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COBOUNDARY EQUATIONS OF EVENTUALLY EXPANDING TRANSFORMATIONS

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Let T be an eventually expansive transformation on the unit interval satisfying the Markov condition. Then T is an ergodic transformation on $(X, \mathcal{B}, \mu)$ where $X = [0, 1)$, $\mathcal{B}$ is the Borel σ-algebra on the unit interval and μ is the T invariant absolutely continuous measure. Let G be a finite subgroup of the circle group or the whole circle group and $\phi : X \to G$ be a measurable function with finite discontinuity points. We investigate ergodicity of skew product transformations T_ϕ on $X \times G$ by showing the solvability of the coboundary equation $\phi(x)g(Tx) = \lambda g(x)$, $|\lambda| = 1$. Its relation with the uniform distribution mod M is also shown.

1. INTRODUCTION

Let $(X, \mathcal{B}, \mu)$ be a probability space and T be a measure preserving transformation on X. A transformation T on X is called ergodic if the constant function is the only T-invariant function and it is called weakly mixing if the constant function is the only eigenfunction with respect to T. A measure preserving transformation T is called exact if $\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B}$ is the trivial σ-algebra consisting of empty set and whole set modulo measure zero sets. So exact transformation are as far from being invertible as possible. Recall that if a transformation is exact then that transformation is weakly mixing ([**11**]).

A piecewise differentiable transformation $T : [0, 1) \to [0, 1)$ is said to be *eventually expansive* if some iterate of T has its derivative bounded away from 1 in modulus, that is, $|(T^n)'| > 1$ everywhere for some n. Let $\{\Delta_i\}$ be a countable (or finite) partition of the unit interval $[0, 1)$ by subintervals. Suppose that an eventually expansive map T on the interval $[0, 1)$ satisfies

(i) $T|_{\operatorname{Int}\Delta_i}$ has a C^2-extension to the closure of Δ_i,

(ii) $T|_{\operatorname{Int}\Delta_i}$ is strictly monotone,

(iii) $\overline{T(\Delta_i)} = [0, 1]$, and in the case that the number of subintervals in the partition is infinite

(iv) $\sup_i \left\{ \sup_{x_1 \in \operatorname{Int}\Delta_i} |T''(x_1)| \Big/ \inf_{x_2 \in \operatorname{Int}\Delta_i} |T'(x_2)|^2 \right\} < \infty.$

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Then it is well known that there exists a measure μ which is (a) T-invariant, (b) exact, and (c) finite and of the form $d\mu = \rho(x)\,dx$ where ρ is continuous and $1/C < \rho < C$ for some $C > 0$ ([**1**, **3**, **4**]).

The conditions of the above fact can be modified in several ways. One modification is called the Markov condition, when the number of sets in the partition is finite and $\overline{T(\Delta_i)}$ is a union of $\overline{\Delta_j}$. In this paper, let $\mathbb{T}$ be the unit circle in the complex plane, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

In [**9**], Siboni consider the skew product transformation $T_{2,a,\omega}$ on the torus, $[0,1) \times [0,1)$ defined by

$$(x, y) \mapsto (2x, y + ax + \omega) \pmod 1.$$

He proved a criterion for the ergodicity of the transformation by accurate estimation of correlations of characteristic functions.

Let T be an eventually expansive transformation on the unit interval satisfying the Markov conditions. Then T is an exact transformation on $(X, \mathcal{B}, \mu)$ where $X = [0, 1)$, $\mathcal{B}$ is Borel σ-algebra on the unit interval and μ is the T invariant absolutely continuous measure. Let G be a finite subgroup of the circle group or the whole circle group $\mathbb{T}$ and $\phi : X \to G$ a measurable function with finite discontinuity points. In this paper we investigate ergodicity of skew product transformations T_ϕ on $X \times G$ by showing the solvability of the coboundary equation $\phi(x)g(Tx) = \lambda g(x)$, $|\lambda| = 1$ and we give a simple proof and generalisations of Siboni's results, see Proposition 3.

Let $X = \{x : 0 \leqslant x < 1\}$ be the compact group of real numbers modulo 1, and let $\theta \in X$ be irrational. The numbers $j\theta$, $j = 0, \pm 1, \ldots$, comprise a dense subgroup of X. For each interval $I \subset X$ and $n > 0$ define $S_n = S_n(\theta, I)$ to be the number of integers j, $0 \leqslant j \leqslant n - 1$, such that $j\theta \in I$. By the Kronecker-Weyl theorem $\lim_{n\to\infty} S_n/n = \mu(I)$, where μ is Lebesgue measure on X ([**6**]). Veech [**10**] was interested in the behaviour of the sequence $\{d_n\}$ of parities of $\{S_n\}$, that is, d_n is 0 or 1 as S_n is even or odd. He investigated the existence of the limit

$$\mu_\theta(I) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n,$$

and he showed that a necessary and sufficient condition for $\mu_\theta(I)$ to exist for every interval $I \subset X$ is that θ has bounded partial quotients. He also showed that d_n is evenly distributed if the length of the interval is not an integral multiple of θ modulo 1.

In this paper, we are interested in the uniform distribution of the sequence $d_n \in \{0, \ldots, M - 1\}$ defined by

$$d_n(x) \equiv \sum_{k=0}^{n-1} 1_E(T^k x) \pmod M$$

for eventually expanding transformations, particularly for generalised L-covering maps (which will be defined in section 3) and for Gauss transformation on the interval. Here $\mathbf{1}_E(x)$ denotes the indicator function of $E \subset X$.

To investigate the sequence $\{d_n(x)\}$, we consider the behaviour of the sequence $\exp\big((2\pi i/M)d_n(x)\big)$ and check whether this sequence is uniformly distributed on the finite group G generated by $\exp\big((2\pi i)/M\big)$. Weyl's criterion for uniform distribution says that the sequence $\exp\big((2\pi i/M)d_n(x)\big)$ is uniformly distributed if and only if $\lim_{N\to\infty} \sum_{n=1}^{N} /N \exp^k\big((2\pi i/M)d_n(x)\big) = 0$ for all $1 \leqslant k \leqslant M-1$ ([6]). We investigate this problem from the viewpoint of spectral theory. Let (X,μ) be a probability space and T be an ergodic measure preserving transformation on X, which is not necessarily invertible. Let $\phi(x)$ be a G-valued function defined by $\phi(x) = \exp\big((2\pi i/M)\mathbf{1}_E(x)\big)$. Consider the skew product transformation T_ϕ on $X \times G$ defined by $T_\phi(x,g) = (Tx, \phi(x)g)$. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp^k\Big(\frac{2\pi i}{M} d_n(x)\Big)\cdot z^k = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} U_{T_\phi} f(x,z)$$

where U_{T_ϕ} is an isometry on $L^2(X\times G)$ induced by T_ϕ and $f(x,z) = z^k$. If T_ϕ is ergodic, we may apply Birkhoff's Ergodic Theorem to f to deduce that $\lim_{N\to\infty} 1/N \sum_{n=1}^{N} \exp^k\big((2\pi i/M)d_n (x)\big) = 0$, Recall that the dual group of G consists of the $\gamma_k(z) = z^k$ for $0 \leqslant k \leqslant M-1$. Hence $L^2(X\times G) = \bigoplus_{k=0}^{M-1} L^2(X)\cdot z^k$ and each $L^2(X)\cdot z^k$ is an invariant subspace of U_{T_ϕ}. If $f(x,z) \in L^2(X,G)$ then $f(x,z) = \sum_{k=0}^{M-1} f_k(x)z^k$, and

$$U_{T_\phi} f(x,z) = \sum_{k=0}^{M-1} \phi^k(x) f_k(Tx)\cdot z^k.$$

Hence, if f is an eigenfunction with eigenvalue λ we have $\phi^k(x)f_k(Tx) = \lambda f_k(x)$ for all k. Recall that a nonconstant function $h(x)$ is called a *coboundary* if $h(x) = \overline{q(Tx)}q(x)$, $|q(x)| = 1$ almost everhwyere on X.

In [**2**], Ahn and Choe considered the case when T is an $(1/L,\dots,1/L)$-Bernoulli transformation and show and that if E is a cylinder set with the same missing initial digit and $M = 2$, then the sequence $\{d_n\}$ is evenly distributed. In this paper, we show that for all generalised L-covering maps and Gauss transformation on the unit interval, the sequence $\{d_n\}$ is uniformly distributed and that compact group extension by $\phi(x)$ is exact.

2. Coboundary equations

Let $(Y,\mathcal{C},\mu)$ be a probability space, $f \in L^1(Y,\mathcal{C},\mu)$ and $\mathcal{B}\subset\mathcal{C}$ a sub σ-algebra. We denote by $E(f \mid \mathcal{B})$ the conditional expectation of f with respect to $\mathcal{B}$. Recall that this

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is a $\mathcal{B}$-measurable function g with the property that $\int_B f d\mu = \int_B g d\mu$ for all $B \in \mathcal{B}$. Let S be a transformation defined on Y and $\mathcal{B}$ be *exhaustive* that is, $S^{-1}\mathcal{B} \subset \mathcal{B}$ and $S^n\mathcal{B} \uparrow \mathcal{C}$ as $n \to +\infty$. The Martingale Convergence Theorem says that $E(f \mid S^n\mathcal{B})$ converges to f almost everywhere and in $L^1(Y, \mathcal{C}, \mu)$ as $n \to +\infty$ for $f \in L^1(Y, \mathcal{C}, \mu)$

Lemma 1. *Let S be a measure preserving transformation on $(Y, \mathcal{C}, \mu)$, and $\mathcal{B}$ be an exhaustive σ-algebra $\mathcal{B} \subset \mathcal{C}$, and let $\phi : Y \to \mathbb{T}$ be a $\mathcal{B}$-measurable map to the circle group $\mathbb{T}$. If $q : Y \to \mathbb{T}$ is a $\mathcal{C}$-measurable solution to the equation $\phi \cdot q \circ S = q$, then q is $\mathcal{B}$-measurable.*

Proof: We follow an idea of Parry in [**8**]. Applying the conditional expectation operator $E(\cdot \mid \mathcal{B})$ to the equation

$$(*) \qquad \phi \cdot q \circ S = q$$

we obtain $\phi \cdot E(q \circ S \mid \mathcal{B}) = E(q \mid \mathcal{B})$ or $\phi \cdot E(q \mid S\mathcal{B}) \circ S = E(q \mid \mathcal{B})$. Multiplying this with the inverse of $(*)$ we have $\overline{q(y)} \cdot E(q \mid \mathcal{B})(y) = \overline{q(Sy)} \cdot E(q \mid S\mathcal{B}) \circ S(y)$ almost everywhere so that $\int_Y \overline{q} \cdot E(q \mid \mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q \mid S\mathcal{B})\, d\mu$. By exactly the same argument, using $S^n\mathcal{B}$ in place of $\mathcal{B}$, we have $\int_Y \overline{q} \cdot E(q \mid S^n\mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q \mid S^{n+1}\mathcal{B})\, d\mu$ so that $\int_Y \overline{q} \cdot E(q \mid \mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q \mid S^n\mathcal{B})\, d\mu$. Taking limits, we get $\int_Y \overline{q} \cdot E(q \mid \mathcal{B})\, d\mu = \int_Y |q|^2\, d\mu$. Thus $E(q \mid \mathcal{B}) = q$ almost everywhere, and q is $\mathcal{B}$-measurable. ∎

Proposition 1. *Let $Y = \prod_{-\infty}^{\infty} \{0, 1, \ldots, L-1\}$ where $L \leqslant \infty$ and let σ be the shift map on Y with σ-invariant measure μ. Let $\mathcal{P}$ denote the state partition $\{P_k : P_k = \{x : x_0 = k\} \quad for \quad 0 \leqslant k \leqslant L-1\}$, and let $\mathcal{B}_l^m = \bigvee_{i=l}^{m} \sigma^{-i}\mathcal{P}$ for $l \leqslant m$. Assume that $\phi(x)$ is a $\mathbb{T}$-valued $\mathcal{B}_l^m$ measurable function. If $g(x)$ is a $\mathbb{T}$-valued solution of the equation, $\phi(x)g(\sigma x) = g(x)$ then $g(x)$ is also a $\mathcal{B}_l^m$ measurable function.*

Proof: Let $\mathcal{B} = \bigvee_{i=l}^{\infty} \sigma^{-i}\mathcal{P}$. Then $\phi(x)$ is $\mathcal{B}$ measurable and $\mathcal{B}$ is exhaustive with respect to σ. Since $\phi(x)g(\sigma x) = g(x)$, $g(x)$ is also $\mathcal{B}$-measurable by the above Lemma. Now let $\mathcal{A} = \bigvee_{i=-m}^{\infty} \sigma^{i}\mathcal{P}$. Then $\phi(\sigma^{-1}x)$ is $\mathcal{A}$ measurable and $\mathcal{A}$ is exhaustive with respect to σ^{-1}. Since $\phi(x)g(\sigma x) = g(x)$ can be rewritten as $\phi(\sigma^{-1}x)g(x) = g(\sigma^{-1}x)$, that is, $\phi(\sigma^{-1}x)\overline{g(\sigma^{-1}x)} = \overline{g(x)}$, $g(x)$ is also $\mathcal{A}$ measurable by applying the above Lemma to the map σ^{-1}. Hence the conclusion follows. ∎

3. The interval maps and symbolic dynamics

In this section we apply the previous result to Markov maps. Consider the behaviour of the iterates of a map τ of the unit interval to itself. We also assume that τ is noninvertible and piecewise continuous. Here are several well-known examples:

(a) $\tau(x) = 2x \pmod 1$;

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(b) $\tau(x) = \beta x \pmod 1$, $\beta = (1+\sqrt{5})/2$;

(c) Gauss transform $\tau(x) = 1/x \pmod 1$.

For each of these transformations explicit formulas are known for absolutely continuous invariant measures:

(a) the Lebesgue measure dx;

(b) $d\mu = \beta\, dx$, $0 \leqslant x < \beta^{-1}$, and $d\mu = dx$, $\beta^{-1} \leqslant x < 1$;

(c) the Gauss measure $d\mu = (1/\log 2)\, dx/(1+x)$.

DEFINITION 1: Let $\tau : [0,1) \to [0,1)$ $0 = a_0 < a_1 < \cdots < a_L = 1$, and let $\{I_j\}_{j=0}^{L-1}$ be a partition of $[0,1)$ with $I_j = [a_j, a_{j+1})$, $0 \leqslant j \leqslant L-1 \leqslant \infty$. Assume that τ satisfies

(1) $\tau|_{\mathrm{Int}\, I_i}$, the restriction of τ to interior points of I_i, has a C^2-extension to the closure of I_i,

(2) $\tau|_{\mathrm{Int}\, I_i}$ is strictly monotone,

(3) $\overline{\tau(I_i)} = [0,1]$, and, in the case where $L = \infty$

(4) $\sup_i \Big\{ \sup_{x_1 \in \mathrm{Int}\, I_i} |\tau''(x_1)| / \inf_{x_2 \in \mathrm{Int}\, I_i} |\tau'(x_2)|^2 \Big\} < \infty$.

Suppose that for some n, $|d\tau^n/dt| \geqslant \theta > 1$ for all t. If we regard the above map τ as being defined on the unit circle, its winding number equals L. We call it a *generalised L-covering map.*

It is known that τ has a finite ergodic measure $\rho(x)\, dx$ where $\rho(x)$ is piecewise continuous and $1/D < \rho < D$ for some $D > 0$. See [**1, 3**].

Given a generalised L-covering map τ, construct an one-sided shift space on L symbols as follows: To each $t \in [0,1)$ there corresponds a one-sided infinite sequence $[a_0, a_1, \ldots, a_n, \ldots]$ such that $\tau^n(t) \in I_{a_n}$. For some $t \in [0,1)$, we can find N such that its representation $t = [a_0, a_1, \ldots, a_n, \ldots]$ satisfies the condition that $a_n = 0$ for all $n \geqslant N$. We call such a t a *generalised L-adic point.* Let X be the set of all such sequences and ψ be the assignment of a sequence to a point. Since τ has a finite absolutely continuous ergodic measure $\rho(t)\, dt$, we can define a shift invariant measure ν on any cylinder set $C \subset \prod_0^\infty \{0, 1, \ldots, L-1\}$ by $\nu(C) = \int_{\psi^{-1}(C)} \rho(x)\; dx$. Note that $\psi^{-1}(C)$ is a union of intervals with generalised L-adic endpoints. The Kolmogorov Extension Theorem guarantes that ν may be uniquely extended to the whole σ-algebra. We call the shift space X the *L-adic symbolic system obtained from τ.* Recall that two measure preserving transformations T_1 and T_2 on X_1 and X_2 are said to be *isomorphic* if there exists a measure preserving transformation $\psi : X_1 \to X_2$ which is one-to-one such that $\psi \circ T_1 = T_2 \circ \psi$ on X_1 modulo sets of measure zero. The mapping ϕ introduced above is an isomorphism between $\big((0,1), \rho\, dt, \tau\big)$ and the one-sided shift space $(X, d\nu, \sigma)$.

Our construction also applies even if the condition (3) in Definition 1 does not hold. For example, the interval map $x \mapsto \beta x \pmod 1$, $\beta = (1+\sqrt{5})/2$ has the following special property: Put $I_0 = [0, \beta^{-1})$, $I_1 = [\beta^{-1}, 1)$. If $x \in I_1$, then $\tau x \in I_0$. In other words, in

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any sequence $[x_0, x_1, \dots, x_n, \dots]$ the symbol 1 does not occur consecutively. Hence in this case X would not be the full shift $\prod_0^\infty\{0,1\}$. In fact, it is a shift of finite type with a forbidden block 11. See [1].

DEFINITION 2: An *L-adic multi-index* $\vec{n}$ is a finite sequence of elements of $\{0\,1\,\dots, L-1\}$ and will denote by $\vec{n} = \langle n_1, \dots, n_k\rangle$ Its length k is denoted by $|\vec{n}|$. If there is no danger of ambiguity we call it a *multi-index*. If $|\vec{n}| = 1$, then $\vec{n} = \langle n_0\rangle$ for some n_0, and we write $\vec{n} = n_0$.

Let τ be a map as given in Definition 1. Define $h_i : [0,1) \to [0,1)$ by letting $h_i(t)$ be the unique element in the set $\tau^{-1}(\{t\}) \cap I_i$, $i = 0, 1, \dots L-1$. Note that $\tau^{-1}(\{t\}) = \{h_0(t), h_1(t), \dots, h_{L-1}(t)\}$. For a multi-index $\vec{n} = \langle n_1, \dots, n_k\rangle$, define $h_{\vec{n}} = h_{n_1} \circ \dots \circ h_{n_k}$.

For example, consider the transformation $\tau(x) = 2x \pmod 1$ defined on the unit interval with the partition $\{[0,(1/2)), [(1/2),1)\}$. Since every $t \in [0,1)$ can be represented as a binary expansion, say $t = [t_1, t_2, \dots]$, $h_0(t) = (1/2)t = [0, t_1, t_2, \dots]$ and $h_1(t) = (1/2) + (1/2)t = [1, t_1, t_2, \dots]$. Hence $h_i(t) = [i, t_1, t_2, \dots]$ for $i = 0, 1$. So $h_{\vec{n}}(t) = [n_1, \dots, n_k, t_1, t_2, \dots]$ where $\vec{n} = \langle n_1, \dots, n_k\rangle$. In particular, $h_{\vec{n}}(0) = [n_1, \dots, n_k] = \sum_{j=1}^{k} n_j 2^{-j}$.

From Definitions 1 and 2 we easily obtain

LEMMA 2. *Put $\vec{n} = \langle n_1, \dots, n_k\rangle$. Let τ and h_i be as given in Definitions 1 and 2. Then*

(1) $\tau^{-k}(\{t\}) = \{h_{\vec{n}}(t) : |\vec{n}| = k\}$,
(2) $\tau^k(h_{\vec{n}}(x)) = x$ *where* $k = |\vec{n}|$,
(3) $h_{\vec{n}}([a_1, a_2, \dots]) = [n_1, \dots, n_k, a_1, a_2, \dots]$, *and*
(4) $\tau^{-k}(E) = \bigcup_{|\vec{n}|=k} h_{\vec{n}}(E)$ *for any subset* E.

For any fixed integer $k > 0$ let $\mathcal{P}_0^k$ be the set of numbers of the form $[a_1, \dots, a_k]$, $a_i = 0, 1, \dots, L-1$ so that the points in $\mathcal{P}_0^k$ partition the whole interval $[0,1)$ into L^k segments. Then (i) $h_{\vec{n}}([0,1))$ is one of the L^k intervals obtained by partitioning the unit interval by the points in $\mathcal{P}_0^k$, $k = |\vec{n}|$ and (ii) if $x \in h_{\vec{n}}([0,1))$, $\vec{n} = \langle n_1, \dots, n_k\rangle$, then the coded sequence for x is $[n_1, \dots, n_k, \dots]$.

PROPOSITION 2. *A complex valued step function $\phi(x)$ with finite generalised L-adic discontinuity points $a_1 \leqslant t_1 < \dots < t_n < 1$, is not a coboundary for any generalised L-covering map.*

PROOF: Let (X, σ_X, ν) be the one-sided shift space which is isomorphic to the given L-covering map. Let $Y = \prod_{-\infty}^{\infty}\{0, 1, \dots, L-1\}, \sigma_Y$ the two-sided shift and μ the unique measure on Y so that (Y, σ_Y, μ) is the natural extension of (X, σ_Y, μ)

Assume that $\phi(x)h(\tau x) = h(x)$. Since $\phi(x)$ is step function with finite L-adic discontinuity points, we can regard $\phi(x)$ is function on Y which is measurable, with respect to $\mathcal{B}_0^m = \bigvee_{i=0}^{m} \sigma^{-i}\mathcal{P}$ for some $m < \infty$ (Lemma 2). Hence $h(x)$ is also $\mathcal{B}_0^m$-measurable by Proposition 1. Thus $h(x)$ is also a step function with finite L-adic discontinuity points. Hence there exists $0 < r \leqslant a_1$ such that $h(x)$ is constant on $[0, r)$. Thus $\phi(x)h(x) = h(x)$ on $[0, r)$, that is, $\phi(x) = 1$ on $[0, t_1)$. Since $\phi(x)h(\tau x) = h(x)$, $t_1 \geqslant a_1$ and $\tau[0, a_1) = [0, 1)$, $h(x)$ is constant on $[0, 1)$. Hence the conclusion follows. □

Example 1. For a transformation $T : [0,1) \to [0,1)$ defined by $x \mapsto 2x \pmod 1$, we consider the following. Let $I = \big[(3/4), 1\big)$, $F = \bigcup_{k=0}^{\infty} (1/2^k)I$ and $E = F \vartriangle T^{-1}F$. Then $\phi(x) = \exp\big(\pi i \mathbf{1}_E(x)\big)$ is a coboundary even if the discontinuity points of $\phi(x)$ are contained in $\big[(1/2), 1\big)$ where the cobounding function $h(x) = \exp\big(\pi i \mathbf{1}_F(x)\big)$. Hence the assumption in Proposition 2 of finite discontinuity points cannot be dropped.

4. A class of skew products of circle endomorphisms

In this section, we investigate the dynamical properties of a class of skew products of circle endomorphisms.

Definition 3: For a positive integer L, let $T_{L,a,\omega}$ be the skew-product transformation on the torus, $[0, 1) \times [0, 1)$ defined by

$$(x, y) \mapsto (Lx, y + ax + \omega) \pmod 1.$$

In [9], Siboni considered the skew product transformation $T_{2,a,\omega}$ and proved a criterion of ergodicity of the transformation by the accurate estimation of correlations of characteristic functions. In this section, we shall give a simple proof and generalisation of his results.

For a fixed natural number L, let T be the transformation on $X = [0, 1)$ defined by $Tx = Lx \pmod 1$.

Lemma 3. *Let S and S' be transformations of the torus defined by*

$$S(x, y) = (Lx, y + ax) \pmod 1 \quad \textit{and}$$

$$S'(x, y) = \left(Lx, y + \sum_{k=0}^{L-1} \frac{ka}{L-1} \mathbf{1}_{[(k/L),(k+1/L))}(x)\right) \pmod 1.$$

Then S and S' are isomorphic.

Proof: We will use the L-adic expansion of x; $x = (x_0/L) + (x_1/L^2) + \cdots$. Let $\phi(x, y) = (x, y + (a/L - 1)x) \pmod 1$. Then $\phi^{-1}(x, y) = \big(x, y - (a/L - 1)x\big) \pmod 1$

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and

$$\begin{aligned}\phi^{-1}S\phi(x,y) &= \phi^{-1}\Big(Lx, y+ax+\frac{a}{L-1}x\Big)\\ &= \Big(Lx, y+ax+\frac{a}{L-1}x-\frac{a}{L-1}(Lx-k)\Big) \quad \text{if } x_0=k\\ &= \Big(Lx, y+\frac{ka}{L-1}\Big) \quad \text{if } x_0=k\\ &= S'(x,y).\end{aligned}$$

□

Hence to investigate the spectral type of S, we only need to study the spectral type of S'.

PROPOSITION 3. *$T_{L,a,\omega}$ is weakly mixing if and only if a is irrational. Further if a is rational then $T_{L,a,\omega}$ is ergodic if and only if ω is irrational.*

PROOF: For convenience, let us denote the 2-torus by $X\times\mathbb{T}$. As before, we use the L-adic expansion of x. Recall that $L^2(X\times\mathbb{T}) = \bigoplus_{n=-\infty}^{\infty} f(x)\cdot z^n$.

Let $\eta(x)=\exp\big(2\pi i(ax+\omega)\big)$, $\psi(x)=\exp(2\pi iax)$ and $\phi(x)=\exp\big(2\pi ia(x_0/L-1)\big)$. Then

$$U_{T_\eta}\big(f(x)\cdot z^n\big) = \exp(2\pi inax)\cdot\exp(2\pi in\omega)\cdot f(Tx)\cdot z^n.$$

We consider the operator

$$U\big(f(x)\big) = \exp(2\pi inax)\cdot\exp(2\pi in\omega)\cdot f(Tx).$$

If $n=0$, then $U\big(f(x)\big)=f(Tx)$. Thus if $f(Tx)=\lambda f(x)$ then $f(x)$ is constant and $\lambda=1$ by the mixing property of T. Hence it remains to consider the case $n\neq 0$. Assume that $U\big(f(x)\big)=\lambda f(x)$. Then

$$\exp(2\pi inax)\exp(2\pi in\omega)f(Tx)=\lambda f(x)$$

and $|\lambda|=1$. Without loss of generality we may assume that $\big|f(x)\big|=1$ almost everywere. So $\exp(2\pi inax)=\lambda'\overline{f(Tx)}f(x)$, where $\lambda'=\lambda\exp(-2\pi in\omega)$. By Lemma 3, U_{T_ψ} is spectrally equivalent to U_{T_ϕ} and U_{T_ϕ} also has an eigenfunction $g(x)$ with eigenvalue λ', that is, $\overline{\lambda'}\exp\Big(2\pi in\sum_{k=0}^{L-1}(ka/L-1)\mathbf{1}_{[(k/L),(k+1/L))}(x)\Big) = \overline{g(Tx)}g(x)$. If a is an irrational number, then there exists the only integer n for mention.

$$\overline{\lambda'}\exp\Big(2\pi in\sum_{k=0}^{L-1}\frac{ka}{L-1}\mathbf{1}_{[(k/L),(k+1/L))}(x)\Big)$$

is constant is $n=0$. Hence by Proposition 2, there exists no eigenfunction for U_{T_ϕ}. If a is rational then there exists n such that $\exp\Big(2\pi in\sum_{k=0}^{L-1}(ka/L-1)\mathbf{1}_{[(k/L),(k+1/L))}(x)\Big)=1$.

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So the problem is reduced to finding $g(x)$ such that $g(Tx) = \lambda' g(x)$. By the mixing property of T we know that $g(x)$ is constant and $\lambda' = \lambda \exp(-2\pi i n\omega) = 1$ for each $n \neq 0$. So U_{T_ϕ} has eigenfunction $g(x,z) = z^n$ with eigenvalue $\lambda = \exp(2\pi i n\omega)$. Hence $T_{L,a,\omega}$ is not weakly mixing. For the ergodicity, we need only consider the case $\lambda = 1$, that is, $\exp(2\pi i n\omega) = 1$. Hence if a is rational, then $T_{L,a,\omega}$ is ergodic if and only if ω is irrational. ∎

REMARK 1. Indeed we have shown that if a is irrational then $T_{L,a,\omega}$ is exact on the torus. In fact, $T_{L,a,\omega}$ is strong mixing. To see this, let $\mathcal{B}_1$ be the Borel σ-algebra on $[0,1)$, $\mathcal{B}_2$ be the Borel σ-algebra on $\mathbb{T}$ and $\mathcal{B}_\infty = \bigcap_{n=0}^{\infty} T_{L,a,\omega}^{-n}(\mathcal{B}_1 \times \mathcal{B}_2)$. We only need to show that $L^2(\mathcal{B}_\infty)$ is the set of constant functions. We use the fact that $T_{L,a,\omega}$ commutes with the circle action $(x,z) \mapsto (x, z\cdot g)$ so that for every n the σ-algebra $T_{L,a,\omega}^{-n}(\mathcal{B}_1 \times \mathcal{B}_2)$ is preserved by the circle action. Hence $\mathcal{B}_\infty$ is invariant with respect to the circle action. If $f \in L^2(\mathcal{B}_\infty)$ then it has a representation

$$f(x,z) = \sum\nolimits_k f_k(x)\cdot z^k$$

and $f(x, z\cdot g) = \sum_k f_k(x)\cdot z^k \cdot g^k$ for all $g \in \mathbb{T}$. Hence $f_k(x)\cdot z^k \in L^2(\mathcal{B}_\infty)$. So $|f_k(x)\cdot z^k| = |f_k(x)|$ is also $\mathcal{B}_\infty$ measurable. But $|f_k(x)|$ depends only on x. Hence $|f_k(x)|$ is constant by the exactness of T. By a similar argument $\overline{f_k}\cdot f_k \circ T_{L,a,\omega}$ is also constant, that is,

$$\exp(2\pi i k a x)\cdot \exp(2\pi i k\omega)\cdot f_k(Tx)\cdot z^k = \lambda f_k(x)\cdot z^k$$

where $\lambda \in \mathbb{C}$. Thus if $T_{L,a,\omega}$ is weakly mixing then $T_{L,a,\omega}$ is exact. For more information on this subject, see [5].

5. MOD M NORMALITY OF L-COVERING MAPS

In this section, let G be the finite subgroup of $\mathbb{T}$ generated by $\exp(2\pi i/M)$.

PROPOSITION 4. *Let T be an ergodic transformation on X and $\phi(x)$ be a G-valued function. Let T_ϕ be the skew product transformation defined by $T_\phi(x,g) = (Tx, \phi(x)\cdot g)$ on $X \times G$. If $\phi(x)h(Tx) = h(x)$, then there exists a G-valued function $q(x)$ such that the following diagram commutes*

$$\begin{array}{ccc} X\times G & \xrightarrow{T_\phi} & X\times G \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ X\times G & \xrightarrow{S} & X\times G \end{array}$$

where $Q(x,g) = (x, q(x)\cdot g)$ and $S(x,g) = (Tx, g)$. Hence T_ϕ has M ergodic components.

PROOF: Since $(\phi(x))^M = 1$, $(\phi(x))^M (h(Tx))^M = (h(x))^M$ is equivalent to $(h(Tx))^M = (h(x))^M$. So we may assume that $(h(x))^M = 1$ by the ergodicity of T. Hence there exist a G-valued function $q(x)$ such that $\phi(x)q(Tx) = q(x)$. For $q(x)$, of this for it is easy to see that the diagram commutes. ∎

LEMMA 4. *Let τ be piecewise, twice continuously differentiable and such that $\inf_{x\in J_1} |\tau'(x)| > 1$ where $J_1 = \{x \in X : \tau'(x) \text{exists}\}$. If the number of discontinuity points of τ of τ' is finite, then there is a finite collection of sets $L_1, \dots, L_n$ and a set of invariant functions $\{f_1, \dots, f_n\}$ such that*

(1) *each $L_i (1 \leqslant i \leqslant n)$ is a finite union of closed intervals;*
(2) *$L_i \cap L_j$ contains at most a finite number of points when $i \neq j$;*
(3) *$f_i(x) = 0$ for $x \notin L_i, 1 \leqslant i \leqslant n$, and $f_i(x) > 0$ for almost everywhere x in L_i;*
(4) *$\int_{L_i} f_i(x)\,dx = 1$ for $1 \leqslant i \leqslant n$;*
(5) *every τ invariant function can be written as $f = \sum_{i=1}^{n} a_i f_i$ with suitable chosen $\{a_i\}$.*

PROOF: For the proof, See [7]. ∎

PROPOSITION 5. *If a G-valued function $\phi(x)$ is a step function with finite discontinuity points $a_1 \leqslant t_1 < \cdots < t_n < 1$, then $\phi(x)$ is not coboundary for any generalised L-covering map.*

PROOF: Assume that $\phi(x)h(Tx) = h(x)$. Without loss of generality assume that $X = [0, 1)$. Since $X \times G = \bigcup_{k=0}^{M-1} \{X \times \exp((2k\pi i)/M)\}$, we may identify $\{X \times \exp((2k\pi i)/M)\}$ with the unit interval $[k, k+1)$, $0 \leqslant k < M$. Since $\phi(x)$ is a G-valued step function with finite discontinuity points, we can regard T_ϕ as a piecewise continuous map on $[0, M)$ satisfying the condition of Lemma 4. So we can say that $h(x)$ is also a G-valued step function with finite discontinuity points by Lemma 4 and Proposition 4. Hence there exists $0 < r \leqslant a_1$ such that $h(x)$ is constant on $[0, r)$. Thus $\phi(x)h(x) = h(x)$ on $[0, r)$. So $h(x)$ is constant on $[0, 1)$ by the argument of the proof of Proposition 2. The conclusion follows. ∎

REMARK 2. In Proposition 5, we have not assumed that the discontinuity points are generalised L-adic points, but rather that the range of $\phi(x)$ is contained in a finite subgroup of $\mathbb{T}$. For generalised L-covering maps, mod M normality holds for finite unions of intervals, when the associated step function $\phi(x)$ step satisfies the condition of Proposition 5. Indeed we may also show that the skew product transformation induced by $\phi(x)$ is also exact on $X \times G$, as in fact strong mixing, using the similar arguments, to those in Remark 1.

6. Mod M normality of Gauss transformation

Recall that the Gauss transformation T on $[0,1)$ is defined by

$$T(x)=\begin{cases} 0 & \text{if} \quad x=0 \\ \dfrac{1}{x} & \text{if} \quad x\neq 0 \pmod 1. \end{cases}$$

It is well known that T preserves the Gauss measure on $[0,1)$ given by

$$\mu(A)=\frac{1}{\log 2}\int_A \frac{1}{1+x}\,dx.$$

Let $\mathcal{P}=\{P_j\}$ be a partition on $[0,1)$ defined by $P_j=\big[(1/j+1),(1/j)\big)$ for $j\in\mathbb{N}$.

Proposition 6. *Let $\{B_i\}$ be a sequence of intervals of $[0,1)$ with rational endpoints and $\{b_i\}$ be a sequence of real numbers. Then a nonconstant function $\phi(x)$ $=\exp\left(2\pi i\sum_{i=1}^{n} b_i\mathbf{1}_{B_i}(x)\right)$ is not a coboundary for the Gauss transformation.*

Proof: Let $Y=\prod_{-\infty}^{\infty}\{1,2,\dots\}$ and $Y^+=\prod_{0}^{\infty}\{1,2,\dots\}$. Consider the following commutative diagram

$$\begin{array}{ccc} [0,1) & \xrightarrow{\;T\;} & [0,1) \\ \psi\big\downarrow & & \big\downarrow\psi \\ Y^+ & \xrightarrow{\;\sigma^+\;} & Y^+ \end{array}$$

where $\big(\psi(x)\big)_i=j$ if $T^ix\in P_j$ for $i\in\mathbb{N}$. Then ψ is a an isomorphism between $\big([0,1),T,\mu\big)$ and (Y^+,σ^+,ν^+) where ν^+ is the induced measure by ψ and σ^+ is the one-sided shift map on Y^+. Let (Y,σ,ν) be the natural extension of (Y^+,σ^+,ν^+) where σ is the two-sided shift map on Y. If $\phi(x)g(Tx)=g(x)$ then $g(x)$ is also step function with rational discontinuity points and there exist an interval I with rational end points such that $g(x)$ is constant on I by the arguments of Proposition 1. Since $T^nI=[0,1)$ for some n, $\phi(x)$ is a function with finite discontinuity points, and $\phi(x)g(Tx)=g(x)$, $g(x)$ is also a function with finite discontinuity points. Since $\phi(x)g(Tx)=g(x)$ can be rewritten as $\phi(x)=g(x)\overline{g(Tx)}$, $\phi(x)$ must be a function with infinite discontinuity points. this is a contradiction. ∎

Remark 3. By Proposition 6, mod M normality holds for finite union of intervals with rational end points on Gauss transformation. By the similar arguments as in Remark 1, the induced skew product transformation on $X\times G$ is also exact.

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