Compositio Mathematica **105:** 65–77, 1997.

Holomorphic vector fields with totally degenerate zeroes

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Received 21 March 1995; accepted in final form 6 November 1995

Abstract. A zero set of a holomorphic vector field is totally degenerate, if the endomorphism of the conormal sheaf induced by the vector field is identically zero. By studying a class of foliations generalizing foliations of $\mathbf{C}^*$-actions, we show that if a projective manifold admits a holomorphic vector field with a smooth totally degenerate zero component, then the manifold is stably birational to that component of the zero set. When the vector field has an isolated totally degenerate zero, we prove that the manifold is rational. This is a special case of Carrell's conjecture.

Key words: holomorphic vector fields, holomorphic foliations, rationality.

1. Introduction

Let X be a complex projective manifold and V be a holomorphic vector field on X. For a component Z of the zero set of V, we have an induced $\mathcal{O}_Z$-linear endomorphism $\mathcal{L}_V$ on the conormal sheaf $\mathcal{I}_Z/\mathcal{I}_Z^2$. We say that Z is *nondegenerate*, if $\mathcal{L}_V$ has non-zero determinant at every point of Z. Using the Bialynicki-Birula decomposition ([BB]), Lieberman showed in [L] that if all zero components of V are nondegenerate, then there exists a component Z of the zero set, such that X is birational to $Z \times \mathbf{P}_r$ for $r = \mathrm{codim}(Z)$.

It is expected that a similar result holds for arbitrary holomorphic vector fields. But without the nondegeneracy assumption, very little is known. In this paper, we will study the other extreme case, namely when the map $\mathcal{L}_V$ on the conormal sheaf of Z is zero. In this case, we say that Z is *totally degenerate*. If V has an isolated zero, it is totally degenerate if the linear part of the vector field at that point vanishes. For example, consider a vector field induced by an additive group action on the projective plane with a single fixed point, i.e. a vector field corresponding to a nilpotent matrix with a single Jordan block. If we blow-up the plane at the single fixed point, we get a vector field with an isolated totally degenerate zero. Our main result is

COROLLARY (to Theorem 2). *Suppose that a projective manifold X has a holomorphic vector field a connected component of whose zero set is smooth and totally degenerate. Then X is stably birational to that component.*

By definition, projective manifolds X, Y are *stably birational*, if $X \times \mathbf{P}_p$ is birational to $Y \times \mathbf{P}_q$ for some positive integers p, q. Similarly, a projective manifold is *stably rational*, if its product with some projective space is rational. As a direct consequence of the above theorem, if a projective manifold has a holomorphic vector field with an isolated totally degenerate zero, it is stably rational. But in this case, we can refine our result to

THEOREM 3 *Let V be a holomorphic vector field on a projective manifold X with an isolated totally degenerate zero P. Then X is rational.*

This result is a special case of Carrell's conjecture ([C]) that a projective manifold having a holomorphic vector field with finite number of zeroes is rational. In fact, it is easy to see that a holomorphic vector field with an isolated totally degenerate zero has no other zero (see the proof of Corollary). So a proof of Carrell's conjecture would imply our result. Note that Carrell's conjecture was proved for dimension $\leqslant 5$ in [H2].

Carrell's conjecture is true if the vector field is induced by a $\mathbf{C}^*$-action. To prove the conjecture, it is enough to prove it under the assumption that the vector field is induced by an additive group action and there exists just a single zero point. So far, most results on Carrell's conjecture are proved by reducing the problem to $\mathbf{C}^*$-action case using some additional assumptions ([C], [K]). For example, Konarski proved it under the assumption that the linear part of the vector field at the isolated zero consists of a single Jordan block ([K]). But such a reduction is very difficult, if the vector field has no linear information. In this sense, our result takes care of a bad case.

The main idea of this paper is, instead of reducing the problem to $\mathbf{C}^*$-case, extracting the key property of the foliation defined by a $\mathbf{C}^*$-action, to define a more general class of foliations 'semisimple foliations'. Then we construct a blow-up procedure for this special class of foliations, which is similar to the blow-up procedure introduced in [H2] for an additive group action. The problem is whether the foliations on the blow-up of the manifold at the totally degenerate zero belong to this class. This can be done based on our earlier result in [H1]. For the isolated totally degenerate case, the rationality can be obtained by studying the structure of a semisimple foliation on the projective space.

2. Birational geometry of semisimple foliations

We recall basic notions in foliations by curves and introduce semisimple foliations.

A 1-dimensional foliation with singularity on a complex projective manifold X is a nontrivial map $\xi\colon L \to T_X$ from a line bundle to the tangent bundle. The subscheme $\mathrm{Sing}(\xi)$ defined by $\xi = 0$ is called the *singular loci* of this foliation. ξ induces a foliation on the complement of $\mathrm{Sing}(\xi)$. If every leaf of this foliation can be compactified to a rational algebraic curve in X, we say that ξ is a *foliation* *by rational curves*. A rational curve compactifying a leaf will be called an *integral curve* of ξ.

A foliation $\xi : L \to T_X$ is said to be *saturated* if Sing(ξ) has no component of codimension 1. If Sing(ξ) contains a divisor D, we get a new foliation $\frac{1}{D}\xi : L(D) \to T_X$ divided by a local defining function of D. Here $L(D)$ is the tensor product of L with the line bundle corresponding to D. If ξ is not saturated, by an irreducible component of Sing(ξ), we mean an irreducible component of either the underlying subvariety of Sing(ξ) or the underlying subvariety of the foliation obtained after divided by the codimension one part.

A foliation ξ by rational curves on a complex projective manifold X is *semisimple*, if there exists an irreducible component Z of Sing(ξ) and a proper subvariety $Z' \subset Z$, with the following properties:

(1) For any point $P \in Z \backslash Z'$, we can find a local coordinate system $z_1, \ldots, z_n, n = \dim(X)$ for X in a neighborhood of P, such that in that neighborhood, the foliation is tangential to a vector field of the form $\lambda_1 z_1 \frac{\partial}{\partial z_1} + \cdots + \lambda_l z_l \frac{\partial}{\partial z_l}$ where $l \leq n$ and λ_i is a positive integer for each $i = 1, \ldots, l$.

(2) Let $\nu : \mathbf{P}_1 \to C$ be the normalization of a generic integral curve. Then the set $\nu^{-1}(Z \setminus Z')$ is a single point.

We will call Z a *semisimple limit variety* of the semisimple foliation. When we say a generic point of Z, we mean a point on $Z \setminus Z'$. Note that if the coordinate system in Condition 1 exists for a point $P \in X$, then in that coordinate neighborhood, the submanifold defined by $z_1 = \cdots = z_l = 0$ is a component of Sing(ξ), and such a coordinate system exists for any point sufficiently close to P lying on the submanifold $z_1 = \cdots = z_l = 0$. It follows that $z_1 = \cdots = z_l = 0$ is precisely the defining equation for the semisimple limit variety Z.

For example, the foliation induced by a $\mathbf{C}^*$-action is semisimple with two semisimple limit varieties, so-called sink and source ([BB]).

For a foliation ξ by rational curves, we can find the subscheme of the Hilbert scheme parametrizing the integral curves. Taking reduction if necessary, we get projective varieties $W, \mathcal{H}$, a birational morphism $\chi : W \to X$ and a proper morphism $\eta : W \to \mathcal{H}$. The generic fibers of η are irreducible rational curves and χ gives an embedding of them into X. Their images are precisely the integral curves of ξ. This pair of morphisms will be called the *Fujiki family associated to* ξ.

For a semisimple foliation, we define the *index* of the foliation as the intersection number of the anti-canonical bundle K_X^{-1} with a generic integral curve. Since the deformations of a generic integral curve sweep out an open set in X, the index is always greater than or equal to 2.

The most important property of a semisimple foliation is that the semisimplicity is preserved under the blow-up of a semisimple limit variety:

PROPOSITION 1 *Let ξ be a semisimple foliation on a projective manifold X with a smooth semisimple limit variety Z. Let $\pi : \tilde{X} \to X$ be the blow-up of X along*

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Z and let $\tilde{\xi}$ be the lifting of the foliation ξ. Then $\tilde{\xi}$ is also semisimple and there exists a semisimple limit variety $\tilde{Z}$ inside the exceptional divisor. Moreover $\tilde{Z}$ is birational to $Z \times \mathbf{P}_m$ for some nonnegative integer m.

Proof. It is obvious that $\tilde{\xi}$ is a foliation by rational curves. So it is enough to find a subvariety $\tilde{Z}$ in the exceptional divisor E with the desired properties.

Let P be a generic point of Z. By assumption, we have local coordinates $z_1, \ldots, z_n, n = \dim(X)$ in a neighborhood U of P, such that ξ is tangential to the vector field $v = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \cdots + \lambda_l z_l \frac{\partial}{\partial z_l}$, where $\lambda_1 \leq \cdots \leq \lambda_l$ are positive integers. Recall that $Z \cap U$ is defined by $z_1 = \cdots = z_l = 0$.

Let us choose the coordinates $u_1 = z_1, u_1 u_2 = z_2, \ldots, u_1 u_l = z_l, u_{l+1} = z_{l+1}, \ldots, u_n = z_n$ for $\tilde{X}$. Then the lifting $\tilde{v}$ of the vector field v is given by

$$\tilde{v} = \lambda_1 u_1 \frac{\partial}{\partial u_1} + (\lambda_2 - \lambda_1) u_2 \frac{\partial}{\partial u_2} + \cdots + (\lambda_l - \lambda_1) u_l \frac{\partial}{\partial u_l}.$$

Let k be the largest integer satisfying $\lambda_k = \lambda_1$. From the expression, it is obvious that in the open subset of $\tilde{X}$, where $u_1, \ldots, u_n$ are defined, Sing$(\tilde{\xi})$ has a component defined by $u_1 = u_{k+1} = \cdots = u_l = 0$. Let $\tilde{Z}$ be the component of Sing$(\tilde{\xi})$ whose intersection with this open set is $u_1 = u_{k+1} = \cdots = u_l = 0$. Certainly, $\tilde{Z}$ is contained in the exceptional divisor E. We claim that $\tilde{Z}$ is a semisimple limit variety of ξ.

First, let us check Condition 1 in the definition of a semisimple foliation. From the above expression for $\tilde{v}$, we can find the desired coordinate system at a generic point of $\tilde{Z}$ over the point $P \in X$. A direct calculation using the coordinates $z_1, \ldots, z_n$ shows that $\tilde{Z}$ is the only irreducible component of Sing$(\tilde{\xi})$ with this property, over the open set U. Since P can be chosen as any generic point of Z, this implies that we can find the desired coordinate system at any generic point of $\tilde{Z}$.

Now, let $\nu\colon \mathbf{P}_1 \to C$ be the normalization of a generic integral curve of $\tilde{\xi}$. Then $\pi \circ \nu\colon \mathbf{P}_1 \to \pi(C)$ is the normalization of a generic integral curve of ξ. Since $\tilde{Z} \subset E = \pi^{-1}(Z)$ and $\pi^{-1}(Z')$ does not contain $\tilde{Z}$, Condition 2 for ξ implies Condition 2 for $\tilde{\xi}$, for a suitable choice of proper subvariety $\tilde{Z}' \subset \tilde{Z}$. This shows that $\tilde{\xi}$ is semisimple and $\tilde{Z}$ is a semisimple limit variety.

In the coordinates chosen above, the intersection of $\tilde{Z}$ with the fiber $\pi^{-1}(P)$ is exactly the submanifold defined by $u_{k+1} = \cdots = u_l = 0$. Since $u_2, \ldots, u_l$ give an inhomogeneous coordinate system on the projective space $\pi^{-1}(P)$, we can see that $\pi|_{\tilde{Z}}\colon \tilde{Z} \to Z$ is generically a $\mathbf{P}_m$-bundle, $m = k - 1$. Now the dual tautological bundle on E which is the projectivization of the normal bundle of $Z \subset X$, gives a line bundle on $\tilde{Z}$, whose restriction on a generic fiber of $\pi|_{\tilde{Z}}\colon \tilde{Z} \to Z$ is the hyperplane bundle for $\mathbf{P}_m$. It follows that $\tilde{Z}$ is birational to $Z \times \mathbf{P}_m$. $\square$

It was proved in [BB] that for a foliation induced by a $\mathbf{C}^*$-action on X, if Z is one of the two semisimple limit varieties, then $Z \times \mathbf{P}_r$ is birational to X where $r =$codim(Z). The following theorem is a generalization of Bialynicki-Birula's result to arbitrary semisimple foliations.

THEOREM 1 *Suppose that Z is a semisimple limit variety of a semisimple foliation on X. Then $Z \times \mathbf{P}_r$ is birational to X for $r =$*codim(Z).

Proof. We will first prove the theorem for $r = 1$, and then introduce a blow-up procedure to reduce the general case to $r = 1$.

(*Proof when* $r = 1$)

Let ξ be the semisimple foliation on X and Z be a semisimple limit variety which is a hypersurface in X. Locally at a generic point of Z, ξ is tangential to a vector field of the form $z_1 \frac{\partial}{\partial z_1}$. This means that ξ is not saturated and has removable singularity at a generic point of Z. From the definition of a semisimple foliation, we conclude that each generic integral curve intersects the generic part of Z transversally at one point.

Let $\eta\colon W \to \mathcal{H}$ and $\chi\colon W \to X$ be the associated Fujiki family of ξ. From the fact that a generic integral curve intersects the generic part of Z transversally at one point, the strict transform $[Z]$ of Z by χ is a reduced Weil divisor in W and each generic fiber of ρ intersects $[Z]$ only at one point. Hence $[Z]$ is birational to $\mathcal{H}$.

By taking normalization of W if necessary, we can assume that a generic fiber of η is a smooth rational curve. Over an affine open subset $\mathcal{U} \subset \mathcal{H}$, we can assume that η is a $\mathbf{P}_1$-bundle and $[Z]$ is a Cartier divisor inducing $\mathcal{O}(1)$ on the generic fibers. It follows that W is birational to $[Z] \times \mathbf{P}_1$, which implies that X is birational to $Z \times \mathbf{P}_1$ because χ is birational.

(*Proof for the general case*)

We introduce a blow-up procedure for any semisimple foliation with the following choice of blow-up center:

(i) If a semisimple limit variety is smooth, then that semisimple limit variety is the blow-up center. By Proposition 1, we can lift the foliation to a semisimple foliation on the blow-up, and the new semisimple limit variety is birational to the product of old semisimple limit variety with $\mathbf{P}_m$ for some nonnegative integer m.

(ii) If all semisimple limit varieties are singular, then the blow-up center is a submanifold of a semisimple limit variety, which is the blow-up center for an embedded resolution of the singularity of that semisimple limit variety. We can lift the given foliation to a semisimple foliation on the blow-up. In fact, a semisimple limit variety for the lifted foliation is just the strict transform of the semisimple limit variety below.

From the existence of the embedded resolution, the choice (ii) can appear consecutively only for a finite number of times. To show that the choice (i) can appear only for a finite number of times, we look at the change of the index under the blow-up $\pi : \tilde{X} \to X$ along a smooth generic limit variety. Note that $K_{\tilde{X}}^{-1} = \pi^* K_X^{-1} - (r-1)E$ where E is the exceptional divisor. Since the generic integral curve on $\tilde{X}$ always intersects E, the index decreases strictly. But we know that the index is always greater than or equal to 2. This shows that the blow-up

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procedure must stop after a finite number of steps, and we get to the situation of $r = 1$. □

3. Semisimplicity of foliations arising from totally degenerate zeroes

Let V be an additive vector field on an irreducible projective variety X, i.e. V is induced by an action of the additive group $\mathbf{C}^+$. It is well-known that each nontrivial orbit of the action can be compactified to a smooth rational curve by adding one point. This rational curve is called an orbital curve and the unique compactifying point is called the limit point of the orbit. We define the *generic limit variety* of an additive vector field as the unique irreducible subvariety of X which is the closure of the limit points of generic points of X ([H2]). In other words, any generic orbital curve intersects the generic limit variety, and the generic limit variety is the irreducible subvariety of the minimal dimension having this property, which is contained in the zero set of V. If the foliation defined by an additive vector field is semisimple, a generic point of the semisimple limit variety is the limit point of a generic point of X. Hence, in this case, the generic limit variety of the additive vector field is the only semisimple limit variety of the semisimple foliation. The following theorem tells us that under some condition on the zero set, an additive vector field gives rise to semisimple foliations.

THEOREM 2 *Let V be an additive vector field on a projective manifold X, which vanishes on a smooth divisor D. Assume that any generic orbital curve intersects D. Then the foliation defined by V on X is a semisimple foliation with a unique semisimple limit variety Z contained in D. Moreover the foliation defined on D by the meromorphic vector field $\frac{1}{D}V$ is semisimple and Z is also a semisimple limit variety of this foliation.*

Proof. We will use the following lemma which follows directly from Lemma 1 in [H1] and its proof. For the reader's convenience, we will give a proof.

LEMMA. *Let X, V and D be as in the assumption of Theorem 2. Then the following is true.*

(i) *At the limit point of a generic point of X, we can find a local coordinate system $z_1, \ldots, z_n, n = \dim(X)$ of X and a local defining function $d(z)$ of D, in terms of which $V = d(z)(\lambda_1 z_1 \frac{\partial}{\partial z_1} + \cdots + \lambda_l z_l \frac{\partial}{\partial z_l})$, where l and λ_i's are positive integers.*
(ii) *The function $d(z)$ satisfies the functional equation $\frac{1}{d(z)}V(d(z)e(z)) = d(z)e(z)$ for some nonvanishing local holomorphic function $e(z)$.*
(iii) *The limit point Q of a generic point has the attracting property, i.e. for any sufficiently small neighborhood U of Q, any point in U has its limit point in U. Here, the limit point of a zero point of V is defined as the zero point itself.*

Proof of lemma. Let $\phi\colon\ F \to X$ and $\psi\colon\ F \to \mathcal{D}$ be the Fujiki family associated to V. Namely, $\mathcal{D}$ is a subvariety of the Hilbert scheme parametrizing invariant curves, and F is the corresponding universal family. Note that ϕ is a birational morphism. We can find an additive vector field W on F, so that W is tangential along the fibers of ψ and $\phi_*(W) = V$. A generic fiber of ψ is $\mathbf{P}_1$. There exists an irreducible hypersurface H in F, so that H is birational to $\mathcal{D}$ by ψ and W vanishes to order 2 along H. At a generic point Q of H, we can find a local coordinate system $y_1, \ldots, y_n$, such that $W = uy_1^2\frac{\partial}{\partial y_1}$, where u is a unit in $\mathcal{O}_Q$. This follows from the standard form of additive vector fields on $\mathbf{P}_1$.

Let $P = \phi(Q)$. By the assumption, P lies on D. Let $h \in \mathcal{O}_P$ be a defining function of D. Suppose that $\frac{1}{h}V$ is tangential along D. Then $Vh = gh^2$ for some $g \in \mathcal{O}_P$. We can write $\phi^*h = vy_1$ for some unit $v \in \mathcal{O}_Q$. From $\phi^*(Vh) = W(vy_1) = uy_1^2(y_1\frac{\partial v}{\partial y_1} + v)$, we have $\phi^*g = \frac{u}{v} + \frac{u}{v^2}y_1\frac{\partial v}{\partial y_1}$. It follows that g is a unit in $\mathcal{O}_P$, and we can consider the local vector field $\frac{1}{gh}V$. By a direct power series calculation, we can find a new coordinate system $x_1, \ldots, x_n$ near Q, so that $\frac{1}{gh}V = \phi_*(x_1\frac{\partial}{\partial x_1})$. It follows that $\frac{1}{gh}V$ is locally of the form $\lambda_1 z_1\frac{\partial}{\partial z_1} + \cdots + \lambda_l z_l\frac{\partial}{\partial z_l}$ for some positive integers $\lambda_1, \ldots, \lambda_l$, in some local coordinates $z_1, \ldots, z_n$ at P, because if a vector field has such a coordinate system, then so does its birational push forward (Proposition 2 in [H1]).

Let $d = gh$. Then (i) is proved. (ii) follows by setting $e = g^{-1}$ and (iii) follows from (i) directly.

It remains to take care of the case when $\frac{1}{h}V$ is not tangential along D. In this case, we can find an invariant curve through any generic point of D, which is not contained in D. Then every point in a neighborhood of P is a limit point. In particular, V vanishes to order 2 along D. Let $\frac{1}{h^2}Vh = g$ and repeat the above argument. □

Now, let us prove Theorem 2, using the lemma. Let $Z \subset D$ be the closure of the limit points of generic points in X, i.e. Z is the generic limit variety of the additive vector field V. Then Z is a semisimple limit variety of the foliation defined by V. In fact, Condition 1 follows from (i) of the lemma, and Condition 2 follows from the fact that an orbital curve of V can intersect D only at one point, because V is additive. So the foliation defined by V is semisimple and Z is the only semisimple limit variety.

Now we claim that Z is also a semisimple limit variety of the foliation defined by $\frac{1}{D}V$ on D. First of all, the leaves of the foliation are limits of leaves of the foliation defined by V on $X \setminus D$. So they can be compactified to algebraic curves. These algebraic curves are specializations of orbital curves of V. So the foliation is a foliation by rational curves.

From (ii) in the lemma, the linear part of the local defining function $d(z)$ of D is a linear combination of $z_1, \ldots, z_l$. Now a suitable choice of $l-1$ functions $w_i = z_{j(i)}$ from $z_1, \ldots, z_l$ and $w_l = z_{l+1}, \ldots, w_{n-1} = z_n$ define a coordinate system on D and with respect to that system, $\frac{1}{d(z)}V$ is of the form $\nu_1 w_1\frac{\partial}{\partial w_1} + \cdots + \nu_{l-1}w_{l-1}\frac{\partial}{\partial w_{l-1}}$ for some positive integers $\nu_i = \lambda_{j(i)}$. This shows that the foliation $\frac{1}{D}V$ satisfies Condition 1 of semisimplicity.

To check Condition 2, assume the contrary. There are two possibilities. First assume that there exists a sufficiently generic integral curve $C \subset D$ of the foliation which intersects the generic part of Z at two or more distinct points. Let $A, B \in Z \cap C$ be two distinct points where coordinate systems with the property given in the lemma exist. Let $\mathcal{E}_A$ be the closure of the constructible set of points in X whose limit point is A. Then from the local expression of V at A, we can see that $\mathcal{E}_A$ is a subvariety in X containing the curve C. In particular, $B \in \mathcal{E}_A$ and there are points arbitrarily close to B whose limit point is A. This is a contradiction to (iii) of the lemma.

The only remaining possibility is when a generic integral curve C of the foliation on D intersects the generic part of Z only at one point R, but the normalization $\nu\colon \mathbf{P}_1 \to C$ is not one-to-one over that point. We will call such a point a multiple point of the curve. We know that C is the underlying curve of an irreducible component of the specialization of smooth rational curves. Consider the invariant subvariety $\mathcal{E}_R$ in X as above. The additive vector field V on $\mathcal{E}_R$ has an isolated attracting point. By choosing a suitable curve in the Hilbert scheme parametrizing integral curves of V in $\mathcal{E}_R$, we get a surface $S \subset \mathcal{E}_R$ which is invariant under V and contains C. We can assume that $V|_S$ is a nontrivial additive vector field and the point R is an isolated attracting point, i.e. R has the attracting property as defined in (iii) of the lemma.

Let $\alpha\colon \hat{S} \to S$ be the normalization of S. We claim that α is one-to-one over R. In fact, V can be lifted to $\hat{S}$, and any inverse image of R is an isolated attracting point. If an additive vector field has an isolated attracting point, the point must be the generic limit variety and there exists no other attracting point. So, there cannot be two or more inverse images of R. In particular, $\hat{S}$ also contains an invariant rational curve with a multiple point. The equivariant resolution $\tilde{S}$ of $\hat{S}$ will contain either an invariant rational curve with a multiple point or a union of invariant rational cuves which forms a cycle, i.e. not simply connected. Any additive group action descends to any minimal model of a surface. Under the contraction of a (-1)-curve, an invariant rational curve with a multiple point is sent to an invariant rational curve with a multiple point. A cycle of rational curves is sent to a cycle of rational curves or a union of rational curves one of whose components has a multiple point. So we get a ruled surface with an additive group action, containing an invariant rational curve with a multiple point or a cycle of rational curves. If there is an invariant rational curve with a multiple point, let us call it C. If there is an invariant cycle of rational curves, we can choose a cycle with a minimal possible number of components. In this case, let C be such a minimal cycle of rational curves. We may assume that C is reduced. In either case, any irreducible component of smooth part of C cannot be biholomorphic to $\mathbf{C}$. This implies that the vector field vanishes on C.

Consider a member of the ruling on this ruled surface. It is always smooth. Hence C cannot be a member of the ruling. This implies that a generic member of the ruling must intersect a component of C. Since the vector field vanishes on

C, the additive group action moves the member of the ruling with some points fixed. But the member of the ruling have trivial self-intersection. It follows that the vector field is tangential along the ruling. From the definition of C, the intersection number of C with a member of the ruling must be greater than or equal to 2. On the other hand, a member of the ruling can have only one zero point of the additive vector field. This is impossible, because C is reduced. □

COROLLARY *Suppose a projective manifold has a holomorphic vector field with the property that a connected component of its zero set is smooth and totally degenerate. Then the manifold is stably birational to that component of zero set.*

Proof. Let X_0 be the projective manifold and let V_0 be the vector field with a smooth totally degenerate component Y. The algebraic subgroup of $\mathrm{Aut}(X_0)$ generated by V_0 is of the form $(\mathbf{C}^*)^p \times (\mathbf{C}^+)^q$, p a nonnegative integer and $q = 0$ or 1 (see [L]). By total degeneracy, the action of this group on the normal bundle of Y is trivial. In particular, the action of $\mathbf{C}^*$ on the tangent space at a point of Y is trivial. But an effective action of $\mathbf{C}^*$ cannot have trivial linear part at a fixed point (see e.g. [BB]). So $p = 0$, $q = 1$, and V_0 is an additive vector field.

Since the zero set of an additive vector field is connected, Y is the only zero set of V_0, and it contains the generic limit variety of V_0. Let $\pi\colon\ X \to X_0$ be the blow-up along Y, D be the exceptional divisor, and V be the additive vector field on X induced by V_0. Then Theorem 2 applies. Let $Z \subset D$ be the semisimple limit variety. It follows that X is stably birational to Z and Z is stably birational to D from Theorem 1. Since D is stably birational to Y, X_0 is stably birational to Y. □

4. Rationality for an isolated totally degenerate zero

From Corollary to Theorem 2, if a holomorphic vector field on a projective manifold has an isolated totally degenerate zero, the manifold is stably rational, i.e. stably birational to the projective space. We can improve this result to the rationality.

THEOREM 3 *Let V be a holomorphic vector field on a projective manifold X with an isolated totally degenerate zero P. Then X is rational.*

Proof. As in the proof of the corollary to Theorem 2, V is an additive vector field with the generic limit variety P. Blowing up P, we get an additive vector field $\tilde{V}$ vanishing on the exceptional divisor E. The line bundle valued vector field $\frac{1}{E}\tilde{V}$ restricted to E corresponds to a section σ of $T_{\mathbf{P}_{n-1}}(1)$ when we identify E with $\mathbf{P}_{n-1}$. From Theorem 1 and 2, X is rational if we can prove that the semisimple limit set Z of the foliation induced by σ is rational.

Choose an inhomogeneous coordinate system $x_1, \ldots, x_{n-1}$ on E centered at a generic point of Z. From the general form of a $\mathcal{O}(1)$-valued vector field, we can write

$$\sigma \;=\; \sum_{i=1}^{n-1}(\sum_{j=1}^{n-1} a_i^j x_j + \sum_{k\le l=1}^{n-1} g_i^{kl} x_k x_l + x_i q(x))\frac{\partial}{\partial x_i},$$

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where a_i^j, g_i^{kl} are complex numbers and $q(x)$ is a homogeneous quadratic polynomial in $x_1, \ldots, x_{n-1}$. From the semisimplicity, the linear part of the vector field at $x_1 = \cdots = x_{n-1} = 0$ is diagonalizable. Hence by a linear coordinate change and multiplying σ by a constant, we get

$$\sigma = \sum_{i=1}^{m}\left(\lambda_i x_i + \sum_{k\leq l=1}^{n-1} b_i^{kl} x_k x_l + x_i Q(x)\right)\frac{\partial}{\partial x_i}$$

$$+ \sum_{j=m+1}^{n-1}\left(\sum_{k\leq l=1}^{n-1} b_j^{kl} x_k x_l + x_j Q(x)\right)\frac{\partial}{\partial x_j}$$

where λ_i's are positive integers, b_i^{kl} is a complex number, and $Q(x)$ is a homogeneous quadratic polynomial. Here, we used the fact that $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is invariant under a linear coordinate change.

By the definition of a semisimple foliation, the foliation defined by σ agrees with the foliation defined by $\sum_{i=1}^m \lambda_i z_i \frac{\partial}{\partial z_i}$, under a suitable local coordinate system $z_1, \ldots, z_{n-1}$. Note that this foliation has the property that for any divisor on the zero set $z_1 = \cdots = z_m = 0$, there exists an invariant hypersurface in a small neighborhood whose restriction to the zero set is precisely the given divisor. In fact, with respect to the coordinates $z_1, \ldots, z_{n-1}$, a divisor on the zero set can be written as a power series in $z_{m+1}, \ldots, z_{n-1}$, and this power series certainly defines an invariant hypersurface in the coordinate neighborhood. Since this property is a property of the foliation, σ has this property. In this coordinate neighborhood, Z is a $(n-1-m)$-dimensional manifold defined by $z_1 = \cdots = z_m = 0$. From the above expression of σ, we can see that Z is defined by the equations $\lambda_i x_i + \sum_{k\leq l=1}^{n-1} b_i^{kl} x_k x_l + x_i Q(x) = 0, 1 \leq i \leq m$ in a small neighborhood. We may assume that $m \leq n-2$ to prove the rationality of Z.

Any linear combination of $x_{m+1}, \ldots, x_{n-1}$ defines a local divisor on Z. It follows that σ has an invariant hypersurface defined by a power series whose linear term is $e^1 x_1 + \cdots + e^m x_m + e^{m+1} x_{m+1} + \cdots + e^{n-1} x_{n-1}$ for any choice of complex numbers $e^{m+1}, \ldots, e^{n-1}$, unless $e^{m+1} = \cdots = e^{n-1} = 0$. Here $e^1, \ldots, e^m$ can take only certain values depending on $e^{m+1}, \ldots, e^{n-1}$. From the expression of σ above

$$\sigma(e^1 x_1 + \cdots + e^{n-1} x_{n-1}) = \lambda_1 e^1 x_1 + \cdots + \lambda_m e^m x_m + O(2).$$

It follows that the numbers $e^1, \ldots, e^m$ must be zero, for any nonzero choice of $e^{m+1}, \ldots, e^n$. Choosing $e^{m+1} = 1, e^{m+2} = \cdots = e^{n-1} = 0$, we can see that $b_{m+1}^{lk} = 0$, if $k \neq m+1, l \neq m+1$. Choosing $e^{m+2} = 1, e^{m+1} = e^{m+3} = \cdots e^{n-1} = 0$, we can see that $b_{m+2}^{kl} = 0$, if $k \neq m+2, l \neq m+2$. Repeating this, we can see that σ has the form

$$\sigma = \sum_{i=1}^{m}(\lambda_i x_i + \sum_{k\leq l=1}^{n-1} b_i^{kl} x_k x_l + x_i Q(x))\frac{\partial}{\partial x_i}$$

https://doi.org/10.1023/A:1000181820373 Published online by Cambridge University Press

$$+\sum_{j=m+1}^{n-1}(L_j(x)+Q(x))x_j\frac{\partial}{\partial x_j},$$

where $L_j(x)$ is a homogeneous linear polynomial in $x_1,\dots,x_n$. Now choosing $e^{m+1},\dots,e^{n-1}$ generically, we get $L_{m+1}(x)=\cdots=L_{n-1}(x)$. So we can write

$$\sigma = \sum_{i=1}^{m}\left(\lambda_i x_i+\sum_{k\le l=1}^{n-1}b_i^{kl}x_kx_l+x_iQ(x)\right)\frac{\partial}{\partial x_i}$$
$$+\sum_{j=m+1}^{n-1}(L(x)+Q(x))x_j\frac{\partial}{\partial x_j}$$

for some homogeneous linear polynomial $L(x)$.

So we get $n-1-m$ linear functions $x_{m+1},\dots,x_{n-1}$, with the property that their linear combinations are all invariant under σ, and they induce a local coordinate system on Z centered at an isolated point of the intersection of Z and the linear subspace defined by $x_{m+1}=\cdots=x_{n-1}=0$. Moreover, $\frac{\sigma(x_l)}{x_l}=\frac{\sigma(x_k)}{x_k}=L(x)+Q(x)$ for any $m+1\le l,k\le n-1$. Applying the same argument at a nearby point P on Z, i.e. translating $x_1,\dots,x_{n-1}$ to P and applying a linear coordinate change to diagonalize the linear part at P, we get $n-1-m$ collection of invariant polynomials of degree ≤ 1, say $x(P)_{m+1},\dots,x(P)_{n-1}$, which induce local coordinates on Z at P. Any linear combination of $x(P)_{m+1},\dots,x(P)_{n-1}$ is invariant under σ and $\frac{\sigma(x(P)_l)}{x(P)_l}=\frac{\sigma(x(P)_k)}{x(P)_k}$ for $m+1\le k,l\le n-1$. Let $x(P)_j=c(P)_j+\sum_{i=1}^{n-1}e(P)_j^ix_i, m+1\le j\le n-1$. For a generic P, we may assume that $c(P)_j\ne 0$ for all $m+1\le j\le n-1$.

From the invariance,

$$\sigma(c(P)_j+\sum_{i=1}^{n-1}e(P)_j^ix_i)$$
$$=\left(\frac{1}{c(P)_j}\sum_{i=1}^{m}\lambda_ie(P)_j^ix_i+Q(x)\right)\left(c(P)_j+\sum_{i=1}^{n-1}e(P)_j^ix_i\right).$$

From $\frac{\sigma(x(P)_l)}{x(P)_l}=\frac{\sigma(x(P)_k)}{x(P)_k}$, we can see that $\frac{e(P)_j^i}{c(P)_j}$ is independent of j. Let us call it $f(P)^i$. Then

$$x(P)_j = c(P)_j(1+f(P)^1x_1+\cdots+f(P)^mx_m)$$
$$+e(P)_j^{m+1}x_{m+1}+\cdots+e(P)_j^{n-1}x_{n-1}$$

Note that the matrix $e(P)_j^i, m+1\le i,j\le n-1$ is nonsingular. For otherwise, we can find a linear combination of $x(P)_j$'s of the form $1+a_1x_1+\cdots+a_mx_m$

which defines a smooth divisor on Z at P. This is impossible, because Z is defined by $\lambda_i x_i + \sum_{k\leq l=1}^{n-1} b_i^{kl} x_k x_l + x_i Q(x) = 0$ for $1 \leq i \leq m$. Now we may assume that

$$x(P)_j = c(P)_j(1 + f(P)^1 x_1 + \cdots + f(P)^m x_m) + x_j$$

Consider the m-dimensional invariant linear subspace W_P defined by $x(P)_j = 0, m+1 \leq j \leq n-1$. Its intersection with the m-dimensional subspace W defined by $x_{m+1} = \cdots = x_{n-1} = 0$, is an invariant hyperplane on W defined by $1 + f(P)^1 x_1 + \cdots + f(P)^m x_m = 0$. Note that σ induces a semisimple foliation on W with an isolated semisimple limit variety. In particular, there can be only a finitely many invariant hyperplanes disjoint from $x_1 = \cdots = x_m = 0$. So we can assume that $f(P)^i$ is independent of P for sufficiently general choice of P. Let us call it f^i.

Now let $X_0, \ldots, X_{n-1}$ be the homogeneous coordinates on E with $x_i = X_i/X_0$. Choose a new homogeneous coordinates $Y_0 = X_0 + f^1 X_1 + \cdots + f^m X_m$ and $Y_k = -X_k$, for $1 \leq k \leq n-1$. Then in terms of the inhomogeneous coordinates $y_i = Y_i/Y_0$, W_P is defined by $y_j = c(P)_j, m+1 \leq j \leq n-1$ for any generic choice of P. In other words, the level hyperplanes of $y_j, m+1 \leq j \leq n-1$, are invariant under σ.

Consider the projection $\rho : (y_1, \ldots, y_{n-1}) \mapsto (y_{m+1}, \ldots, y_{n-1})$ of the affine space $Y_0 \neq 0$ to the $(n-1-m)$-subspace defined by $y_1 = \cdots = y_m = 0$. We claim that ρ gives a birational map from the semisimple limit set Z of σ to the linear subspace $y_1 = \cdots = y_m = 0$. It is enough to show that for any point P in a small open subset in Z, e.g. those points considered above, the intersection of the m-subspace W_P defined by $y_{m+1} = c(P)_{m+1}, \ldots, y_{n-1} = c(P)_{n-1}$, with the generic part of Z consisting of $\tilde{V}$-attracting points, is precisely one point P. As in the proof of Theorem 2, consider the closure $\mathcal{E}_P$ of all points in $\tilde{X}$ whose $\tilde{V}$-limit points are P. Then $\mathcal{E}_P$ contains W_P. It follows that W_P contains no other attracting point of Z. Hence ρ is one-to-one on the Zariski open set of Z contained in the $y_1, \ldots, y_{n-1}$ coordinates cell and consisting of attracting points with respect to the additive vector field $\tilde{V}$. This shows the rationality of Z. □

Acknowledgment

We would like to thank Prof. Sommese for valuable discussions and Prof. Konarski for sending us his preprint [K]. We are very grateful to the referee for many helpful suggestions and pointing out a gap in the earlier version of Corollary to Theorem 2. Research at MSRI was supported in part by NSF grant DMS 9022140.

References

[BB] Bialynicki-Birula, A.: Some theorems on actions of algebraic groups. *Ann. Math.* 98 (1973), 480–497.

[C] Carrell, J.: Holomorphic $\mathbf{C}^*$-actions and vector fields on projective varieties, in *Topics in the theory of algebraic groups*, A. Hahn and W. Wong (eds), (Notre Dame Mathematical Lectures, number 10, pp. 1–37) Notre Dame: University of Notre Dame Press, 1982.

[H1] Hwang, J.-M.: Nondeformability of the complex hyperquadric, *Invent. Math.* 120 (1995), 317–338.

[H2] Hwang, J.-M.: Additive vector fields, algebraicity and rationality, *Math. Ann.* 304 (1996), 757–767.

[K] Konarsky, J.: On Carrell's conjecture in a special case, *J. Algebra* 182 (1996), 38–44.

[L] Lieberman, D.: Holomorphic vector fields on projective varieties. In: Wells, R.O.: Several complex variables. (Proceedings of Symposia in Pure Mathematics, vol. 30, part 1, pp. 273-276) American Mathematical Society 1977.