J. Austral. Math. Soc. **21** (Series A) (1976), 277.

A SHORT PROOF OF A RECENT THEOREM OF G. SZEKERES

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(Received 3 July 1974)

Communicated by G. Szekeres

The purpose of this note is to apply the work of Kasteleyn (1967) on the enumeration of 1-factors of a graph to derive a quick proof of a theorem of Szekeres (1973). In the following, G is understood to be a finite, directed graph. If u, v are adjacent vertices of G, we denote by (u, v) an edge of G directed from u to v. Let $\{f_1, f_2, \cdots, f_m\}$ be a set of 1–factors of G, and for all i write

$$f_i = \{(u_{i1}, v_{i1}), (u_{i2}, v_{i2}), \cdots, (u_{in}, v_{in})\}$$

where n is half the number of vertices of G. (Here we regard a 1-factor as a set of edges). Then Kasteleyn associates with f_i a plus sign if $u_{i1} v_{i1} u_{i2} v_{i2} \cdots u_{in} v_{in}$ is an even permutation of $u_{11} v_{11} u_{12} v_{12} \cdots u_{1n} v_{1n}$, and a minus sign otherwise. The symmetric difference of two 1-factors is a collection of circuits, called alternating circuits. An alternating circuit of G is said to be clockwise even if the number of its edges that are directed in agreement with the clockwise sense is even; otherwise it is clockwise odd. Since the length of any alternating circuit is even, these definitions are not dependent on the sense designated as clockwise. It follows from the work of Kasteleyn that two given 1-factors in a directed graph agree in sign if and only if the number of clockwise even alternating circuits in their symmetric difference is even.

Now let $G = \{G_1, G_2, G_3\}$ be a Tait graph in the sense of Szekeres (1973). Regarding G as a trivalent graph with a Tait colouring, we see trivially that G can be oriented so that G_1, G_2 and G_3 agree in sign. Given such an orientation of G, it follows from Kasteleyn's work that the number k of clockwise odd circuits of G has the same parity as the number $\sigma(G)$ of circuits of G. On the other hand, it is easily seen that k has the same parity as the orientation index $\omega(G)$ of G. Hence we have Szekeres's result that $\omega(G) \equiv \sigma(G) \bmod 2$.

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