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SUMS OF WEIGHTED COMPOSITION OPERATORS ON COP

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Abstract. Let COP $= \mathcal{B}_0 \cap H^\infty$, where $\mathcal{B}_0$ is the little Bloch space on the open unit disk $\mathbb{D}$, and $A(\overline{\mathbb{D}})$ be the disk algebra on $\overline{\mathbb{D}}$. For non-zero functions $u_1, u_2, \ldots, u_N \in A(\overline{\mathbb{D}})$ and distinct analytic self-maps $\varphi_1, \varphi_2, \ldots, \varphi_N$ satisfying $\varphi_j \in A(\overline{\mathbb{D}})$ and $\|\varphi_j\|_\infty = 1$ for every j, it is given characterisations of which the sum of weighted composition operators $\sum_{j=1}^N u_j C_{\varphi_j}$ maps COP into $A(\overline{\mathbb{D}})$.

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1. Introduction. Let D be a domain in $\mathbb{C}$, and X, Y be the Banach spaces consisting of analytic functions on D. Let φ be an analytic self-map of D. Suppose that $f \circ \varphi \in Y$ for every $f \in X$. Then we may define the composition operator $C_\varphi : X \to Y$ by $C_\varphi f = f \circ \varphi$ for $f \in X$. In the recent four decades, there has been much work on composition operators on various spaces of analytic functions (see [**2**, **14**]).

Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk $\mathbb{D}$ with the supremum norm $\|\cdot\|_\infty$ and $M(H^\infty)$ be the space of non-zero multiplicative linear functionals on H^∞ with a weak-*topology. We identify a function in H^∞ with its Gelfand transform. For $x, y \in M(H^\infty)$, let

$$\rho(x, y) = \sup\{|f(y)| : f \in H^\infty, f(x) = 0, \|f\|_\infty \le 1\}$$

and

$$P(x) = \{\zeta \in M(H^\infty) : \rho(x, \zeta) < 1\}.$$

The set $P(x)$ is called the Gleason part containing x. We have $\rho(z, w) = |z - w|/|1 - \overline{w}z|$ for $z, w \in \mathbb{D}$.

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Let $A(\overline{\mathbb{D}})$ be the disk algebra on $\overline{\mathbb{D}}$, that is $A(\overline{\mathbb{D}})$ is the Banach algebra of continuous functions on $\overline{\mathbb{D}}$, which are analytic in $\mathbb{D}$. We denote by $\mathcal{B}_0$ the little Bloch space consisting of analytic functions f on $\mathbb{D}$, provided

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

Then $f \in \mathcal{B}_0 \cap H^\infty$ if and only if f is constant on any Gleason part in $M(H^\infty) \setminus \mathbb{D}$, and for this reason M. Behrens (see [**3**, p. 442]) called this space COP, for constant on parts, that is

$$\mathrm{COP} = \mathcal{B}_0 \cap H^\infty.$$

We also identify a function in H^∞ with its radial limit function $f(e^{i\theta}) = \lim_{r\to 1}(re^{i\theta})$ a.e. on $\partial\mathbb{D}$. Sarason in [**11**] proved that $H^\infty + C$ is a closed subalgebra of $L^\infty(\partial\mathbb{D})$, where C stands for the space of continuous functions on $\partial\mathbb{D}$. Let

$$\mathrm{QA} = \overline{(H^\infty + C)} \cap H^\infty.$$

It is known that

$$\mathrm{QA} = \mathrm{VMO} \cap H^\infty$$

(see [**12**]). We have

$$A(\overline{\mathbb{D}}) \subsetneqq \mathrm{QA} \subsetneqq \mathrm{COP} \subsetneqq H^\infty$$

(see [**3**, **5**, **12**, **13**]).

We denote by $\mathcal{S}(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. For $u \in H^\infty$ and $\varphi \in \mathcal{S}(\mathbb{D})$, we may define the weighted composition operator uC_φ on H^∞ by $(uC_\varphi)f = u(f \circ \varphi)$ for $f \in H^\infty$. It is known that if $\varphi \in \mathrm{QA}$, then C_φ maps QA into QA (see [**15**]), and if $\varphi \in \mathrm{COP}$, then C_φ maps COP into COP (see [**9**]).

Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be distinct functions in $\mathcal{S}(\mathbb{D})$. Let $\mathcal{Z}$ be the family of sequences $\{z_n\}_n$ in $\mathbb{D}$ satisfying the following three conditions:

(a) $\{z_n\}_n$ is a convergent sequence,

(b) $|\varphi_j(z_n)| \to 1$ as $n \to \infty$ for some $1 \le j \le N$ and $\{\varphi_j(z_n)\}_n$ is a convergent sequence for every $1 \le j \le N$,

(c) $\left\{\dfrac{\varphi_i(z_n) - \varphi_j(z_n)}{1 - \overline{\varphi_i(z_n)}\varphi_j(z_n)}\right\}_n$ is a convergent sequence for every $1 \le i, j \le N$.

Let

$$I(\{z_n\}) = \{j : 1 \le j \le N, |\varphi_j(z_k)| \to 1 \ (k \to \infty)\}.$$

For each $t \in I(\{z_n\})$, we write

$$I_0(\{z_n\}, t) = \{j \in I(\{z_n\}) : \rho(\varphi_j(z_k), \varphi_t(z_k)) \to 0 \ (k \to \infty)\}.$$

Then there is a subset $\{t_1, t_2, \ldots, t_\ell\}$ of $I(\{z_n\})$ such that

$$I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_0(\{z_n\}, t_p)$$

and $I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. Izuchi and Ohno in [7] gave a characterisation of compactness of the linear sum of composition operators $\sum_{j=1}^{N} a_j C_{\varphi_j}$ on H^∞. For non-zero functions $u_1, u_2, \ldots, u_N \in H^\infty$, Izuchi and Ohno in [8] have recently shown that $\sum_{j=1}^{N} u_j C_{\varphi_j}$ is compact on H^∞ if and only if

$$\lim_{k\to\infty} \sum_{j \in I_0(\{z_n\}, t)} u_j(z_k) = 0 \tag{1.1}$$

for every $\{z_n\}_n \in \mathcal{Z}$ and $t \in I(\{z_n\})$. Condition (1.1) is called an interior condition, for (1.1) is a condition given in the interior of $\overline{\mathbb{D}}$.

Izuchi and Ohno [8] also showed essentially that if $u_j \in A(\overline{\mathbb{D}})$ and $\varphi_j \in \mathcal{S}(\mathbb{D})$ with $\varphi_j \in A(\overline{\mathbb{D}})$ for every $1 \leq j \leq N$, then $(\sum_{j=1}^{N} u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ for every $f \in H^\infty$ if and only if (1.1) holds (see also [1, 10]). Since QA and COP are the most important spaces between $A(\overline{\mathbb{D}})$ and H^∞, we are interesting in properties of weighted composition operators on QA and COP. Motivated by the above, we have questions when $(\sum_{j=1}^{N} u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ holds for every $f \in$ COP (or QA). In this paper, we answer these questions.

In Section 2, we give interior conditions, and in Section 3 we give boundary conditions.

2. Sum of weighted composition operators. Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be distinct functions in $\mathcal{S}(\mathbb{D})$ satisfying that $\varphi_j \in COP$ and $\|\varphi_j\|_\infty = 1$ for every $1 \leq j \leq N$. Let $u_1, u_2, \ldots, u_N \in$ COP be non-zero functions. Since C_{φ_j} maps COP into COP, $\sum_{j=1}^{N} u_j C_{\varphi_j}$ is an operator on COP. Suppose that $\sum_{j=1}^{N} u_j C_{\varphi_j} : \text{COP} \to \text{COP}$ is compact. Then,

$$\sum_{j=1}^{N} u_j C_{\varphi_j} : A(\overline{\mathbb{D}}) \to \text{COP} \subset H^\infty$$

is compact. By [8], this condition holds if and only if (1.1) holds. Moreover, if $u_j, \varphi_j \in$ QA, then similarly $\sum_{j=1}^{N} u_j C_{\varphi_j} : \text{QA} \to \text{QA}$ is compact if and only if (1.1) holds.

In the rest of this paper, we assume that $\varphi_j \in A(\overline{\mathbb{D}})$ for every $1 \leq j \leq N$. Let $\mathcal{Z}$ be the family of sequences $\{z_n\}_n$ in $\mathbb{D}$ satisfying conditions (a), (b) and (c). Let $\{z_n\}_n \in \mathcal{Z}$. By conditions (a) and (b), $z_n \to e^{i\theta_0}$ as $n \to \infty$ for some $e^{i\theta_0} \in \partial\mathbb{D}$. We have

$$I(\{z_n\}) = \{j : 1 \leq j \leq N, |\varphi_j(e^{i\theta_0})| = 1\}.$$

By (c), we write

$$\beta_{i,j} = \lim_{k\to\infty} \frac{\varphi_i(z_k) - \varphi_j(z_k)}{1 - \overline{\varphi_i(z_k)}\varphi_j(z_k)}, \quad 1 \leq i, j \leq N.$$

We have

$$\lim_{k\to\infty} \rho(\varphi_i(z_k), \varphi_j(z_k)) = |\beta_{i,j}|.$$

For each $t \in I(\{z_n\})$, let

$$I_1(\{z_n\}, t) = \{j \in I(\{z_n\}) : |\beta_{t,j}| < 1\}. \tag{2.1}$$

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For z_0, z_1, z_2 in $\mathbb{D}$, we have

$$\rho(z_0, z_1) \le \frac{\rho(z_0, z_2) + \rho(z_2, z_1)}{1 + \rho(z_0, z_2)\rho(z_2, z_1)}$$

(see [**3**, p. 4]). Hence, for $i, j, t \in I(\{z_n\})$ if $|\beta_{t,i}| < 1$ and $|\beta_{t,j}| < 1$, then $|\beta_{i,j}| < 1$. This shows that for $s, t \in I(\{z_n\})$, we have either $I_1(\{z_n\}, s) = I_1(\{z_n\}, t)$ or $I_1(\{z_n\}, s) \cap I_1(\{z_n\}, t) = \emptyset$, so there is a subset $\{t_1, t_2, \ldots, t_\ell\}$ of $I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_1(\{z_n\}, t_p)$ and $I_1(\{z_n\}, t_p) \cap I_1(\{z_n\}, t_q) = \emptyset$ for $p \ne q$. We note that $|\beta_{t_p,t_q}| = 1$ for $p \ne q$.

THEOREM 2.1. *Let $u_1, u_2, \ldots, u_N \in A(\overline{\mathbb{D}})$ be non-zero functions and $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}(\mathbb{D})$ be distinct functions satisfying that $\varphi_j \in A(\overline{\mathbb{D}})$ and $\|\varphi_j\|_\infty = 1$ for every $1 \le j \le N$. Then the following conditions are equivalent.*

(i) $(\sum_{j=1}^{N} u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ *for every* $f \in$ COP.
(ii) $(\sum_{j=1}^{N} u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ *for every* $f \in$ QA.
(iii) $\lim_{k\to\infty} \sum_{j \in I_1(\{z_n\},t)} u_j(z_k) = 0$ *for every* $\{z_n\}_n \in \mathcal{Z}$ *and* $t \in I(\{z_n\})$.

To prove our theorem, we need some lemmas. By [**6**] (see also [**3**]), one can easily see the following.

LEMMA 2.2. *Let $f \in H^\infty$. Then the following conditions are equivalent:*

(i) $f \in$ COP.
(ii) *For any sequences $\{z_n\}_n, \{w_n\}_n$ in $\mathbb{D}$ satisfying that $|z_n| \to 1$ and $\sup_n \rho(z_n, w_n) < 1$, then $f(z_n) - f(w_n) \to 0$ as $n \to \infty$.*

A sequence $\{z_n\}_n$ in $\mathbb{D}$ is called sparse (or thin) if

$$\lim_{k\to\infty} \prod_{n; n\ne k} \rho(z_n, z_k) = 1.$$

In [**4**], Gorkin showed that for a sequence $\{z_n\}_n$ in $\mathbb{D}$ satisfying $|z_n| \to 1$ as $n \to \infty$, there exists a sparse subsequence of $\{z_n\}_n$. By appropriate modifications of it, we may prove the following.

LEMMA 2.3. *Let $\{z_{t,n}\}_n$ be a sequence in $\mathbb{D}$ satisfying $|z_{t,n}| \to 1$ as $n \to \infty$ for every $1 \le t \le \ell$. Suppose that $\rho(z_{t,n}, z_{s,n}) \to 1$ as $n \to \infty$ for $t \ne s$. Then there is a subsequence $\{n_i\}_i$ such that $\{z_{t,n_i} : 1 \le t \le \ell, i \ge 1\}$ is a sparse sequence.*

In [**16**], Sundberg and Wolff proved the following.

LEMMA 2.4. *If $\{z_n\}_n$ is a sparse sequence in $\mathbb{D}$, then for every bounded sequence $\{a_n\}_n$ of complex numbers there is $f \in$ QA such that $f(z_n) = a_n$ for every $n \ge 1$.*

Proof of Theorem 2.1. (i) $\Rightarrow$ (ii) follows from QA $\subset$ COP.

Suppose that (ii) holds. Let $\{z_n\}_n \in \mathcal{Z}$ and $t \in I(\{z_n\})$. We may write $z_n \to e^{i\theta_0} \in \partial\mathbb{D}$. There is a subset $\{t_1, t_2, \ldots, t_\ell\}$ of $I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_1(\{z_n\}, t_p)$

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and $I_1(\{z_n\}, t_p) \cap I_1(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. By (ii), for every $f \in \mathrm{QA}$ we have $\sum_{j=1}^{N} u_j(z) f(\varphi_j(z)) \in A(\overline{\mathbb{D}})$. Then the above function is continuous at $z = e^{i\theta_0}$. For each $j \notin I(\{z_n\})$, we have $|\varphi_j(e^{i\theta_0})| < 1$, so $u_j(z) f(\varphi_j(z))$ is continuous at $z = e^{i\theta_0}$. Hence,

$$\sum_{j \in I(\{z_n\})} u_j(z) f(\varphi_j(z)) = \sum_{p=1}^{\ell} \sum_{j \in I_1(\{z_n\}, t_p)} u_j(z) f(\varphi_j(z)) \tag{2.2}$$

is continuous at $z = e^{i\theta_0}$.

For each $j \in I_1(\{z_n\}, t_p)$, by (2.1) we have $|\beta_{t_p, j}| < 1$, so

$$\lim_{k \to \infty} \rho(\varphi_{t_p}(z_k), \varphi_j(z_k)) = |\beta_{t_p, j}| < 1.$$

Since $\mathrm{QA} \subset \mathrm{COP}$, by Lemma 2.2 we have $f(\varphi_j(z_k)) - f(\varphi_{t_p}(z_k)) \to 0$ as $k \to \infty$ for every $f \in QA$. By (2.2), there exists the following limit

$$\lim_{k \to \infty} \sum_{p=1}^{\ell} \sum_{j \in I_1(\{z_n\}, t_p)} u_j(z_k) f(\varphi_j(z_k)) = \lim_{k \to \infty} \sum_{p=1}^{\ell} f(\varphi_{t_p}(z_k)) \sum_{j \in I_1(\{z_n\}, t_p)} u_j(z_k)$$

for every $f \in QA$. Since $|\beta_{t_p, t_q}| = 1$, we have $\rho(\varphi_{t_p}(z_k), \varphi_{t_q}(z_k)) \to 1$ as $k \to \infty$ for $p \neq q$. Since $|\varphi_{t_p}(z_k)| \to 1$ as $k \to \infty$, by Lemma 2.3 considering a subsequence we may assume that $\{\varphi_{t_p}(z_k) : k \geq 1, 1 \leq p \leq \ell\}$ is a sparse sequence.

Since $t \in I(\{z_n\})$, $t \in I_1(\{z_n\}, t_{p_0})$ for some $1 \leq p_0 \leq \ell$. By Lemma 2.4, there is $f \in \mathrm{QA}$ such that $f(\varphi_{t_p}(z_k)) = 0$ for every $p \neq p_0$, $f(\varphi_{t_{p_0}}(z_{2k})) = 1$ and $f(\varphi_{t_{p_0}}(z_{2k+1})) = -1$ for every $k \geq 1$. Then there is the following limit

$$\Big(\sum_{j \in I_1(\{z_n\}, t_{p_0})} u_j(e^{i\theta_0}) \Big) (-1)^k \quad (k \to \infty).$$

Consequently we have

$$0 = \sum_{j \in I_1(\{z_n\}, t_{p_0})} u_j(e^{i\theta_0}) = \sum_{j \in I_1(\{z_n\}, t)} u_j(e^{i\theta_0}).$$

Thus, we get (iii).

Next, suppose that (iii) holds. To prove (i), let $f \in \mathrm{COP}$ and $\{z_n\}_n$ be a sequence in $\mathbb{D}$ such that $z_n \to e^{i\theta_0} \in \partial\mathbb{D}$. It is sufficient to prove that $\lim_{n \to \infty} \sum_{j=1}^{N} u_j(z_n) f(\varphi_j(z_n))$ has a limit involving only the point $e^{i\theta_0}$. We may assume that $\{z_n\}_n \in \mathcal{Z}$. There is a subset $\{t_1, t_2, \ldots, t_\ell\}$ of $I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_1(\{z_n\}, t_p)$ and $I_1(\{z_n\}, t_p) \cap I_1(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. We note that $|\varphi_j(e^{i\theta_0})| = 1$ for $j \in I(\{z_n\})$ and $|\varphi_j(e^{i\theta_0})| < 1$ for $j \notin I(\{z_n\})$. We have

$$\sum_{j=1}^{N} u_j(z_k) f(\varphi_j(z_k)) = \sum_{j \in I(\{z_n\})} u_j(z_k) f(\varphi_j(z_k)) + \sum_{j \notin I(\{z_n\})} u_j(z_k) f(\varphi_j(z_k))$$

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and

$$\lim_{k\to\infty} \sum_{j\notin I(\{z_n\})} u_j(z_k)f(\varphi_j(z_k)) = \sum_{j\notin I(\{z_n\})} u_j(e^{i\theta_0})f(\varphi_j(e^{i\theta_0})).$$

We also have

$$\limsup_{k\to\infty} \Big| \sum_{j\in I(\{z_n\})} u_j(z_k)f(\varphi_j(z_k)) \Big| = \limsup_{k\to\infty} \Big| \sum_{p=1}^{\ell} \sum_{j\in I_1(\{z_n\},t_p)} u_j(z_k)f(\varphi_j(z_k)) \Big|.$$

Since $f \in \mathrm{COP}$, by Lemma 2.2 we have $f(\varphi_j(z_k)) - f(\varphi_{t_p}(z_k)) \to 0 \quad (k \to \infty)$ for $j \in I_1(\{z_n\}, t_p)$. Hence,

$$\begin{aligned}
&\limsup_{k\to\infty} \Big| \sum_{j\in I(\{z_n\})} u_j(z_k)f(\varphi_j(z_k)) \Big| \\
&= \limsup_{k\to\infty} \Big| \sum_{p=1}^{\ell} f(\varphi_{t_p}(z_k)) \sum_{j\in I_1(\{z_n\},t_p)} u_j(z_k) \Big| \\
&\le \limsup_{k\to\infty} \sum_{p=1}^{\ell} \|f\circ\varphi_{t_p}\|_\infty \Big| \sum_{j\in I_1(\{z_n\},t_p)} u_j(z_k) \Big| \\
&= 0 \qquad \text{by (iii).}
\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty} \sum_{j=1}^{N} u_j(z_k)f(\varphi_j(z_k)) = \sum_{j\notin I(\{z_n\})} u_j(e^{i\theta_0})f(\varphi_j(e^{i\theta_0})),$$

so we get (i). □

Under the assumptions of Theorem 2.1, generally $\sum_{j=1}^N u_jC_{\varphi_j} : \mathrm{COP} \to A(\overline{\mathbb{D}})$ is not compact in spite of that condition (i) holds. But it is considered that condition (i) leads compactness of $\sum_{j=1}^N u_jC_{\varphi_j} : \mathrm{COP} \to A(\overline{\mathbb{D}})$ in some weak sense. We denote by $B(\mathrm{COP})$ the closed unit ball of COP. We have the following.

THEOREM 2.5. *Let $u_1, u_2, \ldots, u_N \in A(\overline{\mathbb{D}})$ be non-zero functions and $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}(\mathbb{D})$ be distinct functions satisfying that $\varphi_j \in A(\overline{\mathbb{D}})$ and $\|\varphi_j\|_\infty = 1$ for every $1 \le j \le N$. Then the following conditions are equivalent.*

(i) $(\sum_{j=1}^N u_jC_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ *for every* $f \in \mathrm{COP}$.

(ii) $\lim_{k\to\infty} \sum_{j\in I_1(\{z_n\},t)} u_j(z_k) = 0$ *for every* $\{z_n\}_n \in \mathcal{Z}$ *and* $t \in I(\{z_n\})$.

(iii) *If $\{f_m\}_m$ is a sequence in $B(\mathrm{COP})$, which converges uniformly to zero on any compact subset of $\mathbb{D}$, then*

$$\lim_{m\to\infty} \limsup_{k\to\infty} \Big| \Big(\sum_{j=1}^{N} u_jC_{\varphi_j} \Big)(f_m)(z_k) \Big| = 0$$

for every $\{z_n\}_n \in \mathcal{Z}$.

Proof. By Theorem 2.1, we have (i) ⇔ (ii). Suppose that (ii) holds. To show (iii), let $\{f_m\}_m$ be a sequence in $B(\mathrm{COP})$, which converges uniformly to zero on any compact

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subset of $\mathbb{D}$ and $\{z_n\}_n \in \mathcal{Z}$. We have

$$\sum_{j=1}^{N} u_j(z_k)f_m(\varphi_j(z_k)) = \sum_{j\in I(\{z_n\})} u_j(z_k)f_m(\varphi_j(z_k)) + \sum_{j\notin I(\{z_n\})} u_j(z_k)f_m(\varphi_j(z_k)).$$

By the assumption on $\{f_m\}_m$,

$$\lim_{m\to\infty} \sup_{k\geq 1} \Big| \sum_{j\notin I(\{z_n\})} u_j(z_k)f_m(\varphi_j(z_k)) \Big| = 0.$$

Let $\{t_1, t_2, \ldots, t_\ell\} \subset I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_1(\{z_n\}, t_p)$ and $I_1(\{z_n\}, t_p) \cap I_1(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. Since $f_m \in$ COP, we have $f_m(\varphi_j(z_k)) - f_m(\varphi_{t_p}(z_k)) \to 0$ as $k \to \infty$ for $j \in I_1(\{z_n\}, t)$. Then

$$\begin{aligned}
&\limsup_{m\to\infty} \limsup_{k\to\infty} \Big| \Big(\sum_{j=1}^{N} u_j C_{\varphi_j} \Big)(f_m)(z_k) \Big| \\
&= \limsup_{m\to\infty} \limsup_{k\to\infty} \Big| \sum_{p=1}^{\ell} \sum_{j\in I_1(\{z_n\},t_p)} u_j(z_k)f_m(\varphi_j(z_k)) \Big| \\
&\leq \limsup_{k\to\infty} \sum_{p=1}^{\ell} \Big| \sum_{j\in I_1(\{z_n\},t_p)} u_j(z_k) \Big| \\
&= 0 \qquad \text{by (ii).}
\end{aligned}$$

Thus, we get (iii).

Suppose that (iii) holds. To show (ii), let $\{z_n\}_n \in \mathcal{Z}$ and $t \in I(\{z_n\})$. Take $\{t_1, t_2, \ldots, t_\ell\}$ in $I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_1(\{z_n\}, t_p)$ and $I_1(\{z_n\}, t_p) \cap I_1(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. Let $\{f_m\}_m$ be a sequence in B(COP), which converges to 0 uniformly on any compact subset of $\mathbb{D}$. In the same way as the first paragraph of this proof, we have

$$\limsup_{k\to\infty} \Big| \Big(\sum_{j=1}^{N} u_j C_{\varphi_j} \Big)(f_m)(z_k) \Big| = \limsup_{k\to\infty} \Big| \sum_{p=1}^{\ell} f_m(\varphi_{t_p}(z_k)) \sum_{j\in I_1(\{z_n\},t_p)} u_j(z_k) \Big|.$$

By Lemma 2.3, considering a subsequence we may assume that $\{\varphi_{t_p}(z_k) : k \geq 1, 1 \leq p \leq \ell\}$ is a sparse sequence. Note that $t \in I_1(\{z_n\}, t_{p_0})$ for some $1 \leq p_0 \leq \ell$. By Lemma 2.4, there exists $h \in$ QA such that $h(\varphi_{t_p}(z_k)) = 0$ for every $p \neq p_0$ and $h(\varphi_{t_{p_0}}(z_k)) = 1$ for every $k \geq 1$. We may put $\varphi_{t_{p_0}}(z_k) \to e^{i\theta_0} \in \partial\mathbb{D}$ as $k \to \infty$. Let $q(z) \in A(\overline{\mathbb{D}})$ satisfy $q(e^{i\theta_0}) = 1$ and $|q(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}\}$. For each positive integer m, let $f_m = hq^m \in A(\overline{\mathbb{D}})$. Then $\{f_m\}_m$ is a bounded sequence in COP and $f_m \to 0$ uniformly on any compact subset of

$\mathbb{D}$. Thus, we get

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{k\to\infty}\left|\Big(\sum_{j=1}^{N}u_jC_{\varphi_j}\Big)(f_m)(z_k)\right|\\
&=\limsup_{m\to\infty}\limsup_{k\to\infty}\left|f_m(\varphi_{t_{p_0}}(z_k))\sum_{j\in I_1(\{z_n\},t_{p_0})}u_j(z_k)\right|\\
&=\limsup_{m\to\infty}\limsup_{k\to\infty}\left|q^m(\varphi_{t_{p_0}}(z_k))\sum_{j\in I_1(\{z_n\},t_{p_0})}u_j(z_k)\right|\\
&=\limsup_{k\to\infty}\left|\sum_{j\in I_1(\{z_n\},t_{p_0})}u_j(z_k)\right|\\
&=\limsup_{k\to\infty}\left|\sum_{j\in I_1(\{z_n\},t)}u_j(z_k)\right|.
\end{aligned}$$

By conditon (iii), we obtain (ii). □

We denote by m the normalised Lebesgue measure on $\partial\mathbb{D}$. In the same way as the proof of Corollary 2.4 in **[8]**, we have the following.

COROLLARY 2.6. *Let $u_1, u_2, \ldots, u_N \in A(\overline{\mathbb{D}})$ be non-zero functions and $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}(\mathbb{D})$ be distinct functions satisfying that $\varphi_j \in A(\overline{\mathbb{D}})$ and $\|\varphi_j\|_\infty = 1$ for every $1 \le j \le N$. Let $\Gamma(\varphi_j) = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi_j(e^{i\theta})| = 1\}$. If $(\sum_{j=1}^{N} u_jC_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ for every $f \in$ COP, then $m(\Gamma(\varphi_j)) = 0$ for every $1 \le j \le N$.*

3. Boundary conditions. Let $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}(\mathbb{D})$ be distinct functions satisfying that $\varphi_j \in A(\overline{\mathbb{D}})$ and $\|\varphi_j\|_\infty = 1$ for every $1 \le j \le N$. By Corollary 2.6, we may consider a similar concept of $\mathcal{Z}$ on $\partial\mathbb{D}$. Suppose that $m(\Gamma(\varphi_j)) = 0$ for every $1 \le j \le N$. Let $\mathcal{Y}$ be the family of sequences $\{e^{i\theta_n}\}_n$ in $\partial\mathbb{D}$ satisfying

(d) $\{e^{i\theta_n}\}_n$ is a convergent sequence.

(e) $|\varphi_j(e^{i\theta_n})| < 1$ for every $1 \le j \le N$ and $n \ge 1$, $\{\varphi_j(e^{i\theta_n})\}_n$ is a convergent sequence for every $1 \le j \le N$ and $|\varphi_j(e^{i\theta_n})| \to 1$ as $n \to \infty$ for some $1 \le j \le N$.

(f) $\left\{\dfrac{\varphi_i(e^{i\theta_n}) - \varphi_j(e^{i\theta_n})}{1 - \overline{\varphi_i(e^{i\theta_n})}\varphi_j(e^{i\theta_n})}\right\}_n$ is a convergent sequence for every $1 \le i, j \le N$.

Let

$$J(\{e^{i\theta_n}\}) = \{j : 1 \le j \le N, |\varphi_j(e^{i\theta_k})| \to 1 \ \ (k \to \infty)\}.$$

By (f), we write

$$\beta_{i,j} = \lim_{k\to\infty}\frac{\varphi_i(e^{i\theta_k}) - \varphi_j(e^{i\theta_k})}{1 - \overline{\varphi_i(e^{i\theta_k})}\varphi_j(e^{i\theta_k})}, \quad 1 \le i, j \le N.$$

We have $\lim_{k\to\infty} \rho(\varphi_i(e^{i\theta_k}), \varphi_j(e^{i\theta_k})) = |\beta_{i,j}|$. For each $t \in J(\{z_n\})$, let

$$J_1(\{e^{i\theta_n}\}, t) = \{j \in J(\{e^{i\theta_n}\}) : |\beta_{t,j}| < 1\}.$$

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Then in the same way as in Section 2, there is a subset $\{t_1, t_2, \dots, t_\ell\}$ of $J(\{z_n\})$ such that $J(\{e^{i\theta_n}\}) = \bigcup_{p=1}^{\ell} J_1(\{e^{i\theta_n}\}, t_p)$ and $J_1(\{e^{i\theta_n}\}, t_p) \cap J_1(\{e^{i\theta_n}\}, t_q) = \emptyset$ for $p \neq q$. In the similar way as the proof of Theorem 2.1, we may show the following.

THEOREM 3.1. *Let $u_1, u_2, \dots, u_N \in A(\overline{\mathbb{D}})$ be non-zero functions and $\varphi_1, \varphi_2, \dots, \varphi_N \in \mathcal{S}(\mathbb{D})$ be distinct functions satisfying that $\varphi_j \in A(\overline{\mathbb{D}})$ and $\|\varphi_j\|_\infty = 1$ for every $1 \leq j \leq N$. We assume that $m(\Gamma(\varphi_j)) = 0$ for every $1 \leq j \leq N$. Then the following conditions are equivalent:*

(i) $(\sum_{j=1}^N u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ *for every* $f \in$ COP.
(ii) $\lim_{k\to\infty} \sum_{j\in I_1(\{z_n\},t)} u_j(z_k) = 0$ *for every* $\{z_n\}_n \in \mathcal{Z}$ *and* $t \in I(\{z_n\})$.
(iii) $\lim_{k\to\infty} \sum_{j\in J_1(\{e^{i\theta_n}\},t)} u_j(e^{i\theta_k}) = 0$ *for every* $\{e^{i\theta_n}\}_n \in \mathcal{Y}$ *and* $t \in J(\{e^{i\theta_n}\})$.

Proof. (i) $\Leftrightarrow$ (ii) is proven in Theorem 2.1.

Suppose that (ii) holds. To show (iii), let $\{e^{i\theta_n}\}_n \in \mathcal{Y}$ and $t \in J(\{e^{i\theta_n}\})$. For each positive integer n, let $\{r_{n,k}\}_k$ be a sequence of numbers such that $0 < r_{n,k} < 1$ and $r_{n,k} \to 1$ as $k \to \infty$. Put $z_{n,k} = r_{n,k}e^{i\theta_n}$. We may choose a sequence $\{k_n\}_n$ such that

$$\lim_{n\to\infty} z_{n,k_n} = \lim_{n\to\infty} e^{i\theta_n}, \quad \lim_{n\to\infty} \varphi_j(z_{n,k_n}) = \lim_{n\to\infty} \varphi_j(e^{i\theta_n}) \quad (1 \leq j \leq N)$$

and

$$\lim_{n\to\infty} \frac{\varphi_i(z_{n,k_n}) - \varphi_j(z_{n,k_n})}{1 - \overline{\varphi_i(z_{n,k_n})}\varphi_j(z_{n,k_n})} = \lim_{n\to\infty} \frac{\varphi_i(e^{i\theta_n}) - \varphi_j(e^{i\theta_n})}{1 - \overline{\varphi_i(e^{i\theta_n})}\varphi_j(e^{i\theta_n})} \quad (1 \leq i, j \leq N).$$

Put $z_n = z_{n,k_n}$. Then $\{z_n\}_n \in \mathcal{Z}$, $t \in I(\{z_n\})$ and $I_1(\{z_n\}, t) = J_1(\{e^{i\theta_n}\}, t)$. By condition (ii), we have

$$\lim_{k\to\infty} \sum_{j\in J_1(\{e^{i\theta_n}\},t)} u_j(e^{i\theta_k}) = \lim_{k\to\infty} \sum_{j\in I_1(\{z_n\},t)} u_j(z_k) = 0.$$

Suppose that (iii) holds. To show (i), let $f \in$ COP. We have

$$\Big(\sum_{j=1}^N u_j C_{\varphi_j}\Big)(f)(z) = \sum_{j=1}^N u_j(z) f(\varphi_j(z)), \quad z \in \overline{\mathbb{D}} \setminus \bigcup_{j=1}^N \Gamma(\varphi_j).$$

Hence, $(\sum_{j=1}^N u_j C_{\varphi_j})(f)$ is well defined and continuous on $\overline{\mathbb{D}} \setminus \bigcup_{j=1}^N \Gamma(\varphi_j)$. To show $(\sum_{j=1}^N u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$, since $(\sum_{j=1}^N u_j C_{\varphi_j})(f) \in H^\infty$, it is sufficient to show that the function $(\sum_{j=1}^N u_j C_{\varphi_j})(f)$ on $\partial\mathbb{D} \setminus \bigcup_{j=1}^N \Gamma(\varphi_j)$ is continuously extendable to $\partial\mathbb{D}$. Since $m(\Gamma(\varphi_j)) = 0$ for $1 \leq j \leq N$, it is sufficient to show that for a sequence $\{e^{i\theta_n}\}_n$ in $\partial\mathbb{D} \setminus \bigcup_{j=1}^N \Gamma(\varphi_j)$ satisfying $e^{i\theta_n} \to e^{i\theta_0} \in \bigcup_{j=1}^N \Gamma(\varphi_j)$,

$$\lim_{n\to\infty} \sum_{j=1}^N u_j(e^{i\theta_n}) f(\varphi_j(e^{i\theta_n}))$$

has a limit involving only the point $e^{i\theta_0}$. The remaining is the same as the proof of Theorem 2.1. □

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We do not know a direct proof of (iii) ⇒ (ii).

The following is a boundary version of Theorem 2.5, which may be proven in the same way of it.

THEOREM 3.2. *Let* $u_1, u_2, \ldots, u_N \in A(\overline{\mathbb{D}})$ *be non-zero functions and* $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}(\mathbb{D})$ *be distinct functions satisfying that* $\varphi_j \in A(\overline{\mathbb{D}})$ *and* $\|\varphi_j\|_\infty = 1$ *for every* $1 \le j \le N$. *We assume that* $m(\Gamma(\varphi_j)) = 0$ *for every* $1 \le j \le N$. *Then the following conditions are equivalent.*

(i) $(\sum_{j=1}^N u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ *for every* $f \in$ COP.

(ii) $\lim_{k\to\infty} \sum_{j\in I_1(\{z_n\},t)} u_j(z_k) = 0$ *for every* $\{z_n\}_n \in \mathcal{Z}$ *and* $t \in I(\{z_n\})$.

(iii) *If* $\{f_m\}_m$ *is a sequence in* $B(\text{COP})$, *which converges uniformly on any compact subset of* $\mathbb{D}$, *then*

$$\lim_{m\to\infty} \limsup_{k\to\infty} \Big|\Big(\sum_{j=1}^N u_j C_{\varphi_j}\Big)(f_m)(e^{i\theta_k})\Big| = 0$$

for every $\{e^{i\theta_n}\}_n \in \mathcal{Y}$.

We may also give a boundary condition, which is equivalent to condition (1.1). Let $\{e^{i\theta_n}\}_n \in \mathcal{Y}$. For each $t \in J(\{e^{i\theta_n}\})$, let

$$J_0(\{e^{i\theta_n}\}, t) = \{j \in J(\{e^{i\theta_n}\}) : \rho(\varphi_j(e^{i\theta_k}), \varphi_t(e^{i\theta_k})) \to 0\ (k \to \infty)\}.$$

Then there is a subset $\{t_1, t_2, \ldots, t_\ell\}$ of $J(\{e^{i\theta_n}\})$ such that $J(\{e^{i\theta_n}\}) = \bigcup_{p=1}^{\ell} J_0(\{e^{i\theta_n}\}, t_p)$ and $J_0(\{e^{i\theta_n}\}, t_p) \cap J_0(\{e^{i\theta_n}\}, t_q) = \emptyset$ for $p \ne q$.

THEOREM 3.3. *Let* $u_1, u_2, \ldots, u_N \in A(\overline{\mathbb{D}})$ *be non-zero functions and* $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}(\mathbb{D})$ *be distinct functions satisfying that* $\varphi_j \in A(\overline{\mathbb{D}})$ *and* $\|\varphi_j\|_\infty = 1$ *for every* $1 \le j \le N$. *We assume that* $m(\Gamma(\varphi_j)) = 0$ *for every* $1 \le j \le N$. *Then the following conditions are equivalent.*

(i) $\sum_{j=1}^N u_j C_{\varphi_j}$ *is compact on* H^∞.

(ii) $(\sum_{j=1}^N u_j C_{\varphi_j})(f) \in A(\overline{\mathbb{D}})$ *for every* $f \in H^\infty$.

(iii) $\lim_{k\to\infty} \sum_{j\in I_0(\{z_n\},t)} u_j(z_k) = 0$ *for every* $\{z_n\}_n \in \mathcal{Z}$ *and* $t \in I(\{z_n\})$.

(iv) $\lim_{k\to\infty} \sum_{j\in J_0(\{e^{i\theta_n}\},t)} u_j(e^{i\theta_k}) = 0$ *for every* $\{e^{i\theta_n}\}_n \in \mathcal{Y}$ *and* $t \in J(\{e^{i\theta_n}\})$.

Sketch of Proof. In **[8]**, equivalencies of (i) ⇔ (ii) ⇔ (iii) were proven. In the same way as the proof of Theorem 3.1, we may prove that (iii) ⇒ (iv) ⇒ (ii). □

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