Canad. Math. Bull. Vol. **41** (3), 1998 pp. 267–278

ON THE NONEMPTINESS OF THE ADJOINT LINEAR SYSTEM OF POLARIZED MANIFOLDS

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ABSTRACT. Let (X, L) be a polarized manifold over the complex number field with $\dim X = n$. In this paper, we consider a conjecture of M. C. Beltrametti and A. J. Sommese and we obtain that this conjecture is true if $n = 3$ and $h^0(L) \geq 2$, or $\dim \operatorname{Bs}|L| \leq 0$ for any $n \geq 3$. Moreover we can generalize the result of Sommese.

0. **Introduction.** Let X be a smooth projective variety over the complex number field $\mathbb{C}$ with $\dim X = n$ and let L be a (Cartier) divisor on X. Then (X, L) is called a *polarized* (resp. *quasi-polarized*) *manifold* if L is ample (resp. nef-big). Beltrametti and Sommese conjectured the following in their book (Conjecture 7.2.7 in [BS]):

CONJECTURE A. *Let* (X, L) *be a polarized manifold with* $\dim X = n$. *If* $K_X + (n-1)L$ *is nef, then* $h^0\big(K_X + (n-1)L\big) > 0$.

This conjecture is true if $n = 2$ or L is spanned. But in general, it is unknown whether this conjecture is true or not. In order to solve this conjecture it is necessary to consider the case in which (X, L) is a quasi-polarized manifold. If L is ample and Conjecture A is true, then we can prove that $K_X + (n-1)L$ is nef if and only if $h^0\big(K_X + (n-1)L\big) > 0$. But if L is nef-big, then there exists an example such that $K_X + (n-1)L$ is not nef but $h^0\big(K_X + (n-1)L\big) > 0$. So we propose the following conjecture for any quasi-polarized manifold:

CONJECTURE NB. *Let* (X, L) *be a quasi-polarized manifold with* $\dim X = n \geq 2$. *If* $\kappa\big(K_X + (n-1)L\big) \geq 0$, *then* $h^0\big(K_X + (n-1)L\big) > 0$.

We remark that Conjecture A is equivalent to Conjecture NB for any polarized manifold. In this paper, we will prove that Conjecture A is true if one of the following is satisfied:

(1) $n = 3$ and $h^0(L) \geq 2$,
(2) $\operatorname{Bs}|L|$ is finite,

and by this result, we can generalize a result of Sommese (Theorem 4.1 in [So2]). I think that in order to prove Conjecture A for $\dim X = n$ it is necessary to consider Conjecture NB for $\dim X = n-1$.

Received by the editors September 18, 1996.
The author is a Research Fellow of the Japan Society for the Promotion of Science.
AMS subject classification: Primary: 14C20; secondary: 14J99.
Key words and phrases: Polarized manifold, adjoint bundle.

Furthermore we will propose a conjecture (Conjecture 3.8) which gives the relationship between $h^0\big(K_X + (n-1)L\big)$ and $g(L)$. We use the customary notations in algebraic geometry.

ACKNOWLEDGMENT. The author would like to express his hearty gratitude to Professors Takao Fujita and Masanori Kobayashi for giving some useful comments.

1. **Preliminaries.**

DEFINITION 1.1. Let X be a smooth projective variety with $\dim X > \dim Y \geq 1$. Then a morphism $f: X \longrightarrow Y$ is a fiber space if f is surjective with connected fibers. Let L be a Cartier divisor on X. Then (f, X, Y, L) is called a *polarized* (resp. *quasi-polarized*) *fiber space* if $f: X \longrightarrow Y$ is a fiber space and L is ample (resp. nef-big).

DEFINITION 1.2. Let X be a smooth projective variety with $\dim X = n$ and let L be a line bundle on X. Then we say that (X, L) is a scroll over Y if there exists a fiber space $\pi: X \longrightarrow Y$ such that any fiber of π is isomorphic to $\mathbb{P}^{n-m}$ and $L|_F = \mathcal{O}_{\mathbb{P}^{n-m}}(1)$, where $1 \leq m = \dim Y < \dim X$. A quasi-polarized fiber space (f, X, Y, L) is called a *scroll* if $(F, L_F) \cong \big(\mathbb{P}^{n-m}, \mathcal{O}_{\mathbb{P}^{n-m}}(1)\big)$ for any fiber F of f, where $\dim X = n$ and $\dim Y = m$.

NOTATION 1.3. Let D_1 and D_2 be divisors on a smooth projective manifold X. We denote $D_1 \geq D_2$ if $D_1 - D_2$ is linearly equivalent to an effective divisor on X.

DEFINITION 1.4 (SEE [FK1]). (1) Let (X, L) be a quasi-polarized surface. Then (X, L) is called *L-minimal* if $LE > 0$ for any (-1)-curve E on X.

(2) For any quasi-polarized surface (X, L), there is a quasi-polarized surface (X_1, L_1) and a birational morphism $\mu: X \longrightarrow X_1$ such that $L = \mu^*(L_1)$ and (X_1, L_1) is L_1-minimal. Then we call (X_1, L_1) an *L-minimalization* of (X, L).

DEFINITION 1.5. (1) Let (X, L) and (X', L') be polarized manifolds and $\mu: X \longrightarrow X'$ a birational morphism. Then μ is called a *simple blowing up* if μ is a blowing up at one point on X' and $L = \mu^* L' - E$, where E is the μ-exceptional effective reduced divisor.

(2) Let (X, L) be a polarized manifold. Then (X, L) is called a *minimal reduction model* if (X, L) is not obtained by a finite number of simple blowing ups of another polarized manifold. If (X, L) is not a minimal reduction model, then there exist a smooth projective variety Y, an ample divisor A on Y, and a finite number of simple blowing ups $\mu: X \longrightarrow Y$ such that (Y, A) is a minimal reduction model. We call (Y, A) a minimal reduction of (X, L).

REMARK 1.5.1. Let (X, L) be a polarized manifold with $\dim X = n$ and let (Y, A) be a minimal reduction of (X, L). Then $h^0\Big(m\big(K_X + (n-1)L\big)\Big) = h^0\Big(m\big(K_Y + (n-1)A\big)\Big)$ for any natural number m.

THEOREM 1.6. *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that $K_X + (n-1)L$ is nef. If $K_X + (n-2)L$ is not nef, then (X, L) is one of the following types.*

a) (X, L) is obtained by a simple blowing up of another polarized manifold.

b0) (X, L) *is a Del Pezzo manifold with* $b_2(X) = 1$, $(\mathbb{P}^3, \mathcal{O}(j))$ *with* $j = 2$ *or* 3, $(\mathbb{P}^4, \mathcal{O}(2))$, *or a hyperquadric in* $\mathbb{P}^4$ *with* $L = \mathcal{O}(2)$.

b1) There is a fibration $\Phi: X \longrightarrow C$ *over a curve* C *with one of the following properties:*

b1-v) $(F, L_F) \cong (\mathbb{P}^2, \mathcal{O}(2))$ *for any fiber* F *of* Φ.

b1-q) Every fiber F *of* Φ *is an irreducible hyperquadric in* $\mathbb{P}^n$ *having only isolated singularities.*

b2) (X, L) *is a scroll over a smooth surface* S.

PROOF. See [Fj1] or [I].

LEMMA 1.7. *Let* (S, A) *be a quasi-polarized surface. Then the following are equivalent:*

(1) $h^0(K_S + A) = 0$.

(2) $h^0(m(K_S + A)) = 0$ *for any natural number* m.

PROOF (*cf.* PROPOSITION 3.5 IN [LP]). It is sufficient to prove that condition (1) implies (2). By Riemann-Roch Theorem, Serre duality, and Kawamata-Viehweg Vanishing Theorem, we obtain $h^0(K_S + A) = g(A) - q(S) + h^0(K_S)$. If $h^0(K_S + A) = 0$, then $g(A) = q(S) - h^0(K_S)$. If $\kappa(S) \geq 0$, then $q(S) - h^0(K_S) \leq 1$ and so we have $g(A) \leq 1$. But this is impossible since $\kappa(S) \geq 0$. Hence $\kappa(S) = -\infty$. Let (S_1, A_1) be an A-minimalization of (S, A) and let $\mu: S \longrightarrow S_1$ be its birational morphism. We remark that $A = (\mu)^*(A_1)$.

CLAIM 1.8. $h^0(m(K_S + A)) = h^0(m(K_{S_1} + A_1))$ *for any natural number* m.

PROOF.

$$\begin{aligned} h^0(m(K_S + A)) &= h^0(m(\mu^*(K_{S_1} + A_1) + E_\mu)) \\ &= h^0(m(K_{S_1} + A_1)), \end{aligned}$$

where E_μ is an effective μ-exceptional divisor such that $K_S = \mu^*(K_{S_1}) + E_\mu$. This completes the proof of Claim 1.8.

Assume that $h^0(K_S + A) = 0$. Then by Claim 1.8, we obtain $h^0(K_{S_1} + A_1) = 0$. On the other hand, by Riemann-Roch Theorem, Serre duality, and Kawamata-Viehweg Vanishing Theorem, we obtain $h^0(K_{S_1} + A_1) = g(A_1) - q(S_1)$ since $\kappa(S_1) = -\infty$. Since $h^0(K_{S_1} + A_1) = 0$, we have $g(A_1) = q(S_1)$. Hence (S_1, A_1) is isomorphic to $(\mathbb{P}^2, \mathcal{O}(r))$ for $r = 1$ or 2, or a scroll over a smooth curve by Theorem 3.1 in [Fk1].

(A) The case in which $(S_1, A_1) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$ for $r = 1$ or 2.

Then $K_{S_1} + A_1 = \mathcal{O}_{\mathbb{P}^2}(-3) + \mathcal{O}_{\mathbb{P}^2}(r) = \mathcal{O}_{\mathbb{P}^2}(r-3)$. Hence $h^0(m(K_{S_1} + A_1)) = 0$ for any natural number m since $r \leq 2$.

(B) The case in which (S_1, A_1) is a scroll over a smooth curve.

Let $\pi: S_1 \longrightarrow B$ be the $\mathbb{P}^1$-bundle, where B is a smooth curve. Let $\mathcal{E}$ be a locally free sheaf of rank 2 on B such that $\mathcal{E}$ is normalized and $S_1 = \mathbb{P}_B(\mathcal{E})$. Let C_0 be a section of π such that $C_0 \in |\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)|$, where $\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$ is the tautological line bundle on S_1, and let F_π be a fiber of π. We put $e = -C_0^2$. Then $A_1 \equiv C_0 + bF_\pi$, where $\equiv$ denotes the numerical equivalence and b is an integer. On the other hand, $K_{S_1} \equiv -2C_0 + (2g(C) - 2 - e)F_\pi$.

https://doi.org/10.4153/CMB-1998-039-9 Published online by Cambridge University Press

Hence $K_{S_1}+A_1 \equiv -C_0+(2g(C)-2-e+b)F_\pi$. If $\kappa(K_{S_1}+A_1) \geq 0$, then $(K_{S_1}+A_1)F_\pi \geq 0$ since F_π is nef. But $(K_{S_1}+A_1)F_\pi = -1$. So we obtain $\kappa(K_{S_1}+A_1) = -\infty$, that is, $h^0\big(m(K_{S_1}+A_1)\big) = 0$ for any natural number m. By Claim 1.8, this completes the proof of Lemma 1.7. ■

By Lemma 1.7, we can prove the following:

COROLLARY 1.9. *Let (X,L) be a quasi-polarized manifold. Assume that L is spanned. Then Conjecture NB is true.*

LEMMA 1.10. *Let L be a nef-big Cartier divisor on a normal projective variety X. Then*

(1) $H^i(X,-L) = 0$ for $i < \min\{\dim X, 2\}$,

(2) $H^i(X, K_X+L) = 0$ for $i > \max\{0, \dim \operatorname{Irr}(X)\}$,

where $\operatorname{Irr}(X)$ denotes the irrational locus of X.

PROOF. See Theorem 0.2.1 in [So2]. ■

DEFINITION 1.11. Let X be a normal projective variety. Let $r: X_r \longrightarrow X$ be a resolution of X. Then we say that the Albanese mapping is defined for X if there is a morphism $\beta: X \longrightarrow \operatorname{Alb}(X_r)$ such that $\alpha = \beta \circ r$, where $\operatorname{Alb}(X_r)$ denotes the Albanese variety of X_r and $\alpha: X_r \longrightarrow \operatorname{Alb}(X_r)$ is the Albanese map of X_r. In this case, β and $\operatorname{Alb}(X_r)$ are independent of the resolution of X.

LEMMA 1.12. *Let X be a normal projective variety and let X_r be a resolution of X. If $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{X_r})$, then the Albanese mapping is defined for X.*

PROOF. See Lemma 0.3.3 in [So2] or Lemma 2.4.1 and Remark 2.4.2 in [BS]. ■

2. **The case in which** $\dim X = 3$ **and** $h^0(L) \geq 2$.

THEOREM 2.1. *Let (X,L) be a quasi-polarized 3-fold with $h^0(L) \geq 2$. If K_X+L is nef, then $h^0(K_X+2L) > 0$.*

PROOF. Let $|M|$ be the movable part of $|L|$, and let Z be the fixed part of $|L|$. Let $\mu: X' \longrightarrow X$ be a birational morphism such that $\operatorname{Bs}|M'| = \phi$, where M' is the movable part of μ^*M. Let $L' = \mu^*L$.

Since $\operatorname{Bs}|M'| = \phi$, by Bertini's theorem, a general member D' of $|M'|$ is smooth. We remark that D' is not irreducible in general. Let S' be one irreducible component of D' such that $(L')^2S' > 0$. (We can take this S' since $(L')^2M' > 0$.) We also remark that $\mathcal{O}(mD')|_{S'} = \mathcal{O}(mS')|_{S'}$ for any natural number m because D' is smooth.

For any natural number m,

$$\begin{aligned} m(K_{S'}+L'|_{S'}) &= \big(m(K_{X'}+S'+L')\big)|_{S'} \\ &= \big(m(K_{X'}+D'+L')\big)|_{S'} \\ &= (\mu^*\big(m(K_X+L)\big)+mD'+mE_\mu)|_{S'}, \end{aligned}$$

https://doi.org/10.4153/CMB-1998-039-9 Published online by Cambridge University Press

where E_μ is an effective μ-exceptional divisor such that $K_{X'} = \mu^* K_X + E_\mu$.

By base point free theorem ([KMM]), $\operatorname{Bs}|m(K_X + L)| = \phi$ for some $m > 0$ because $K_X + L$ is nef. Hence $\operatorname{Bs}\left|\mu^*\bigl(m(K_X+L)\bigr)\right| = \phi$. Since $\operatorname{Bs}\left|\mu^*\bigl(m(K_X+L)\bigr)\right| = \phi$ and $\operatorname{Bs}|mD'| = \phi$, we obtain $h^0\Bigl(\mu^*\bigl(m(K_X+L)\bigr)|_{S'}\Bigr) > 0$, $h^0((mD')|_{S'}) > 0$, and $h^0((mE_\mu)|_{S'}) > 0$. Therefore $h^0\bigl(m(K_{S'} + L'|_{S'})\bigr) > 0$ for some $m > 0$.

We remark that $(S', L'|_{S'})$ is a quasi-polarized surface. Indeed the nefness of $L'|_{S'}$ is trivial, and $L'|_{S'}$ is big because $(L'|_{S'})^2 = (L')^2 S' > 0$.

By Lemma 1.7, we obtain $h^0(K_{S'} + L'|_{S'}) > 0$.

Next we consider the following exact sequence:

$$0 \longrightarrow H^0(K_{X'} + L') \longrightarrow H^0(K_{X'} + L' + S') \longrightarrow H^0(K_{S'} + L'|_{S'}) \longrightarrow H^1(K_{X'} + L').$$

By Kawamata-Viehweg vanishing Theorem, we have $h^1(K_{X'} + L') = 0$. Therefore $h^0(K_{X'} + L' + S') > 0$ since $h^0(K_{S'} + L'|_{S'}) > 0$. On the other hand,

$$\begin{aligned} K_{X'} + L' + S' &\le K_{X'} + L' + D' \\ &\le K_{X'} + 2L' \\ &= \mu^*(K_X + 2L) + E_\mu. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &< h^0(K_{X'} + L' + S') \\ &\le h^0\bigl(\mu^*(K_X + 2L) + E_\mu\bigr) \\ &= h^0(K_X + 2L). \end{aligned}$$

This completes the proof of Theorem 2.1. ∎

REMARK 2.2. By the same argument as the proof of Theorem 2.1, we can prove the following: Let (X, L) be a quasi-polarized manifold with $\dim X = n$ and $h^0(L) \ge 2$. Assume that Conjecture NB is true for any quasi-polarized manifold (Y, A) with $\dim Y = n-1$ and $h^0(A) > 0$. If $K_X + (n-2)L$ is nef, then $h^0\bigl(K_X + (n-1)L\bigr) > 0$.

THEOREM 2.3. *Let (X, L) be a polarized manifold with $\dim X = n$. If (X, L) is the type b0), b1), and b2) in Theorem 1.6, then $h^0\bigl(K_X + (n-1)L\bigr) > 0$.*

PROOF. We use Theorem 1.6 and its notations.

(b0-1) The case in which (X, L) is a Del Pezzo manifold: Then $K_X + (n-1)L \sim \mathcal{O}_X$ and $h^0\bigl(K_X + (n-1)L\bigr) = 1$.

(b0-2) The case in which $(X, L) = \bigl(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(j)\bigr)$ for $j = 2$ or 3: Then $K_X + 2L = \mathcal{O}_{\mathbb{P}^3}(-4) + 2\mathcal{O}_{\mathbb{P}^3}(j) = \mathcal{O}_{\mathbb{P}^3}(2j-4)$. Hence $h^0(K_X + 2L) \ge 1$ since $j = 2$ or 3.

(b0-3) The case in which X is a hyperquadric in $\mathbb{P}^4$ with $L = \mathcal{O}_X(2)$: Then $K_X + 2L = \mathcal{O}_X(-3) + 2\mathcal{O}_X(2) = \mathcal{O}_X(1)$. Therefore $h^0(K_X + 2L) > 0$.

(b0-4) The case in which $(X, L) = \bigl(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)\bigr)$: Then $K_X + 3L = \mathcal{O}_{\mathbb{P}^4}(-5) + 3\mathcal{O}_{\mathbb{P}^4}(2) = \mathcal{O}_{\mathbb{P}^4}(1)$. Hence $h^0(K_X + 3L) = 5$.

(b1-1) The case in which (X, L) is the type b1-v) in Theorem 1.6: (See (13.10) in [Fj0], or § 3 in [Is].) Let $H = K_X + 2L$. Then (X, H) is a scroll over a smooth curve C.

Let $\mathcal{E}$ be a locally free sheaf of rank 3 on C such that $X = \mathbb{P}_C(\mathcal{E})$ and $H = \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$, where $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is the tautological line bundle on $\mathbb{P}_C(\mathcal{E})$. Let $e = \deg \mathcal{E}$. Then $L = 2\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1) + \pi^*(D)$, where $\pi: X = \mathbb{P}_C(\mathcal{E}) \longrightarrow C$ is the natural projection and D is a divisor on C such that $\deg D = -g(C) + 1 - (e/2)$. In particular e is even. By the above construction,

$$h^0(K_X + 2L) = h^0\big(\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)\big) = h^0(\mathcal{E}).$$

By Riemann-Roch Theorem, we have

$$\begin{aligned} h^0(\mathcal{E}) &= h^1(\mathcal{E}) + 3\big(1 - g(C)\big) + e \\ &\geq 3\big(1 - g(C)\big) + e. \end{aligned}$$

(b1-1-1) The case in which $g(C) \geq 1$: We remark that

$$\begin{aligned} 0 < L^3 &= (2\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1) + \pi^* D)^3 \\ &= 8\big(\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)\big)^3 + 12\big(\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)\big)^2 \pi^* D \\ &= 8e - 12g(C) + 12 - 6e \\ &= 2e - 12g(C) + 12. \end{aligned}$$

Hence $e > 6g(C) - 6$.

Therefore

$$h^0(K_X + 2L) \geq 3\big(1 - g(C)\big) + e > 3\big(g(C) - 1\big) \geq 0.$$

(b1-1-2) The case in which $g(C) = 0$.

CLAIM 2.4. $e \geq 2$.

PROOF. We remark that $2K_X + 3L$ is nef in the case (b1-1). On the other hand,

$$\begin{aligned} 2K_X + 3L &= \big(2\pi^*(K_C + \det \mathcal{E}) + 3\pi^*(D)\big) \\ &= \pi^*(2K_C + 2\det \mathcal{E} + 3D). \end{aligned}$$

Since $2K_X + 3L$ is nef, we obtain that $\deg(2K_C + 2\det \mathcal{E} + 3D) \geq 0$. Hence $e \geq 2$ since $g(C) = 0$. This completes the proof of Claim 2.4.

Therefore

$$h^0(K_X + 2L) \geq 3\big(1 - g(C)\big) + e \geq 5.$$

(b1-2) The case in which (X, L) is the type b1-q) in Theorem 1.6: Let $f: X \longrightarrow C$ be the hyperquadric fibration, where C is a smooth curve. Then there is an embedding $\iota: X \longrightarrow \mathbb{P}_C(\mathcal{E})$ such that $\iota^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1) = L$, where $\mathcal{E} = f_*L$ is a locally free sheaf of rank n+1 and $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. Then X is a divisor on $\mathbb{P}_C(\mathcal{E})$ and is a member of $|2\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1) + \pi^*B|$, where $\pi: \mathbb{P}_C(\mathcal{E}) \longrightarrow C$ is the projection and $B \in \operatorname{Pic} C$. Then $K_X = -(n-1)L + f^*A$, where $A = K_C + \det \mathcal{E} + B$ (see (3.5) in [Fj2]). Let $e = \deg \mathcal{E}$ and $b = \deg B$.

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CLAIM 2.5. $e + b > 0$.

PROOF. By (3.3) in [Fj2], we obtain $2e + (n+1)b \geq 0$. By (3.4) in [Fj2], we have $2e + b > 0$. By these inequalities, we have $2e + 2b \geq 0$.

If $e+b = 0$, then $b < 0$ because $2e+b > 0$. But then $2e+(n+1)b = (2e+2b)+(n-1)b < 0$. This is a contradiction. Therefore $e + b > 0$. This completes the proof of Claim 2.5.

By Riemann-Roch Theorem,

$$h^0\big(-(\det \mathcal{E} + B)\big) = h^1\big(-(\det \mathcal{E} + B)\big) + 1 - g(C) + (-e - b).$$

By Claim 2.5, $h^0\big(-(\det \mathcal{E} + B)\big) = 0$.

Therefore by Serre duality,

$$h^0(K_C + \det \mathcal{E} + B) = g(C) - 1 + e + b.$$

On the other hand,

$$\begin{aligned}\big(K_X + (n-1)L\big)L^2 &= f^*(K_C + \det \mathcal{E} + B)L^2 \\ &= 2\big(2g(C) - 2 + e + b\big).\end{aligned}$$

CLAIM 2.6. $g(L) > g(C)$.

PROOF. Let $s = 2e + (n+1)b$. Then $s \geq 0$ by (3.3) in [Fj2]. On the other hand, we obtain $(n-1)d + s + 4ng(C) = 2n\big(g(L) + 1\big)$ by easy calculation, where $d = L^n$. Assume that $g(L) = g(C)$. Then $(n-1)d + s + 2ng(C) = 2n$. Since $K_X + (n-1)L$ is nef, we have $g(C) = g(L) \geq 1$. But this is a contradiction since $(n-1)d + s + 2ng(C) > 2n$. This completes the proof of Claim 2.6. ∎

By Claim 2.6, we obtain $2\big(2g(C) - 2 + e + b\big) = \big(K_X + (n-1)L\big)L^2 = 2g(L) - 2 > 2g(C) - 2$, and hence $g(C) - 1 + e + b > 0$.

Therefore

$$\begin{aligned}h^0\big(K_X + (n-1)L\big) &= h^0\big(f^*(K_C + \det \mathcal{E} + B)\big) \\ &= h^0(K_C + \det \mathcal{E} + B) \\ &= g(C) - 1 + e + b \\ &> 0.\end{aligned}$$

(b2) The case in which (X, L) is the type b2) in Theorem 1.6.

Let $\pi: X \longrightarrow S$ be the $\mathbb{P}^{n-2}$-bundle, where S is a smooth surface. Let $X = \mathbb{P}_S(\mathcal{E})$ such that $L = \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$, where $\mathcal{E}$ is a locally free sheaf of rank n-1 and $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ is the tautological line bundle of $\mathbb{P}_S(\mathcal{E})$. Then $\mathcal{E}$ is ample. By the canonical bundle formula, $K_X = \pi^*(K_S + \det \mathcal{E}) - (n-1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. Hence $K_X + (n-1)L = \pi^*(K_S + \det \mathcal{E})$ and we have

$$\begin{aligned}h^0\big(K_X + (n-1)L\big) &= h^0\big(\pi^*(K_S + \det \mathcal{E})\big) \\ &= h^0(K_S + \det \mathcal{E}).\end{aligned}$$

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Since $K_X + (n-1)L$ is nef, so is $K_S + \det E$. Hence $\kappa(K_S + \det E) \geq 0$ and so we obtain $h^0(K_S+\det E) > 0$ by Lemma 1.7 since $\det E$ is ample. Therefore $h^0\big(K_X+(n-1)L\big) > 0$. This completes the proof of Theorem 2.3. ∎

COROLLARY 2.7. *Let (X, L) be a polarized 3-fold with $h^0(L) \geq 2$. If $K_X + 2L$ is nef, then $h^0(K_X + 2L) > 0$.*

PROOF. (A) The case in which (X, L) is a minimal reduction model: If $K_X + L$ is nef, then $h^0(K_X + 2L) > 0$ by Theorem 2.1. If $K_X + L$ is not nef, then $h^0(K_X + 2L) > 0$ by Theorem 1.6 and Theorem 2.3.

(B) The case in which (X, L) is not a minimal reduction model: Let (Y, A) be a minimal reduction of (X, L) and let $\mu: X \longrightarrow Y$ be its morphism. Then $K_Y + 2A$ is nef because $K_X + 2L = \mu^*(K_Y + 2A)$ and $K_X + 2L$ is nef. But then $h^0(K_Y + 2A) > 0$ by the above case (A). Therefore $h^0(K_X + 2L) > 0$. This completes the proof of Corollary 2.7. ∎

REMARK 2.8. By the same argument as the proof of Corollary 2.7, we can prove the following (see Remark 2.2): Let (X, L) be a polarized manifold with $\dim X = n$ and $h^0(L) \geq 2$. Assume that Conjecture NB is true for any quasi-polarized manifold (Y, A) with $\dim Y = n - 1$ and $h^0(A) > 0$. If $K_X + (n-1)L$ is nef, then $h^0\big(K_X + (n-1)L\big) > 0$.

COROLLARY 2.9. *Let (X, L) be a polarized 3-fold with $h^0(L) \geq 2$. Then the following are equivalent:*

(1) $\Delta(L) = 0$ *or* (X, L) *is a scroll over a smooth curve.*
(2) $h^0\big(m(K_X + 2L)\big) = 0$ *for any natural number* m*.*
(3) $h^0(K_X + 2L) = 0$.
(4) $K_X + 2L$ *is not nef.*
(5) $K_X + 2L$ *is not semiample.*

Moreover if $h^0(L) \geq 3$, then the following is equivalent to the above;
(6) $g(L) = q(X)$.

PROOF. It is easy to prove that (1) ⇒ (6), (1) ⇒ (3), (1) ⇒ (2), (2) ⇒ (3), and (1) ⇔ (4) ⇔ (5) without the assumption that $h^0(L) \geq 2$. By Corollary 2.7, we obtain that (3) implies (4) if $h^0(L) \geq 2$. By Theorem 2.12 in [Fk3], we obtain that (6) implies (1) if $h^0(L) \geq 3$. This completes the proof of Corollary 2.9. ∎

3. The case in which $\operatorname{Bs}|L|$ is finite.

In this section, we consider the case in which $\operatorname{Bs}|L|$ is finite. First we fix the notations used later.

NOTATION 3.1. Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that $\operatorname{Bs}|L|$ is finite. Let $S_1 \in |L|$ be a general member. Then S_1 is a normal Gorenstein projective variety with $\dim S_1 = n - 1$ and $\dim \operatorname{Sing}(S_1) \leq 0$, where $\operatorname{Sing}(S_1)$ denotes the singular locus of S_1. We remark that the base locus of $L_1 = L_{S_1}$ is finite. For $i = 2, \ldots, n-2$, let $S_i \in |L_{i-1}|$ be a general member. Then S_i is a normal Gorenstein projective variety with $\dim S_i = n - i$ and $\dim \operatorname{Sing}(S_i) \leq 0$ by Bertini's Theorem, and the base locus of $L_i = L_{i-1}|_{S_i}$ is finite.

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We remark that (S_{n-2}, L_{n-2}) is a polarized surface, where S_{n-2} is a normal Gorenstein projective surface. Let $r: S'_{n-2} \longrightarrow S_{n-2}$ be a minimal resolution of S_{n-2} and $L'_{n-2} = r^*L_{n-2}$.

THEOREM 3.2. *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that $\mathrm{Bs}\,|L|$ is finite. Then $g(L) \geq q(X)$. If $g(L) = q(X)$, then (X, L) satisfies one of the following:*

(1) $\Delta(L) = 0$.

(2) (X, L) *is a scroll over a smooth curve.*

PROOF. We use Notation 3.1. First we have $g(L) = g(L_1) = \cdots = g(L_{n-2})$ by construction. By Lemma 1.10 (2) and Serre duality, we obtain that $q(X) = h^1(\mathcal{O}_{S_1}) = \cdots = h^1(\mathcal{O}_{S_{n-2}})$. On the other hand, $g(L_{n-2}) = g(L'_{n-2}) \geq q(S'_{n-2}) \geq h^1(\mathcal{O}_{S_{n-2}})$ since $h^0(L'_{n-2}) > 0$. Therefore $g(L) = g(L_{n-2}) \geq h^1(\mathcal{O}_{S_{n-2}}) = q(X)$.

Assume that $g(L) = q(X)$. If $q(X) = 0$, then $g(L) = 0$ implies $\Delta(L) = 0$ by Corollary 1 in [Fj1]. So we assume $q(X) \geq 1$. Then $g(L_{n-2}) = g(L'_{n-2}) = q(S'_{n-2}) = h^1(\mathcal{O}_{S_{n-2}}) \geq 1$ by the above inequalities. Hence by Lemma 1.12, the Albanese mapping is defined for S_{n-2}.

CLAIM 3.3. $\kappa(S'_{n-2}) = -\infty$.

PROOF. By the above inequalities, $g(L) = q(X)$ implies $g(L'_{n-2}) = q(S'_{n-2})$. On the other hand, since $\mathrm{Bs}\,|L_{n-2}|$ is finite, we have $h^0(L'_{n-2}) = h^0(L_{n-2}) \geq 2$. Hence $\kappa(S'_{n-2}) = -\infty$. This completes the proof of Claim 3.3. ∎

Since S'_{n-2} is a minimal resolution of S_{n-2} and L_{n-2} is ample, (S'_{n-2}, L'_{n-2}) is L'_{n-2}-minimal. So by Theorem 3.1 in [Fk1], (S'_{n-2}, L'_{n-2}) is a scroll over a smooth curve since $q(S'_{n-2}) \geq 1$.

CLAIM 3.4. *S_{n-2} is smooth.*

PROOF. Let $\pi: S'_{n-2} \longrightarrow B$ be the $\mathbb{P}^1$-bundle structure, where B is a smooth curve. Let $\mathcal{E}$ be a locally free sheaf of rank 2 on B such that $\mathcal{E}$ is normalized and $S'_{n-2} = \mathbb{P}_B(\mathcal{E})$. Let C_0 be a section of π such that $C_0 \in |\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)|$ and $e = -C_0^2$, where $\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$ is the tautological line bundle on S'_{n-2}. Then $K_{S'_{n-2}} \equiv -2C_0 + \big(2g(B) - 2 - e\big)F_\pi$, where F_π is a fiber of π and $\equiv$ denotes the numerical equivalence. We put $L'_{n-2} \equiv C_0 + bF_\pi$, where b is an integer.

(1) The case in which $e < 0$: Then L'_{n-2} is nef-big if and only if L'_{n-2} is ample. So L'_{n-2} is ample. But since $L'_{n-2} = r^*L_{n-2}$, we obtain $r = \mathrm{id}$, that is, S_{n-2} is smooth.

(2) The case in which $e \geq 0$: Then $b \geq e$ since L'_{n-2} is nef-big. If $b > e$, then L'_{n-2} is ample. So we obtain that S_{n-2} is smooth by the same argument as the case (1).

If $b = e$, then $L'_{n-2}C_0 = 0$. So C_0 is an r-exceptional curve. But if C_0 is contracted by r, then the Albanese mapping is not defined for S_{n-2} because C_0 is not contained in a fiber of π. This is a contradiction. This completes the proof of Claim 3.4. ∎

By Claim 3.4, (S_{n-2}, L_{n-2}) is scroll over a smooth curve since $g(L_{n-2}) = h^1(\mathcal{O}_{S_{n-2}}) \geq 1$ and $h^0(L_{n-2}) \geq 2$. Hence $K_{S_{n-2}} + L_{n-2}$ is not nef. Therefore $K_X + (n-1)L$ is not nef. By

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Theorem 2 in [Fj1], (X, L) is a scroll over a smooth curve since $q(X) \geq 1$. This completes the proof of Theorem 3.2. ■

THEOREM 3.5. *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$. *Assume that* $\mathrm{Bs}\,|L|$ *is finite. If* $K_X + (n-1)L$ *is nef, then* $h^0\big(K_X + (n-1)L\big) > 0$.

PROOF. We use Notation 3.1. Then S_{n-2} is a normal Gorenstein surface. By the Riemann-Roch Theorem for normal Gorenstein surfaces (see Theorem 0.6.2 in [So1]), Serre duality, and Lemma 1.10, we obtain that $h^0(K_{S_{n-2}} + L_{n-2}) = g(L_{n-2}) - h^1(\mathcal{O}_{S_{n-2}}) + h^0(K_{S_{n-2}})$. If $h^0(K_{S_{n-2}} + L_{n-2}) > 0$, then $h^0\big(K_X + (n-1)L\big) > 0$ is easily proved by Lemma 1.10. So we may assume $h^0(K_{S_{n-2}} + L_{n-2}) = 0$. Then $h^0(K_{S_{n-2}}) = 0$ since $h^0(L_{n-2}) \geq 2$. Hence by the above equality, $g(L_{n-2}) = h^1(\mathcal{O}_{S_{n-2}})$. Since $g(L) = g(L_{n-2})$ and $h^1(\mathcal{O}_{S_{n-2}}) = q(X)$, we obtain that $g(L) = q(X)$. By Theorem 3.2, $K_X + (n-1)L$ is not nef. But this contradicts the hypothesis. This completes the proof of Theorem 3.5. ■

By the above Theorems we can prove the following:

COROLLARY 3.6. *Let* (X, L) *be a polarized n-fold with* $\dim \mathrm{Bs}\,|L| = 0$. *Then the following are equivalent:*

(1) $g(L) = q(X)$.
(2) $\Delta(L) = 0$ *or* (X, L) *is a scroll over a smooth curve.*
(3) $h^0\Big(m\big(K_X + (n-1)L\big)\Big) = 0$ *for any natural number m.*
(4) $h^0\big(K_X + (n-1)L\big) = 0$.
(5) $K_X + (n-1)L$ *is not nef.*
(6) $K_X + (n-1)L$ *is not semiample.*

In fact, we can prove the following theorem.

THEOREM 3.7. *Let* (X, L) *be a quasi-polarized manifold. Assume that* $\mathrm{Bs}\,|L|$ *is finite. Then Conjecture NB is true.*

PROOF. We use Notation 3.1. Assume that $h^0\Big(m\big(K_X + (n-1)L\big)\Big) > 0$ for some $m \in \mathbb{N}$. By taking a general element $S_1 \in |L|$, we obtain $h^0\Big(m\big(K_{S_1} + (n-2)L_1\big)\Big) = h^0\Big(m\big(K_X + (n-1)L\big)|_{S_1}\Big) > 0$ since $\dim \mathrm{Bs}\,|L| = 0$. Since $\dim \mathrm{Bs}\,|L_{i-1}| \leq 0$, we obtain

$$h^0\Big(m\big(K_{S_i} + (n-i-1)L_i\big)\Big) = h^0\Big(m\big(K_{S_{i-1}} + (n-i)L_{i-1}\big)|_{S_i}\Big) > 0.$$

In particular, $h^0\big(m(K_{S_{n-2}} + L_{n-2})\big) > 0$.

We assume $h^0(K_{S_{n-2}} + L_{n-2}) = 0$. Since $h^0(L_{n-2}) > 0$, we have $h^0(K_{S_{n-2}}) = 0$. So we obtain $g(L_{n-2}) = h^1(\mathcal{O}_{S_{n-2}})$ by the same argument as the proof of Theorem 3.5. If $h^1(\mathcal{O}_{S_{n-2}}) = 0$, then $g(L_{n-2}) = 0$ and $g(L) = 0$. But then $\kappa\big(K_X + (n-1)L\big) = -\infty$. So $h^1(\mathcal{O}_{S_{n-2}}) > 0$. Since $h^0(L'_{n-2}) = h^0(L_{n-2}) \geq 2$, we have $g(L'_{n-2}) \geq h^1(\mathcal{O}_{S'_{n-2}})$. Therefore $h^1(\mathcal{O}_{S_{n-2}}) = h^1(\mathcal{O}_{S'_{n-2}}) = g(L'_{n-2}) = g(L_{n-2})$. Hence by Lemma 1.12, the Albanese

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mapping is defined for S_{n-2}. Moreover $\kappa(S'_{n-2}) = -\infty$ since $h^0(L'_{n-2}) \geq 2$ and $g(L'_{n-2}) = h^1(\mathcal{O}_{S'_{n-2}})$. Let $\alpha' \colon S'_{n-2} \longrightarrow B$ be the Albanese fibration of S'_{n-2} and let $\alpha \colon S_{n-2} \longrightarrow B$ be the Albanese fibration of S_{n-2} such that $\alpha' = \alpha \circ r$, where B is a smooth curve. Let $K_{S'_{n-2}} = r^*(K_{S_{n-2}}) - E_r$, where E_r is an r-exceptional effective divisor. Since $\alpha' = \alpha \circ r$, E_r is contained in a fiber of α'. Let $F_{\alpha'}$ be a general fiber of α' such that $F_{\alpha'} \cong r(F_{\alpha'})$. Since $g(L'_{n-2}) = h^1(\mathcal{O}_{S'_{n-2}}) \geq 1$ and $h^0(L'_{n-2}) \geq 2$, an L'_{n-2}-minimalization of (S'_{n-2}, L'_{n-2}) is a scroll over a smooth curve by Theorem 3.1 in [Fk1]. Hence $(K_{S'_{n-2}} + L'_{n-2})F_{\alpha'} = -1$. On the other hand, $(K_{S_{n-2}} + L_{n-2})F_\alpha = (K_{S'_{n-2}} + L'_{n-2})F_{\alpha'}$, where F_α is a general fiber of α. Hence $(K_{S_{n-2}} + L_{n-2})F_\alpha = -1$. Since F_α is nef, we have $\kappa(K_{S_{n-2}} + L_{n-2}) = -\infty$. But this is a contradiction since $h^0\big(m(K_{S_{n-2}} + L_{n-2})\big) > 0$.

Hence $h^0(K_{S_{n-2}} + L_{n-2}) > 0$. Therefore $h^0\big(K_X + (n-1)L\big) > 0$ by Lemma 1.10. ∎

By considering the above results and their proofs, I think that there is some relationship between $h^0\big(K_X + (n-1)L\big)$ and $g(L)$. So we propose the following conjecture:

CONJECTURE 3.8. *Let (X, L) be a polarized manifold with $\dim X = n$. Then $h^0\big(K_X + (n-1)L\big) \geq g(L) - q(X)$.*

This conjecture is true if one of the following is satisfied;

(1) $\dim \mathrm{Bs}\,|L| \leq 0$.

(2) $\dim X = 2$.

(3) (X, L) is a minimal reduction model and $K_X + (n-2)L$ is not nef.

We will study this conjecture in a future paper.

REFERENCES

[BS] M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties.* de Gruyter Expositions in Math. **16**, Walter de Gruyter, Berlin, New York.

[Fj0] T. Fujita, *Classification Theories of Polarized Varieties.* London Math. Soc. Lecture Note Series **155** (1990).

[Fj1] ______, *On polarized manifolds whose adjoint bundles are not semipositive.* Adv. Stud. Pure Math. **10**(1985), 167–178.

[Fj2] ______, *Classification of polarized manifolds of sectional genus two.* In: Algebraic Geometry and Commutative Algebra, in Honor of Masayoshi Nagata, Kinokuniya, 1987, 73–98.

[Fk1] Y. Fukuma, *A lower bound for the sectional genus of quasi-polarized surfaces.* Geom. Dedicata **64**(1997), 229–251.

[Fk2] ______, *A lower bound for sectional genus of quasi-polarized manifolds.* J. Math. Soc. Japan **49**(1997), 339–362.

[Fk3] ______, *On sectional genus of quasi-polarized* 3*-folds.* Trans. Amer. Math. Soc., to appear.

[I] P. Ionescu, *Generalized adjunction and applications.* Math. Proc. Cambridge Philos. Soc. **99**(1986), 457–472.

[Is] H. Ishihara, *On polarized manifolds of sectional genus three.* Kodai Math. J. **18**(1995), 328–343.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem.* Adv. Stud. Pure Math. **10**(1987), 283–360.

[LP] A. Lanteri and M. Palleschi, *About the adjunction process for polarized algebraic surfaces.* J. Reine Angew. Math. **352**(1984), 15–23.

[So1] A. J. Sommese, *Ample divisors on normal Gorenstein surfaces.* Abh. Math. Sem. Univ. Hamburg **55** (1985), 151–170.

[So2] ______, *On the adjunction theoretic structure of projective varieties.* In: Proc. Complex Analysis and Algebraic Geometry Conf. 1985, Lecture Notes in Math., Springer, 1986, 175–213.

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