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CUSP DENSITIES OF HYPERBOLIC 3-MANIFOLDS

COLIN ADAMS

Department of Mathematics, Williams College, Williamstown, MA 01267, USA (Colin.Adams@williams.edu)

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Abstract The cusp density of a hyperbolic 3-manifold is the ratio of the largest possible volume in a set of cusps with disjoint interiors to the volume in the manifold. It is known that all cusp densities fall in the interval $[0, 0.853\dots]$. It is shown that the cusp densities of finite-volume orientable hyperbolic 3-manifolds are dense in this interval.

Keywords: hyperbolic 3-manifold; cusp volume; hyperbolic volume; cusp density

AMS 2000 *Mathematics subject classification:* Primary 57M50

1. Introduction

By a hyperbolic 3-manifold, we will always mean an orientable 3-manifold M together with a covering map p from hyperbolic 3-space H^3 to M such that all of the covering translations are isometries. A finite-volume non-compact orientable hyperbolic 3-manifold is known to decompose along disjoint tori into a compact piece and a finite set of cusps. A cusp C in M has preimage $p^{-1}(C)$ consisting of a set of horoballs in H^3 with disjoint interiors. Topologically, a cusp is homeomorphic to $T^2 \times [0, \infty)$, possibly with some pairs of points identified on the boundary torus. A maximal cusp is a cusp contained in no other cusp. We think of obtaining a maximal cusp by expanding a given cusp until it touches itself on the boundary. Such a cusp will lift to a horoball packing in H^3, where some horoballs are tangent to one another.

If a manifold has more than one cusp, the maximal disjoint cusp volume, denoted $\mathrm{cv}(M)$, is the volume contained within a set of cusps in the manifold with disjoint interiors and with as large a volume as possible. The cusp density of a 3-manifold M, denoted $\mathrm{cd}(M)$, is the maximal disjoint cusp volume divided by the volume of the manifold. By horoball packing results of Böröczky, as applied by Meyerhoff (cf. [**4**]), it is known that the cusp density for any non-compact finite-volume hyperbolic 3-manifold or 3-orbifold must lie in the interval $(0, 0.853\dots]$, where $0.853\dots = \sqrt{3}/(2(1.01494\dots))$. The value $1.014\,94\dots$ is the Gieseking constant, which is the volume of a regular ideal hyperbolic tetrahedron. The upper bound on cusp density is realized by the figure-eight knot complement, which has maximal cusp volume $\sqrt{3}$ and volume $2.029\,88\dots$. The main result of this paper is to show that the set of cusp densities for hyperbolic 3-manifolds is dense in the interval $[0, 0.853\ldots]$.

If we take a set of hyperbolic manifolds with bounded volumes, the corresponding cusp densities cannot be dense. In fact, they are bounded away from 0, since in any n-cusped hyperbolic 3-manifold, there always exists a set of cusps with disjoint interiors that has volume at least $n(\sqrt{3}/2)$ (see [**2**]). So the cusp density of a manifold with n cusps and volume at most V is at least $n\sqrt{3}/2V$.

It remains to determine if the cusp densities of hyperbolic knot complements are dense in the interval. If not, perhaps one-cusped hyperbolic 3-manifolds have dense cusp densities. Or perhaps n-cusped hyperbolic 3-manifolds for some fixed n have dense cusp densities.

2. Results

We begin with a discussion of two manifolds that will be useful to us. As mentioned in the introduction, it is known that the maximum possible cusp density for a hyperbolic 3-manifold is $0.853\ldots$. The link complement L appearing in Figure 1a has density $0.853\ldots$ and is known as the minimally twisted 5-chain. It has recently appeared in a variety of contexts (for instance [**3**] and [**5**]). It is obtained by gluing together the faces of two regular ideal hyperbolic cubes as in Figure 1b, each of dihedral angle $\pi/3$.

Each cube can in turn be constructed by gluing a regular ideal tetrahedron to each face of a regular ideal tetrahedron. The five cusps appear at the ideal vertices as C_1, C_2, C_3, C_4, C_5. Any one of the five can be maximized to a cusp volume of $4\sqrt{3}$. The remaining four cusps, when maximized relative to it, each has volume $\sqrt{3}/4$. For the decomposition shown here, we can take C_1 to be maximized, meaning that in each cube there is a horoball centred at each of the four vertices labelled with C_1, and those four horoballs are mutually tangent in pairs. Horoballs centred at the remaining vertices are then expanded until they touch the three horoballs corresponding to C_1 at the adjacent vertices.

Although it appears that C_1 is playing a special role here, we could retriangulate the whole picture so any one of the other cusps appears eight times at the ideal vertices of the resulting pair of cubes.

Each component in L bounds an obvious twice-punctured disk (which can be considered as a thrice-punctured sphere). Incompressible boundary-incompressible thrice-punctured spheres such as these are known to be totally geodesic in the 3-manifold (see, for example, [**1**]).

There are an additional five thrice-punctured spheres in the link complement, each bounded by two adjacent link components and the component opposite to where they cross one another. The total of 10 thrice-punctured spheres appear in the cubical decomposition as the six faces of either cube together with four additional thrice-punctured spheres. Each of the last four is obtained by taking two ideal triangles, one inside each cube, such that the three vertices of the triangle are the three vertices of the cube that are adjacent on the cube to a vertex with three different edge labels.

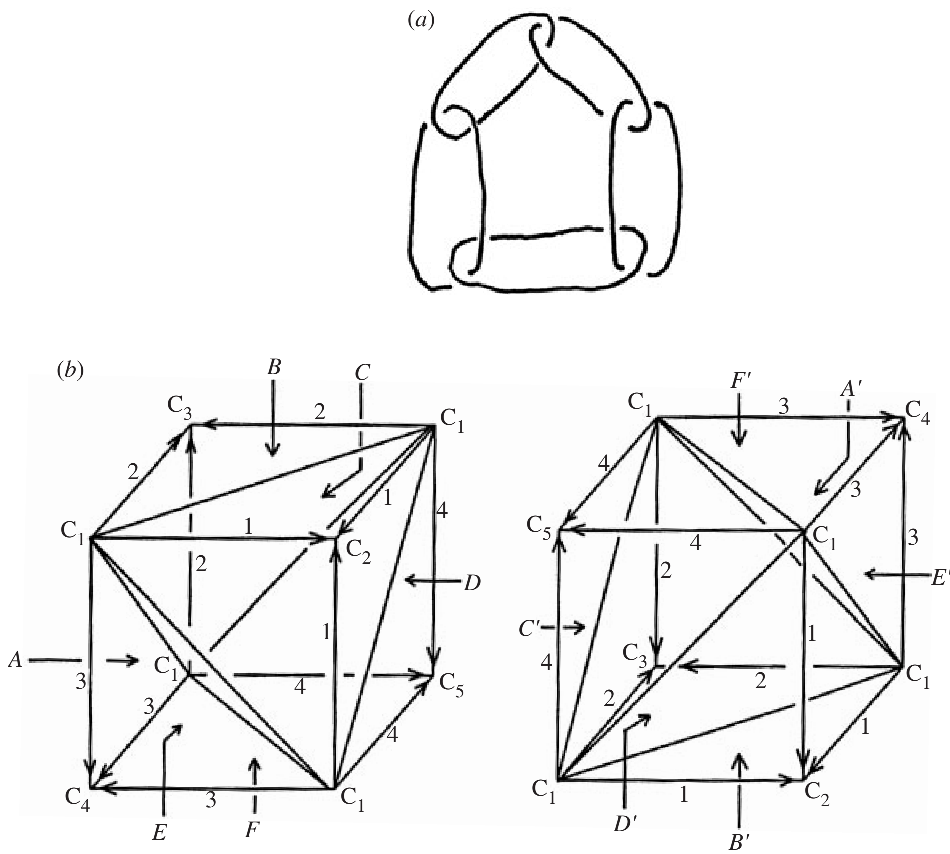

Figure 1. The minimally twisted 5-chain and the hyperbolic structure on its complement.

Choosing the twice-punctured disk D bounded by C_1, we see that it can be realized as a face which appears on each cube. When any one of the five cusps is maximized, it will intersect this face. The maximal cusp corresponding to C_1 is tangent to itself within the disk D. Therefore it restricts to a maximal cusp on the thrice-punctured sphere, which must have length exactly 4. So the longitude of any of the five maximal cusps must have length exactly 4. The two maximal cusps corresponding to the two components that puncture D intersect D only at the punctures. The remaining two cusps, when maximized, intersect D by poking through it away from its boundary, but each pokes through only a finite distance.

The second manifold that will be useful is the complement of the alternating daisy chain with n components, as appears in Figure 2a, denoted D_n. We will assume n even for convenience. The hyperbolic structure of the complement of D_n is realized by two drums as in Figure 2b, and was first discussed in [**6**, Chapter 6]. The dihedral angles α and β on the drums can be computed to be $\alpha = \arccos(\cos(\pi/n)/\sqrt{2})$ and $\beta = \pi - 2\alpha$.

Note that as n approaches infinity, the volume in a single maximal cusp approaches the volume in a single cusp in the Borromean rings, which is 4. We can see this from the

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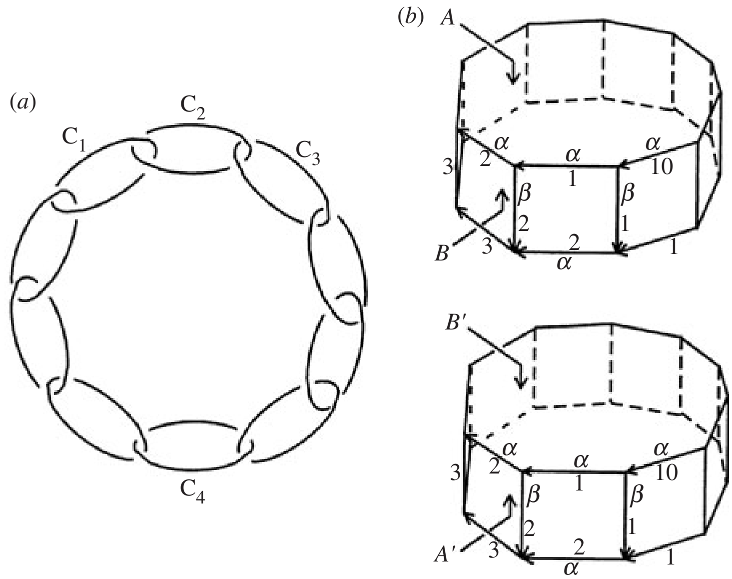

Figure 2. The daisy chain D_n (D_{10} depicted) and the hyperbolic structure on its complement.

symmetry δ that is rotation by $2\pi/n$ about an axis perpendicular to the plane of the page. If we take the quotient of the manifold by the subgroup of isometries generated by δ^2, the resulting orbifolds will have geometric structure approaching the geometric structure on the Borromean rings. As n increases, the single maximal cusp in D_n approaches the structure of a maximal cusp in the Borromean rings. It is necessary to use δ^2 rather than δ, as under the quotient by δ, the cusp is clasped with itself, unlike what happens in D_n. This causes the maximal cusp volume in the quotient, which is the Whitehead link, to be lower than the maximal cusp volume that the cusps in D_n are approaching.

The volume of the complements is approaching ∞ as n increases, by the fact that an n-cusped manifold has volume at least $n(1.014\,94\dots)$ (see [**2**]). So the ratio of the volume in any one cusp to the total volume approaches 0.

We will use these two manifolds to prove that cusp densities are dense in the interval $[0, 0.853\dots]$. The basic idea is to do high surgery on most of the cusps of D_n in order to obtain manifolds with arbitrarily low cusp density. A set of m of the resulting manifolds will be conjoined along twice-punctured spheres with a set of k manifolds with density $0.853\dots$ coming from the twisted 5-chains. By appropriate choice of m and k, we will be able to approach in cusp density any real value in the interval $[0, 0.853\dots]$. However, we must also account for the effect on cusp density of the surgery coefficients and for the interaction of the cusps from each side when the manifolds are glued together. This is achieved by replacing the two manifolds with appropriate covers, and choosing the parameters appropriately.

Theorem 2.1. *The set of all cusp densities for finite-volume orientable hyperbolic 3-manifolds is dense in the interval* $[0, 0.853\dots]$.

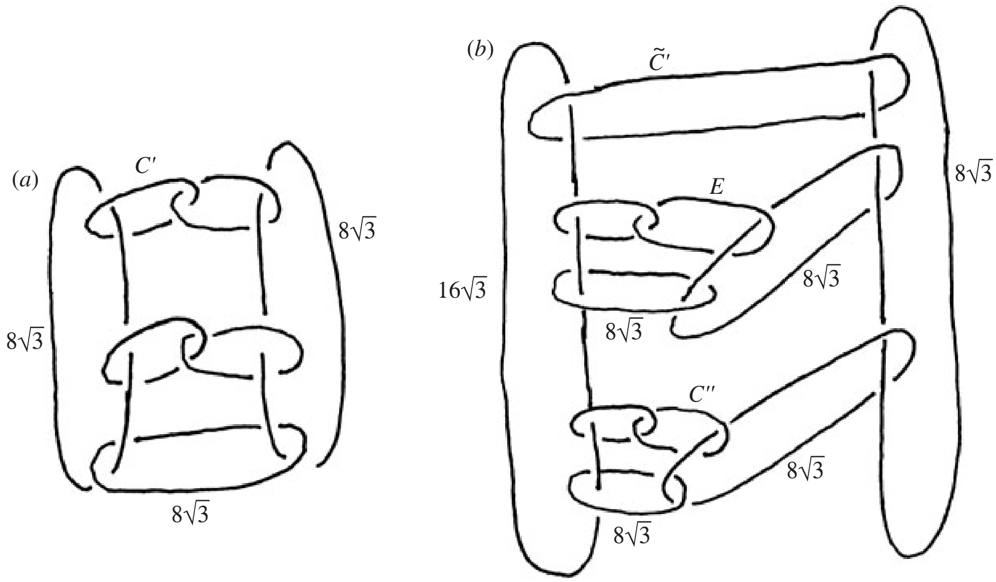

Figure 3. A double cover L' of L and a double cover L'' of L'.

Proof. Let L' be the 2-fold cyclic cover of L, as we unwind the meridian of the component C_1. The corresponding link complement appears in Figure 3*a*. Components are labelled with their individual maximal cusp volume, with unlabelled components having maximal cusp volume $4\sqrt{3}$. Let L'' be the link complement that results from taking the 2-fold cyclic cover of L' as we unwind the meridian of the component labelled C', as in Figure 3*b*.

Note that the twice-punctured disk D' bounded by the component E is not touched by any of the maximal cusps corresponding to the components adjacent to the component C''. Let L''_k be the link complement obtained by taking the k-fold cyclic cover of L'', unwinding a meridian of the component labelled C''. The resulting link complement has cusp density $\mathrm{cd}_0 = \sqrt{3}/(2(1.014\,94\ldots)) = 0.853\ldots$ and volume $4k(10.194\ldots)$. The lifts of the component E border twice-punctured disks which are intersected by exactly five maximal cusps, none of which have a volume greater than $16\sqrt{3}$.

For $n > 4$, let $\mathcal{C}$ be a set of four cusps in D_n, where C_1, C_2 and C_3 are three consecutive cusps in D_n and C_4 is the cusp that is opposite C_2 in D_n (see, for example, Figure 2*a*). Note that the cusp density of the set of cusps $\mathcal{C}$ in D_n goes to 0 as n goes to infinity. Also, the maximal cusp corresponding to C_4 does not intersect any of the three maximal cusps corresponding to C_1, C_2 and C_3. Let $D'_{n,m}$ be the link complement that is obtained by taking an m-fold cyclic cover of D_n, unwinding the meridian of C_4, as in Figure 4.

Let $\mathcal{C}'$ be the set of cusps that cover $\mathcal{C}$. The cusp density of $\mathcal{C}'$ in $D'_{n,m}$ is the same as the cusp density of $\mathcal{C}$ in D_n. This is because the sets of three maximal cusps that are adjacent to one another and that cover C_1, C_2 and C_3 do not touch the other sets of three maximal cusps covering C_1, C_2 and C_3, and they do not intersect the maximal

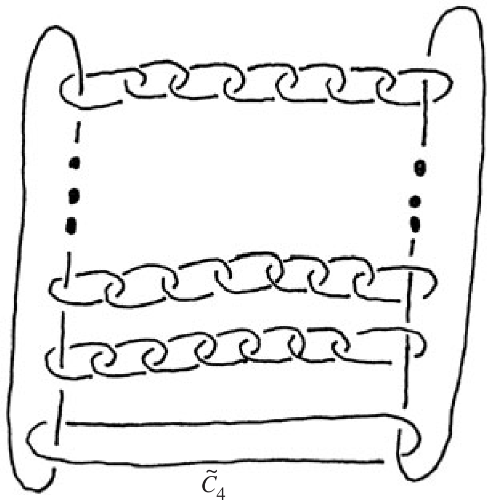

Figure 4. The m-fold cyclic cover of D_n.

cusp corresponding to the cover of C_4. So one cannot increase the cusp density of $\mathcal{C}'$ in $D'_{n,m}$ by adjusting these cusps.

Both of the link complements L''_k and $D'_{n,m}$ contain incompressible boundary-incompressible twice-punctured disks. Choose one in L''_k bounded by one of the components that covers E and one in $D'_{n,m}$ bounded by a cusp covering C_2. By [**1**], these manifolds can be cut open along these twice-punctured disks and then glued together along the copies of the resulting twice-punctured disks to obtain a manifold that has volume the sum of their volumes. Call this manifold $F_{k,n,m}$. Define $F_{k,n,m,p}$ to be the manifold obtained by doing $(1, p)$ surgery on all cusps of the manifold that come from $D'_{n,m}$ other than the ones in $\mathcal{C}'$. For large enough p, the resulting link complement will have cusp density arbitrarily close to the cusp density of the subset of cusps in $F_{k,n,m}$ on which no surgery was performed. So it is enough to show that the cusp density of those cusps in $F_{k,n,m}$ are dense in the interval. Define the restricted cusp density of $F_{k,n,m}$, denoted $\mathrm{cd}_R(F_{k,n,m})$, to be the cusp density restricted to these particular cusps, which is to say the cusps coming from L''_k together with the cusps coming from $\mathcal{C}'$.

First note that by taking a sequence of values with k and m fixed and n approaching ∞, we obtain a set of manifolds with restricted cusp density approaching 0. This occurs because the volume of the manifolds is approaching infinity but the number of cusps in the restricted set is bounded by $8k + 3m - 2$ and all cusps have bounded volume.

Now let x be a value in $(0, 0.853\ldots]$. We will find manifolds with restricted cusp densities arbitrarily close to x. Choose n large enough that the cusp density of $\mathcal{C}$ in D_n is less than x, and that the maximal cusp volume in each cusp is within 0.1 of 4. We will fix n at this value. When we form $F_{k,n,m}$, other than the five cusps in L''_k that touch the twice-punctured disk we are gluing along, the other cusps of L''_k need not be shrunk out

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of the way. Since these other cusps correspond to a density of $0.853\ldots$, one could not increase the overall cusp density of the resulting manifold by allowing some to grow and others to shrink.

Then for some p such that $0 \leqslant p < 81.6$,

$$\begin{aligned}\operatorname{cd}_R(F_{k,n,m}) &\\ &= \frac{k\operatorname{cv}(L'') + m\operatorname{cv}_R(D_n) - p}{k\operatorname{vol}(L'') + m\operatorname{vol}(D_n)}\\ &= \frac{k\operatorname{cv}(L'')}{k\operatorname{vol}(L'') + m\operatorname{vol}(D_n)} + \frac{m\operatorname{cv}_R(D_n)}{k\operatorname{vol}(L'') + m\operatorname{vol}(D_n)} - \frac{p}{k\operatorname{vol}(L'') + m\operatorname{vol}(D_n)}.\end{aligned}$$

The decrease in cusp volume due to having to match up the cusp sizes when gluing the two manifolds together is the negative contribution of p. Because of the way we have set the manifolds up, at most five cusps from L''_k might need to be shrunk, two of maximal cusp volume $4\sqrt{3}$, two of maximal cusp volume $8\sqrt{3}$ and one of maximal cusp volume $16\sqrt{3}$. At most three cusps from the restricted set in $D'_{n,m}$ might need to be shrunk, all of which have maximal cusp volume within 0.1 of 4. Hence, we may assume that $p < 81.6$. By making sure that our choice of $k + m$ is large enough, we can assume that the last term is arbitrarily small, so we will represent it by ϵ:

$$\operatorname{cd}_R(F_{k,n,m}) = \frac{(k/m)\operatorname{cv}(L'')}{(k/m)\operatorname{vol}(L'') + \operatorname{vol}(D_n)} + \frac{\operatorname{cv}_R(D_n)}{(k/m)\operatorname{vol}(L'') + \operatorname{vol}(D_n)} - \epsilon.$$

Replacing k/m by t, we can write this expression as

$$f(t) = \frac{t\operatorname{cv}(L'')}{t\operatorname{vol}(L'') + \operatorname{vol}(D_n)} + \frac{\operatorname{cv}_R(D_n)}{t\operatorname{vol}(L'') + \operatorname{vol}(D_n)} - \epsilon.$$

Note that as m gets large relative to k, t approaches 0, so $f(t)$ approaches $\operatorname{cd}_R(D_n)$, which is less than x. As k gets large relative to m, t goes to ∞, so $f(t)$ approaches $\operatorname{cd}(L'') = 0.853\ldots$.

Since $f(t)$ is a continuous function taking values from $\operatorname{cd}_R(D_n)$ to $\operatorname{cd}(L'')$, there exists a t value such that $f(t) = x$. We can choose positive integers a and b such that a/b is arbitrarily close to t. Letting $k = ca$ and $m = cb$, where c is a large enough positive integer, will ensure that $k + m$ is large enough and that k/m is close enough to t to cause the density of the resulting manifold to be arbitrarily close to x. □

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