropic risk measures. Finally, we solve the problem of the optimal structure.

2. Static and dynamic risk measures

Our random variables and processes will be defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathrm{P})$. Our processes will be defined on $[0, T]$, where T is finite and deterministic.

2.1. Static risk measures

Suppose that $\mathcal{X}$ denotes a set of financial positions, that is, $\mathcal{X}$ is the set of bounded, $\mathcal{F}_T$-measurable random variables. Following [9], a *static risk measure* $\rho(\cdot)$ is a mapping

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$\rho\colon \mathcal{X} \to \mathbb{R}$, which satisfies some of the following properties for all X, Y in $\mathcal{X}$.

- *Monotonicity:* $\rho(X) \geq \rho(Y)$ if $X \leq Y$ almost surely (a.s.).
- *Convexity:* $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$ for all $\alpha \in (0, 1)$.
- *Positivity:* $X \geq 0$ a.s. implies that $\rho(X) \leq \rho(0)$.
- *Constancy:* $\rho(\alpha) = -\alpha$ for all $\alpha \in \mathbb{R}$.
- *Translatability:* $\rho(X + \beta) = \rho(X) - \beta$ for all $\beta \in \mathbb{R}$.
- *Subadditivity:* $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
- *Lower semicontinuity:* $\{X \in \mathcal{X}\colon \rho(X) \leq \gamma\}$ is closed in $\mathcal{X}$ for any $\gamma \in \mathbb{R}$.

A functional ρ is called a convex risk measure if it satisfies monotonicity, convexity, lower semicontinuity, and $\rho(0) = 0$. The convexity property implies that diversification of investment strategies should not increase risk [10].

Example 2.1. For any X in $\mathcal{X}$, an important example of a convex risk measure is the *entropic risk measure*:

$$e^{\gamma}(X) = \sup_{\mathrm{Q}\in\mathcal{M}_1} (\mathrm{E_Q}[-X] - \gamma h(\mathrm{Q} \mid \mathrm{P})) = \gamma \ln \mathrm{E_P}\bigg[\exp\bigg(-\frac{1}{\gamma}X\bigg)\bigg]. \tag{2.1}$$

Here γ is the risk tolerance coefficient, $\mathcal{M}_1$ is the set of all probability measures on the considered space, and $h(\mathrm{Q} \mid \mathrm{P})$ is the relative entropy of Q with respect to the probability of P, which is defined as

$$h(\mathrm{Q} \mid \mathrm{P}) = \begin{cases} \mathrm{E_P}\bigg[\dfrac{\mathrm{dQ}}{\mathrm{dP}} \ln \dfrac{\mathrm{dQ}}{\mathrm{dP}}\bigg] & \text{if } \mathrm{Q} \ll \mathrm{P}, \\ +\infty & \text{otherwise.} \end{cases}$$

Another convex risk measure is the inf-convolution of convex functionals. This is established in the following theorem. For the proof, see [1].

Theorem 2.1. *Let ρ_1 and ρ_2 be two convex risk measures. The inf-convolution of ρ_1 and ρ_2, $\rho_{1,2}$, is defined as*

$$\rho_{1,2}(X) = \rho_1 \square \rho_2(X) = \inf_{S\in\mathcal{X}}\{\rho_1(X - S) + \rho_2(S)\}.$$

We assume that $\rho_{1,2}(0) > -\infty$. Then $\rho_{1,2}$ is a convex risk measure, which is finite for all $X \in \mathcal{X}$.

2.2. Dynamic risk measures

A static risk measure as described above applies to a single-stage portfolio allocation problem. However, most investors make portfolio allocations dynamically over time. Consequently, they need time-consistent dynamic risk measures which are appropriate not only for the final time horizon but also for intermediate times as the process evolves. In fact, dynamic risk measures can be defined using backward stochastic differential, or in discrete time difference, equations.

A dynamic risk measure is a map satisfying some of the following conditions.

- $\rho_t\colon \mathcal{X} \to \mathcal{L}^0(\mathcal{F}_t)$ for all $t \in [0, T]$.
- ρ_0 is a static risk measure.

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- $\rho_T(X) = -X$ for all $X \in \mathcal{X}$.
- *Convexity:* for all $t \in [0, T]$, ρ_t is a convex risk measure.
- *Positivity:* $X \geq 0$ implies that, for all $t \in [0, T]$, $\rho_t(X) \leq \rho_t(0)$ a.s.
- *Constancy:* for all $t \in [0, T]$ and all $c \in \mathbb{R}$, $\rho_t(c) = -c$.
- *Translatability:* for all $t \in [0, T]$ and all $X \in \mathcal{X}$, $\rho_t(X + a) = \rho_t(X) - a$ a.s.
- *Subadditivity:* for all $t \in [0, T]$ and $X, Y \in \mathcal{X}$, $\rho_t(X + Y) \leq \rho_t(X) + \rho_t(Y)$.

3. Backward stochastic differential equations for the single jump process

Suppose that g is an $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted process

$$g = g(\omega, t, y, z) \colon \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$$

satisfying suitable conditions.

A backward stochastic differential equation (BSDE) is an equation of the form

$$Y_t = Y_0 - \int_{[0,t]} g(u, Y_u, Z_u)\,\mathrm{d}u + \int_{[0,t]} Z_u\,\mathrm{d}M_u \quad \text{for all } t \in [0, T], \qquad Y_T = X. \tag{3.1}$$

In the work of Peng [12] and Pardoux and Peng [11], M is a Brownian motion. A solution of (3.1) is a pair (Y, Z) of adapted processes. In [11] it was shown that, for a given terminal condition $X \in \mathcal{L}^2(\mathcal{F}_T)$, (3.1) has a unique solution (Y, Z) if g satisfies some regular conditions. The solution (Y, Z) is required to be adapted to the forward filtration, and Z is required to be predictable. More general martingales M were considered by El Karoui and Huang [7]. Cohen and Elliott [4], [5] discussed backward stochastic differential and difference equations when the martingale term M is related to a finite-state Markov chain or some other finite-state processes.

For appropriate coefficients g, a general dynamic risk measure ρ can be defined using the solutions of the BSDE (3.1) by putting $\rho_t(X) = -Y_t$. The dynamic risk measure $(\rho_t)_{t\in[0,T]}$ then provides a measure of risk of a position X at intermediate times t between the initial time 0 and the final time T. Depending on the properties of g, ρ_t will be a dynamic risk measure. Furthermore, ρ_T will be the opposite of the final risky position, i.e. $\rho_T(X) = -X$. See [3].

3.1. The continuous finite-time single jump process

Consider a continuous finite-time single jump process $W(\omega) = \{W_t(\omega),\ t \in [0, L]\}$, where L is a finite deterministic terminal time. In fact, $W(\omega)$ remains at 0 until a random time $\tau(\omega)$ (where $0 < \tau(\omega) \leq L$ a.s.), when it jumps to 1. The random time τ might model the time of an insurance event or a default.

Then W can be defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq L}, \mu)$, where $\Omega = [0, L]$, $\mathcal{F} = \mathcal{B}([0, L])$, $\mathcal{F}_t$ is the completed σ-field generated by $\{W_s,\ s \leq t\}$, and $\mu \colon \mathcal{F} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F})$. We suppose that the probability of a jump at time zero is zero.

Write F_t for the probability that $\tau \in (t, L]$. Then F_t is monotonic and nonincreasing. We suppose that F_t is continuous.

For $t \in (0, L]$, write

$$p_t = \mathbf{1}_{\{\tau(\omega)\leq t\}}, \qquad \tilde{p}_t = \int_{(0,\tau(\omega)\wedge t]} \frac{1}{F_s}\,\mathrm{d}(-F_s).$$

Then $q_t = p_t - \tilde{p}_t$ is an $\mathcal{F}_t$-martingale. (See [6] and [8].)

Consider the set $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ of functions such that $\int_\Omega |f| \,\mathrm{d}\mu < +\infty$. For $g \in \mathcal{L}^1(\mu)$, we define the Stieltjes integrals, with $\Omega = [0, L]$,

$$\int_{[0,L]} g(u)\,\mathrm{d}p_u = g(\tau(\omega)), \qquad \int_{[0,L]} g(u)\,\mathrm{d}\tilde{p}_u = \int_{(0,\tau(\omega)]} g(u)\frac{1}{F_u}\,\mathrm{d}(-F_u).$$

Put

$$\int_{[0,L]} g(u)\,\mathrm{d}q_u = \int_{[0,L]} g(u)\,\mathrm{d}p_u - \int_{[0,L]} g(u)\,\mathrm{d}\tilde{p}_u$$

and

$$\int_{(0,t]} g(u)\,\mathrm{d}q_u = \int_{[0,L]} \mathbf{1}_{\{u\le t\}}\, g(u)\,\mathrm{d}q_u.$$

We have the following martingale representation theorem (see [6] and [8]).

Theorem 3.1. *For any $\mathcal{F}_t$-martingale $\mathcal{M}_t$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le L}, \mu)$, there exists $g \in \mathcal{L}^1(\mu)$ such that $\mathcal{M}_t = \mathcal{M}_0 + \mathcal{M}_t^g$ a.s., where $\mathcal{M}_t^g = \int_{(0,t]} g(u)\,\mathrm{d}q_u$.*

3.2. BSDEs

A BSDE based on the martingale random measure q is an equation of the form

$$Y_t + \int_{(t,L]} H(\omega, u, Z_u(\cdot))\,\mathrm{d}(-F_u) + \int_{(t,L]} Z_u\,\mathrm{d}q_u = Q \tag{3.2}$$

for $t \in [0, L]$. Here, H is an adapted function $H\colon \Omega \times [0, L] \times \mathbb{R} \to \mathbb{R}$. A solution of the BSDE (3.2) is a pair of adapted processes (Y, Z) which satisfies (3.2) with $Y_L(\omega) = Q(\omega)$ for $\omega \in \Omega$. We assume that Y_u is left continuous. Also, we suppose that $H(\omega, u, Z_u(\cdot)) \in \mathcal{L}^2(\mathcal{F}_u)$ for all u.

Theorem 3.2. *Assume that H is Lipschitz continuous as follows: there exists $c \in \mathbb{R}^+$ such that, for all $u \in [0, L]$,*

$$|H(\omega, u, Z_u^1(\cdot)) - H(\omega, u, Z_u^2(\cdot))| \le c|Z_u^1 - Z_u^2|.$$

Then, for any $\mathcal{F}_L$-measurable terminal condition Q, the BSDE (3.2) has an adapted unique solution (Y, Z). (See [13].)

4. Optimal design problem

In the following we focus on an optimal transaction between two economic agents, denoted respectively by A and B, who exist in an uncertain universe modeled by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le L}, \mu)$. In the work of Barrieu and El Karoui [2], a problem is considered where uncertainty is modeled by a Brownian filtration.

Suppose that agent A invests a dollars in a defaultable zero-coupon bond with maturity T $(0 < T < L)$ at time 0. Agent A is exposed towards a nonhedgeable risk associated with the possible default. Default might occur at a random time τ (where τ is defined on the above probability space $(\Omega, \mathcal{F}, \mu)$ and $0 < \tau \le L$). For $t \in (0, T]$, the time-t value X of the defaultable zero-coupon bond, with maturity T, deterministic interest rate $(r(s); s \ge 0)$, and constant rebate δ $(0 < \delta < 1)$, is defined as

- the payment of $a \exp(\int_{(0,T]} r(s)\,\mathrm{d}s)$ at time T if default τ has not occurred before time T;
- a payment of $a\delta \exp(\int_{(0,T]} r(s)\,\mathrm{d}s)$, made at maturity, if the default time $\tau \le T$.

That is,

$$X = a \exp\biggl(\int_{(0,T]} r(s)\,\mathrm{d}s\biggr)(\mathbf{1}_{\{\tau>T\}} + \delta\,\mathbf{1}_{\{\tau\le T\}}).$$

Agent A wishes to issue a financial product $S(\tau)$ and sell it to agent B for a forward price paid at time T, denoted by π, to reduce his exposure.

4.1. Optimal structure in a static framework

Suppose that both agents assess the risk associated with their respective positions using an entropic risk measure as defined by (2.1), denoted respectively by e^{γ} and $e^{\gamma'}$. Here agents A and B have risk tolerances γ and γ', respectively.

The issuer, agent A, wants to determine the structure (S, π) in order to minimize his/her global risk measure

$$\inf_{S,\pi} e^{\gamma}(X - S + \pi)$$

with the constraint

$$e^{\gamma'}(S - \pi) \le e^{\gamma'}(0) = 0.$$

Using the translatability property in Section 2.1 and binding the constraint at the optimum, the pricing rule of the S-structure is fully determined by the buyer as

$$\pi^* = -e^{\gamma'}(S).$$

Using the translatability property again, the optimization program simply becomes

$$\inf_{S}(e^{\gamma}(X - S) + e^{\gamma'}(S)).$$

4.2. Solving the inf-convolution in a dynamic framework

We extend the notion of static entropic risk measure defined by (2.1) to a dynamic one on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le L}, \mu)$.

For $t \in (0, T]$, consider the martingale

$$M_t^{\gamma} = \mathrm{E}\biggl[\exp\biggl(-\frac{X - S}{\gamma}\biggr)\biggm| \mathcal{F}_t\biggr],$$

where the risk tolerance coefficient γ is strictly positive. Define the dynamic entropic risk measure associated with receiving X and paying S at time T by

$$e_t^{\gamma}(X - S) = \gamma \log M_t^{\gamma}.$$

We now prove the following result.

Theorem 4.1. *$(-e_t^{\gamma}(X - S), Z_t^{\gamma})$ is the solution of the BSDE*

$$-e_t^{\gamma}(X - S) + \int_{(t,T]} H^{\gamma}(\omega, u, Z_u^{\gamma}(\cdot))\,\mathrm{d}(-F_u) + \int_{(t,T]} Z_u^{\gamma}\,\mathrm{d}q_u = X - S, \qquad (4.1)$$

where

$$H^{\gamma}(\omega, t, Z_t^{\gamma}(\cdot)) = \frac{\mathbf{1}_{\{t\le\tau\}}}{F_t}\biggl(Z_t^{\gamma} + \gamma \exp\biggl(-\frac{Z_t^{\gamma}}{\gamma}\biggr) - \gamma\biggr).$$

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Proof. We prove that $-e^\gamma_t(X-S)$ is the solution of the BSDE (4.1). Clearly, $e^\gamma_T(X-S) = -(X-S)$.

By the martingale representation theorem, for the single jump process (see [6] and [8]), there exists a unique $\varphi^\gamma \in \mathcal{L}^1(\mu)$ such that, for $t \in (0, T]$,

$$M^\gamma_t = M^\gamma_0 + \int_{(0,t]} \varphi^\gamma_s \, \mathrm{d}q_s.$$

For the expression for φ^γ, see Appendix A. By the Itô formula (see [8]),

$$\begin{aligned}
e^\gamma_t(X-S) &= \gamma \log M^\gamma_0 + \gamma \int_{(0,t]} \frac{1}{M^\gamma_{u-}} \varphi^\gamma_u (\mathrm{d}p_u - \mathrm{d}\tilde{p}_u) \\
&\quad + \gamma \sum_{0<u\le t} \bigg(\log M^\gamma_u - \log M^\gamma_{u-} - \frac{1}{M^\gamma_{u-}} \triangle M^\gamma_u \bigg) \\
&= \gamma \log M^\gamma_0 + \gamma \bigg(\mathbf{1}_{\{\tau \le t\}} \frac{\varphi^\gamma_\tau}{M^\gamma_{\tau-}} - \int_{(0,\tau\wedge t]} \frac{\varphi^\gamma_u}{M^\gamma_{u-}} \frac{1}{F_u} \, \mathrm{d}(-F_u) \bigg) \\
&\quad + \gamma \, \mathbf{1}_{\{\tau \le t\}} \bigg(\log\bigg(1 + \frac{\varphi^\gamma_\tau}{M^\gamma_{\tau-}}\bigg) - \frac{\varphi^\gamma_\tau}{M^\gamma_{\tau-}} \bigg) \\
&= \gamma \log M^\gamma_0 - \gamma \int_{(0,\tau\wedge t]} \frac{\varphi^\gamma_u}{M^\gamma_{u-}} \frac{1}{F_u} \, \mathrm{d}(-F_u) + \gamma \int_{(0,t]} \log\bigg(1 + \frac{\varphi^\gamma_u}{M^\gamma_{u-}}\bigg) \mathrm{d}p_u \\
&= \gamma \log M^\gamma_0 + \gamma \int_{(0,\tau\wedge t]} \log\bigg(1 + \frac{\varphi^\gamma_u}{M^\gamma_{u-}}\bigg) \frac{1}{F_u} - \frac{\varphi^\gamma_u}{M^\gamma_{u-}} \frac{1}{F_u} \, \mathrm{d}(-F_u) \\
&\quad + \gamma \int_{(0,t]} \log\bigg(1 + \frac{\varphi^\gamma_u}{M^\gamma_{u-}}\bigg) \mathrm{d}q_u.
\end{aligned}$$

Define

$$Z^\gamma_u = -\gamma \log\bigg(1 + \frac{\varphi^\gamma_u}{M^\gamma_{u-}}\bigg).$$

Then

$$\frac{\varphi^\gamma_u}{M^\gamma_{u-}} = \exp\bigg(-\frac{Z^\gamma_u}{\gamma}\bigg) - 1.$$

Hence,

$$\begin{aligned}
e^\gamma_t(X-S) &= \gamma \log M^\gamma_0 - \int_{(0,\tau\wedge t]} \bigg(\frac{Z^\gamma_u}{F_u} + \frac{\gamma}{F_u} \bigg(\exp\bigg(-\frac{Z^\gamma_u}{\gamma}\bigg) - 1 \bigg) \bigg) \mathrm{d}(-F_u) \\
&\quad - \int_{(0,t]} Z^\gamma_u \, \mathrm{d}q_u \\
&= \gamma \log M^\gamma_0 - \int_{(0,t]} \frac{\mathbf{1}_{\{u\le\tau\}}}{F_u} \bigg(Z^\gamma_u + \gamma \exp\bigg(-\frac{Z^\gamma_u}{\gamma}\bigg) - \gamma \bigg) \mathrm{d}(-F_u) \\
&\quad - \int_{(0,t]} Z^\gamma_u \, \mathrm{d}q_u.
\end{aligned}$$

Write

$$H^\gamma(\omega, u, Z^\gamma_u(\cdot)) = \frac{\mathbf{1}_{\{u\le\tau\}}}{F_u} \bigg(Z^\gamma_u + \gamma \exp\bigg(-\frac{Z^\gamma_u}{\gamma}\bigg) - \gamma \bigg).$$

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Then

$$e_t^\gamma(X-S)=\gamma\log M_0^\gamma-\int_{(0,t]}H^\gamma(\omega,u,Z_u^\gamma(\cdot))\,\mathrm{d}(-F_u)-\int_{(0,t]}Z_u^\gamma\,\mathrm{d}q_u.$$

Since

$$e_T^\gamma(X-S)=-(X-S)=\gamma\log M_0^\gamma-\int_{(0,T]}H^\gamma(\omega,u,Z_u^\gamma(\cdot))\,\mathrm{d}(-F_u)-\int_{(0,T]}Z_u^\gamma\,\mathrm{d}q_u,$$

then

$$-e_t^\gamma(X-S)+\int_{(t,T]}H^\gamma(\omega,u,Z_u^\gamma(\cdot))\,\mathrm{d}(-F_u)+\int_{(t,T]}Z_u^\gamma\,\mathrm{d}q_u=X-S.$$

By Theorem 3.2, $(-e_t^\gamma(X-S),Z_t^\gamma)$ is the unique solution of the BSDE (4.1) with terminal condition $X-S$. This completes the proof.

We now discuss the inf-convolution of two entropic risk measures.

Similarly to the above, for γ', define

$$M_t^{\gamma'}=\mathrm{E}\bigg[\exp\bigg(-\frac{S}{\gamma'}\bigg)\;\bigg|\;\mathcal{F}_t\bigg]$$

and

$$e_t^{\gamma'}(S)=\gamma'\log M_t^{\gamma'}.$$

Then, as above, there exists a unique $\varphi^{\gamma'}\in\mathcal{L}^1(\mu)$ such that, for $t\in(0,T]$,

$$M_t^{\gamma'}=M_0^{\gamma'}+\int_{(0,t]}\varphi_s^{\gamma'}\,\mathrm{d}q_s.$$

For the expression for $\varphi^{\gamma'}$, see Appendix A. Also, from Theorem 4.1,

$$-e_t^{\gamma'}(S)+\int_{(t,T]}H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))\,\mathrm{d}(-F_u)+\int_{(t,T]}Z_u^{\gamma'}\,\mathrm{d}q_u=S,\tag{4.2}$$

where

$$Z_u^{\gamma'}=-\gamma'\log\bigg(1+\frac{\varphi_u^{\gamma'}}{M_{u-}^{\gamma'}}\bigg)$$

and

$$H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))=\frac{\mathbf{1}_{\{u\le\tau\}}}{F_u}\bigg(Z_u^{\gamma'}+\gamma'\exp\bigg(-\frac{Z_u^{\gamma'}}{\gamma'}\bigg)-\gamma'\bigg).$$

Here $e_t^{\gamma'}(S)$ is the dynamic entropic risk measure of S when the risk tolerance is γ'.

We now study, for any $t\in(0,T]$, the inf-convolution of the dynamic entropic risk measures e_t^γ and $e_t^{\gamma'}$. As in Theorem 2.1, this is defined as

$$(e^\gamma\Box e^{\gamma'})_t(X)=\inf_S(e_t^\gamma(X-S)+e_t^{\gamma'}(S)).$$

This quantity describes the optimum minimal total remaining risk for the two investors if B buys an insurance product of value S from A.

Write $Z_u = Z_u^{\gamma} + Z_u^{\gamma'}$. Then we have

$$H^{\gamma}(\omega, u, Z_u^{\gamma}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)) = H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)).$$

Define

$$H^{\gamma}\Box H^{\gamma'}(\omega, u, Z_u(\cdot)) = \inf_{Z_u^{\gamma'}}(H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot))).$$

We now prove the following theorem.

Theorem 4.2. *It holds that*

$$H^{\gamma}\Box H^{\gamma'}(\omega, u, Z_u(\cdot)) = H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot)). \tag{4.3}$$

Also,

$$\begin{aligned}(e^{\gamma}\Box e^{\gamma'})_t(X) &= \int_{(t,T]} H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot))\,\mathrm{d}(-F_u) + \int_{(t,T]} Z_u\,\mathrm{d}q_u - X \\ &= (e^{\gamma+\gamma'})_t(X).\end{aligned} \tag{4.4}$$

Proof. Adding (4.1) and (4.2), we have

$$\begin{aligned}e_t^{\gamma}(X-S) + e_t^{\gamma'}(S) = {}& \int_{(t,T]} (H^{\gamma}(\omega, u, Z_u^{\gamma}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)))\,\mathrm{d}(-F_u) \\ &+ \int_{(t,T]} (Z_u^{\gamma} + Z_u^{\gamma'})\,\mathrm{d}q_u - X.\end{aligned}$$

With $Z_u = Z_u^{\gamma} + Z_u^{\gamma'}$, then

$$\begin{aligned}e_t^{\gamma}(X-S) + e_t^{\gamma'}(S) = {}& \int_{(t,T]} (H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)))\,\mathrm{d}(-F_u) \\ &+ \int_{(t,T]} Z_u\,\mathrm{d}q_u - X.\end{aligned} \tag{4.5}$$

Consider the functional

$$\begin{aligned}&H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)) \\ &\quad = \frac{\mathbf{1}_{\{u\le\tau\}}}{F_u}\bigg(Z_u + \gamma\exp\bigg(-\frac{Z_u - Z_u^{\gamma'}}{\gamma}\bigg) + \gamma'\exp\bigg(-\frac{Z_u^{\gamma'}}{\gamma'}\bigg) - \gamma - \gamma'\bigg).\end{aligned} \tag{4.6}$$

This is a convex function with respect to $Z_u^{\gamma'}$, since the second derivative of (4.6) with respect to $Z_u^{\gamma'}$ is, for each ω,

$$\frac{\mathbf{1}_{\{u\le\tau\}}}{F_u}\bigg(\frac{1}{\gamma}\exp\bigg(-\frac{Z_u - Z_u^{\gamma'}}{\gamma}\bigg) + \frac{1}{\gamma'}\exp\bigg(-\frac{Z_u^{\gamma'}}{\gamma'}\bigg)\bigg) \ge 0.$$

Therefore, for each ω, the minimum of (4.6) with respect to $Z_u^{\gamma'}$ occurs when the first derivative of (4.6) with respect to $Z_u^{\gamma'}$ is 0. That is, when

$$\frac{\mathbf{1}_{\{u\le\tau\}}}{F_u}\bigg(\exp\bigg(-\frac{Z_u - Z_u^{\gamma'}}{\gamma}\bigg) - \exp\bigg(-\frac{Z_u^{\gamma'}}{\gamma'}\bigg)\bigg) = 0.$$

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Write $Z_u^{*\gamma'}$ for the value at which the minimum is attained. Clearly, $Z_u^{*\gamma'}$ is unique, and

$$Z_u^{*\gamma'} = \frac{\gamma'}{\gamma+\gamma'} Z_u.$$

Therefore,

$$\begin{aligned} H^{\gamma} \Box H^{\gamma'}(\omega, u, Z_u(\cdot)) &= \inf_{Z_u^{\gamma'}} (H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot))) \\ &= H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{*\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{*\gamma'}(\cdot)) \\ &= \frac{\mathbf{1}_{\{u \le \tau\}}}{F_u} \bigg(Z_u + (\gamma+\gamma') \exp\bigg(-\frac{Z_u}{\gamma+\gamma'}\bigg) - (\gamma+\gamma') \bigg) \\ &= H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot)). \end{aligned}$$

This establishes (4.3).

By (4.3) and (4.5), we obtain

$$e_t^{\gamma}(X-S) + e_t^{\gamma'}(S) \ge \int_{(t,T]} (H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot)))\, \mathrm{d}(-F_u) + \int_{(t,T]} Z_u \, \mathrm{d}q_u - X;$$

therefore,

$$\inf_S (e_t^{\gamma}(X-S) + e_t^{\gamma'}(S)) \ge \int_{(t,T]} (H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot)))\, \mathrm{d}(-F_u) + \int_{(t,T]} Z_u \, \mathrm{d}q_u - X.$$

Take $S^* = (\gamma'/(\gamma+\gamma'))X$. We will show that

$$e_t^{\gamma}(X-S^*) + e_t^{\gamma'}(S^*) = \int_{(t,T]} (H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot)))\, \mathrm{d}(-F_u) + \int_{(t,T]} Z_u \, \mathrm{d}q_u - X.$$

In fact, with $S^* = (\gamma'/(\gamma+\gamma'))X$,

$$\frac{X-S^*}{\gamma} = \frac{S^*}{\gamma'}.$$

Therefore, the martingales

$$M_t^{\gamma} = \mathrm{E}\bigg[\exp\bigg(-\frac{X-S^*}{\gamma}\bigg) \,\bigg|\, \mathcal{F}_t\bigg] \quad \text{and} \quad M_t^{*\gamma'} = \mathrm{E}\bigg[\exp\bigg(-\frac{S^*}{\gamma'}\bigg) \,\bigg|\, \mathcal{F}_t\bigg]$$

are equal, as well the integrands φ^{γ} and $\varphi^{*\gamma'}$ in their martingale representations. Then, with

$$Z_u^{\gamma} = -\gamma \log\bigg(1 + \frac{\varphi_u^{\gamma}}{M_{u-}^{\gamma}}\bigg) \quad \text{and} \quad Z_u^{*\gamma'} = -\gamma' \log\bigg(1 + \frac{\varphi_u^{*\gamma'}}{M_{u-}^{*\gamma'}}\bigg),$$

we have

$$\gamma Z_u^{*\gamma'} = \gamma' Z_u^{\gamma}.$$

With

$$Z_u = Z_u^{\gamma} + Z_u^{*\gamma'} = \frac{\gamma+\gamma'}{\gamma'} Z_u^{*\gamma'},$$

we have

$$Z_u^{*\gamma'} = \frac{\gamma'}{\gamma + \gamma'} Z_u.$$

Consequently, as in (4.3), this $Z_u^{*\gamma'}$ is such that

$$\begin{aligned}
&\inf_{Z_u^{\gamma'}} (H^\gamma(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot))) \\
&\qquad = H^\gamma(\omega, u, Z_u(\cdot) - Z_u^{*\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{*\gamma'}(\cdot)) \\
&\qquad = H^\gamma \square H^{\gamma'}(\omega, u, Z_u(\cdot)) \\
&\qquad = H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot)).
\end{aligned}$$

This establishes (4.4).

We have established that, for all $t \in (0, T]$, when $S = S^*$ and $Z_t^{\gamma'} = Z_t^{*\gamma'}$, we have

$$\begin{aligned}
&\inf_S (e_t^\gamma(X - S) + e_t^{\gamma'}(S)) \\
&\quad = e_t^\gamma(X - S^*) + e_t^{\gamma'}(S^*) \\
&\quad = \int_{(t,T]} \inf_{Z_u^{\gamma'}} (H^\gamma(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)))\, \mathrm{d}(-F_u) \\
&\qquad + \int_{(t,T]} Z_u dq_u - X \\
&\quad = \int_{(t,T]} H^\gamma(\omega, u, Z_u(\cdot) - Z_u^{*\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{*\gamma'}(\cdot))\, \mathrm{d}(-F_u) \\
&\qquad + \int_{(t,T]} Z_u \,\mathrm{d}q_u - X.
\end{aligned}$$

By Theorem 3.2, $(-e_t^{\gamma'}(S^*), Z_t^{*\gamma'})$ is the unique solution of the BSDE (4.2) with terminal condition S^*. We note that, for any constant c,

$$e_t^\gamma(X - S - c) = e_t^\gamma(X - S) + c \quad \text{and} \quad e_t^{\gamma'}(S + c) = e_t^{\gamma'}(S) - c.$$

Therefore, $S^* + c$ is also optimal.

5. Conclusion

We obtained an optimal solution for the inf-convolution problem of the dynamic entropic risk measures. This is the minimum remaining risk if investor B buys an insurance product of value S from A.

Appendix A. The expressions for φ^γ and $\varphi^{\gamma'}$

Clearly, S is $\mathcal{F}_T$-measurable; therefore, S is defined as

$$S = h(\tau)\, \mathbf{1}_{\{\tau \le T\}} + b\, \mathbf{1}_{\{\tau > T\}},$$

where $h \in \mathcal{L}^1(\mu)$ and b is constant.

As in [6], for all $t \in (0, T]$, the integrands have the form

$$\varphi_t^{\gamma} = \exp\biggl(-\frac{1}{\gamma}\biggl(a\delta \exp\biggl(\int_{(0,T]} r(s)\,\mathrm{d}s\biggr) - h(t)\biggr)\biggr) - \frac{1}{F_t}\int_{(t,T]} \exp\biggl(-\frac{1}{\gamma}\biggl(a\delta \exp\biggl(\int_{(0,T]} r(s)\,\mathrm{d}s\biggr) - h(u)\biggr)\biggr)\,\mathrm{d}(-F_u) - \frac{F_T}{F_t}\exp\biggl(-\frac{1}{\gamma}\biggl(a \exp\biggl(\int_{(0,T]} r(s)\,\mathrm{d}s\biggr) - b\biggr)\biggr)$$

and

$$\varphi_t^{\gamma'} = \exp\biggl(-\frac{h(t)}{\gamma'}\biggr) - \frac{1}{F_t}\int_{(t,T]} \exp\biggl(-\frac{h(u)}{\gamma'}\biggr)\,\mathrm{d}(-F_u) - \frac{F_T}{F_t}\exp\biggl(-\frac{b}{\gamma'}\biggr).$$

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